Introduction to My Annotated Bibliography on the Topic of 'Longevity Risk and Portfolio Sustainability'

The following document is primarily intended to act as a scholarly reference source. Over the last fifty years, there has been an accretion of research on this topic from scholars and practitioners in diverse fields:

<u>Actuaries</u> are interested in the factors that determine pricing of contracts promising lifetime income;

<u>Financial Economists</u> are interested in building models that reflect factors determining the evolution of retirement portfolios under the stress of expenses and withdrawals, and on using models to optimize outcomes expressed in both dollar-wealth and utility terms;

<u>Investment Advisors</u> are interested in how best to advise clients on a variety of retirement and intergenerational wealth management issues;

<u>Trustees</u> charged with providing lifetime income to current beneficiaries and terminal wealth to remaindermen are interested in how to discharge prudently and impartially their fiduciary duties; and,

<u>Investors</u> are interested in how much money they can safely spend or bequeath from their retirement portfolio.

Not only is the volume of research vast, but the range of publications reflects pedagogy from academic fields that traditionally have had little overlap in readership. It is both interesting and beneficial to examine, from the perspectives of various professions, what questions are raised, how hypotheses are formulated and tested, and why strategies designed to secure a safe, substantial and sustainable income are in some cases recommended or, in others, rejected.

From time-to-time, a seemingly "new" idea pops up in a journal aimed towards one audience when, in fact, the idea has been well developed in a journal written for an entirely different audience. Indeed, in some cases, the "new" idea was anticipated in journals published a decade or more earlier. It is a rare idea that, like Botticelli's Venus, springs forward fully developed *ab initio*. Indeed, tracing through the historical scholarship record can be particularly annoying if one encounters individuals with the effrontery to have published our unique and proprietary ideas before us. Furthermore, important papers may suffer neglect simply because they appear in publications that are off the beaten track to many potential readers. Articles on longevity risk management in actuarial journals, for example, are rarely read by attorneys providing opinions on the prudence of trust portfolio administration despite the fact that Grantors may direct trustees to provide lifetime income to a trust's beneficiary.

Also of interest is the chronology of advice, strategies and solutions. Although the annotated bibliography does not rise to a narrative intellectual history—it was never intended to do so—nevertheless, if read from beginning to end, it traces the introduction and development of important ideas and research methods. The bibliography lacks a topical index because it seeks to provide a sense of how ideas and insights from a variety of professions developed, over time, and, perhaps independently from each another. Chronological presentation substitutes for a narrative history of ideas.

This observation leads straight away to a primary motivation for creating the annotated bibliography. It is both difficult and time consuming to survey the relevant source materials. Research, however, demands that the scholar know the work of those who have preceded him. This bibliography should facilitate what has now become a herculean task of gathering and investigating published studies and commentary. This said, the research and publication vistas are vast, and no representation can be made regarding the completeness of this bibliography except that it is incomplete. One is reminded of the effort needed to survey the relevant literature on, say, Shakespeare or Napoleon—absolute comprehensiveness would require a lifetime and more. It is hoped that this bibliography will assist those with the temerity to put pen to paper to understand the nature and scope of previous inquiry and investigation.

Many of the bibliographical entries summarize complex models in which the solution path aims towards maximization of a utility function. Summaries of normative articles, sometimes only slightly less mathematically complex, also appear. These articles provide insights in the areas of asset allocation, spending, and other financial strategies under a variety of assumptions such as complete markets, log-normal probability distributions of financial asset returns, constant relative risk aversion, and so forth. Finally, a generous sampling of practitioner-oriented articles appears.

A bird's-eye look at the literature reveals a plethora of different research methods, modeling assumptions, and portfolio allocation/spending preferencing criteria, all of which may produce significantly different outputs even given the same empirical data. Conclusions are subject to model risk; and, from time-to-time, practitioners may translate the output from academic model building—an exercise designed to explore quantitative relationships among variables of interest—into prescriptive statements for investors. But the mathematical assumptions required for tractable model building often diverge in both their character and form from common investor utility functions or from the process underlying the distribution of empirical asset price evolutions. Conversely, from time-to-time, practitioner-oriented articles may resort to pure empiricism in an attempt to parse historical return evolutions to find patterns which can be turned into rules for safe and sustainable portfolio withdrawals.

Of course, taken to an extreme, this is mere data mining. Hence, the final purpose of this bibliography. Conclusions are a function of the research methodology employed by the investigator; and, the intelligent advisor realizes critical distinctions between the investigative/academic outcomes and the needs, goals, circumstances, and purposes of an investor. One benefit of reviewing historical research is to recognize that investors should not be confused with models.

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Patrick Collins

Longevity Risk and Portfolio Sustainability

Chronological Summary of Articles [1965 – 2014]

DATE	TITLE / AUTHOR(S)	THESIS	COMMENTS
1965	"Uncertain Lifetime, Life Insurance, and the Theory of the Consumer," Menahem E. Yaari The Review of Economic Studies Vol. 32, No.2. pp. 137 – 150.	The article is an early analysis of optimization of discounted expected utility under the conditions of an uncertain life span. Investors lacking a bequest objective will maximize a utility function for consumption only (c) where the function has a positive first and negative second derivative: $$V(c) = \int_0^T a(t)g[c(t)]dt$$ Where' a' is the subjective discount rate—i.e., the investor's time preference rate. For investors with a bequest objective, the function to be maximized is: $$U(c) = \int_0^T a(t)g[c(t)]dt + \beta(T)\varphi[S(T)]$$ Where Beta is a subjective weighting function for bequest 'S' that is itself a function of the time of the bequest. The problem is to find the optimal feasible consumption plan when the planning horizon is uncertain. Yaari notes that the feasibility problem can be solved either through a "chance-constrained programming" methodology or through a "penalty function" procedure. "The chance-constrained programming approach requires that the constraint (in this case the wealth constraint) be met with probability λ or more, where λ is some number fixed in advance, say .95….This approach is, of course very common in statistics where many	If the choice of a value for λ in a chance-constrained programming approach is to be optimized, then the coefficient of success (success = likelihood of portfolio sustainability throughout the remainder of life) involves a tradeoff between "safety-first" (maximize safety by putting all assets into a risk-free investment) and opportunity for future growth of the portfolio above the risk-free rate. Is maximizing expected utility inconsistent with minimizing the risk of ruin? Prudence is something more than selecting the lowest failure rate probability. See later articles reconciling shortfall risk metric with traditional utility preferencing metric. For example, "Annuities vs. Safe Withdrawal Rates: Comparing Floor-with-Upside Approaches Michael Kitces [2012]. Under the penalty function approach, penetration of a floor value equal to funding for a minimum standard of living may produce disutility approaching infinity. In such cases a shortfall risk metric converges to a utility-based risk metric. If an uncertain lifespan results in future cash flows being more heavily discounted, a smoothed consumption pattern may not be preferred over a front-loaded pattern. Or, a smoothed consumption

		of the standard tests are based on the idea of maximizing some criterion subject to the constraint that the probability of type I error be less than, say .05." However, Yaari points out that the choice of λ is itself a decision problem: '…one might want to choose λ optimally rather than arbitrarily." Note: the decision problem was articulated by I. Fisher who advanced the proposition that income early in retirement had greater utility than the equivalent inflation-adjusted income in late retirement. See "The 7 most important equations for your retirement—Milevsky (2012)]. With the penalty function approach the consumer himself guards against violating the wealth constraint "…by assuming that a violation of the constraint carries a penalty, i.e., a loss of utility." Yaari considers several economies where annuities may or may not be available to the investor. A central point is that the optimal utility-of-consumption plan contains a term for survival probability. The uncertainty of future survival means that the future is discounted more heavily. He demonstrates that absent a bequest objective in a (complete) market with fair-valued annuities available to the consumer, all wealth will be held in an "actuarial note" that is equivalent to an annuity and which pays a rate of return equal to the commercial interest rate plus an extra factor reflecting the fact that the investor's estate must forfeit further income at death (Terminal Portfolio value = $0). "…positive assets will always be held in the form of actuarial notes." [Note: the "extra factor" is the mortality premium offered by annuity contracts] When a bequest motive exists, the investor's problem is to solve for optimization of two decision values: a feasible consumption plan and a feasible "saving plan." Assuming no labor income, the consumption plan will be funded entirely with annuities while the saving plan is a function of available investment returns. Ideally, the marginal utility of the consumption plan will exactly equal that of the saving plan. Yaari makes	plan (smooth = constant marginal utility of consumption) may lead to a front-end loaded retirement income stream. This is the "Fisher Utility" argument—probability of failure at an advanced age is not as onerous as at an early age. The case of the irrevocable family trust is akin to a case where annuities are available *and* there exists a positive bequest motive. Is it prudent to fund the current beneficiary's income right with an annuity and the remaindermen's share with an investment portfolio? Note: The opportunity for an improved future budget constraint may be offset by a higher discount rate for future income. This constitutes and important rationale for a portfolio monitoring program that reflects the preferences of the investor rather than performance relative to an outside benchmark. This is a variation on the 2-fund [Tobin] trust solution where consumption is financed by an annuity and the remainder interest by an investment fund. The duty of impartiality is defined in terms of equalizing marginal utility between beneficiary classes.

		the important observation that when annuities are available, "…the consumer can separate the consumption decision from the bequest decision." In the absence of annuities, such a separation is not possible.	
1977	"Savings and Consumption with an Uncertain Horizon," David Levhari and Leonard J. Mirman, Journal of Political Economy Vol. 85, No. 2 (1977), pp. 265 - 281	The Levhari/Mirman article demonstrates that a (mean-preserving) change in the "distribution of lifetime uncertainty" may motivate either a decrease in current consumption in the face of a higher probability of a longer life; or, an increase in consumption because sure current consumption is preferred over a more uncertain probability for future spending. The effect of uncertainty on an individual investor depends on which motivation is stronger: "…if uncertainty of lifetime is considered as part of a Fisherian lifetime optimization problem, with a risk-averse consumer, consumption in the form of an uncertain lifetime has the same effect as increasing the rate of discount. This is one of the more important results found in the contribution of Yaari (1965)." The risk-averse investor, confronted by an uncertain life span, may elect to consume more in the present. The article points out that Champernowne [Uncertainty and Estimation in Economics Vol. 3 (Holden-Day), 1969—not included in this bibliography] advances the opposite hypothesis by stressing the precautionary savings aspect of optimization in the face of uncertainty in the distribution of mortality: "…'the effect of not knowing when' the consumer 'will die is to lower the initial value of consumption.'" Hence, as Levhari & Mirman point out: "The desire to provide for a longer life together with the desire for more certainty by consuming now pull in opposite directions." Levhari and Mirman compare two individuals each having the same tastes and attributes except with respect to the distribution of lifetime—i.e., the survival distribution of one investor is more risky than the other. Their model assumes constant relative risk aversion: $u(x) = x^{1-\gamma}/(1-\gamma)$ with $u(x) = \log x$ for $\gamma = 1$. Not surprisingly, it suggests that consumption decisions depend heavily on the utility function, investment returns, and the investor's subjective discounting (impatience-to-consume). Although the model can incorporate earned income, its primary focus is on the retirement period beginning at time 't.'. Investor wealth [W] at time	It is interesting to compare and contrast Levhari and Mirman's analysis of spending from their utility optimization model reflecting a dynamic programming approach, to prescriptive recommendation such as the 4% spending rule. The Levhari & Mirman study emphasizes how utility-optimizing retirement spending decisions are dynamic and reflect information regarding time horizon, returns, risk preferences, etc. By contrast, many empirical "rules" assume that the myopic investor selects a "safe" spending level and maintains the level under the assumption that past results act as a condition precedent for future periods (economies). A choice among approaches—mathematical modeling approaches include (linear and nonlinear) dynamic programming, maximum likelihood, optimum (stochastic) control theory, etc.; empirical/numerical approaches include data mining of time series, simulation, etc.—often depends on (1) tractability, and (2) the structural characteristics of the problem—e.g., dimensionality, and other characteristics of both the 'problem' and the model selected to represent the 'problem.' The 2012 article by Huang, Milevsky and Salisbury ["Optimal retirement consumption with a stochastic force of mortality" Insurance: Mathematics and Economics, Vol 51, pp. 282-291]] extends the work of Levhari and Mirman and James Davies ["Uncertain Lifetime, Consumption, and Dissaving in Retirement," James B. Davies, Journal of Political

		‘t+1’ equals $(W_t – C_t)r_t$ where ‘r’ = return on investments and ‘C’ = Consumption during period ‘t.’ Each investor’s goal is to maximize a Von-Neumann-Morgenstern expected (additive and separable) utility function with respect to lifetime consumption where consumption during any future period ‘t’ is subject to a discount factor reflective of each investor’s personal impatience to consume. Adding a term for the probability of survival [P], the mathematical expression for maximizing the lifetime stream of expected utility-of-consumption becomes: $E_r = \sum_{t=0}^{T} a^t P_t u(C_t)$ Using dynamic programming methods, the model indicates that consumption at time 0 is: $C_0(W) = \frac{W}{\sum_{i=0}^{T} P_i^{\frac{1}{\gamma}} (ar^{1-\gamma})^{\frac{i}{\gamma}}}$ If ‘r’ is random, the term ‘ar’ in the above equation’s denominator becomes ‘aE(r).’ Holding all else equal, an increase in risk aversion (γ) decreases consumption in the initial period: “In other words, the individual, being more sensitive to the possibility of lower consumption in the future, saves more in the initial period.” However, the primary goal is to consider an investor who faces an uncertain lifetime. In order to isolate the influence of uncertainty (the “riskiness of life”), the model assumes a nonrandom return. The central question is whether consumption is an increasing or decreasing function of uncertainty in life span. It turns out, under very restrictive conditions on available return and subjective discounting, that key factor are the mortality probability distribution [P] and the consumption risk aversion factor 1/γ. When γ > 1, a riskier distribution of life span reduced current consumption; when 0	Economy Vol. 89, No.3 (1981), pp. 561 – 577] by incorporating stochastic morality into the consumption utility-based model.

		$< \gamma < 1$, current consumption increases. Under more realistic conditions, "…a straightforward relationship between riskiness and optimal consumption does not exist…." In some cases, uncertainty elicits greater consumption; in other cases, greater savings. For example, a high return on available savings motivates less current consumption; a high utility discounting factor motivates greater current consumption: "In other words a small rate of return combined with a riskier horizon will increase consumption." The article concludes by considering an uncertain rate of return. The model preserves the return's mean but increases its variance while keeping lifetime uncertainty unchanged. The effect on consumption depends on γ. If $\gamma > 1$, $r^{1-\gamma}$ is a convex function of r and hence will reduce consumption. If $0 < \gamma < 1$, then $r^{1-\gamma}$ is a concave function of r and "…increases the proportion of wealth consumed."	
1977	"Mean-Risk Analysis with Risk Associated with Below-Target Returns," Peter C. Fishburn The American Economic Review, Vol. 67, No. 2 (March, 1977), pp. 116 – 126.	Fishburn reviews a number of preferencing models in the face of uncertain outcomes. He contrasts and compares "mean – risk trade off" and "mean-risk dominance" models to a "mean-risk utility" model. Although there is a substantial body of research using mean-variance or mean-semivariance models a la Markowitz, Fishburn believes that such modeling does not always capture the risk attitudes of investors. The reason for this belief is two-fold: 1. Mean-variance analysis "…should not be taken very seriously unless the probability distributions used in the analysis satisfy certain restrictions."—i.e., distributions should be IID normal; and, 2. Investors "…very frequently associate risk with failure to attain a target return. To the extent that this contention is correct, it casts serious doubt on variance—or, for that matter, on any measure of dispersion taken with respect to a parameter(for example, mean) which changes from distribution to distribution—as a suitable measure of risk." Thus, Fishburn is interested in developing mathematical expressions for	An early and important survey of the mathematical approaches commonly found in the literature of investment decision making. Fishburn provides examples of how the form of the utility function can reflect both a wide value of risk-aversion parameters—as opposed to the assumption of quadratic risk aversion in Markowitz—and can accommodate a shortfall risk measure.

		a variety of models that are based on either mean/variance, or loss probability/shortfall magnitudes, or expected utility including stochastic dominance models. The specific model of interest is the α-t model where α is a risk aversion parameter and t is an investment target wealth level or rate of return. The Markowitz model is a special case where the exponent α takes on a value of 2 (quadratic utility) and t is the investor's optimal portfolio location on the efficient frontier. Fishburn asserts that generalized α-t models can accommodate a range of risk aversion parameters and specific target returns or reference levels. Given a cumulative probability distribution function where F(x) specifies the probability of a return not exceeding x (the area under the graph of the function to the left of x), the mathematical form of such a mean-risk dominance model is a probability-weighted function of investment results below the specified target return t: $$F_\alpha(t) = \int_{-\infty}^{t} (t-x)^\alpha dF(x)$$ Such a flexible α-t model, where α can accommodate many attitudes towards risk and t can accommodate the investor's desire to avoid unacceptable shortfall probabilities and magnitudes, is compatible with both stochastic dominance models and with von Neumann-Morgenstern utility-based models. He observes: "The idea of a mean-risk dominance model in which risk is measured by probability-weighted dispersions below a target seems rather appealing since it recognizes the desire to come out well in the long run while avoiding potentially disastrous setbacks or embarrassing failures to perform up to standard in the short run." The mean-risk dominance model expresses the investor's preference criterion as follows: "F dominates G if an only if μ(F) ≥ μ(G) and ρ(F) ≤ ρ(G) with at least one strict inequality. Rho of F is defined by: $$\rho(F) = \int_{-\infty}^{t} \varphi(t-x) dF(x)$$	

		In which $\varphi(y)$ for $y \geq 0$ is a nonnegative nondecreasing function in y with $\varphi(0) = 0$ that expresses the 'riskiness' of getting a return that is y units below the target." Fishburn develops several theorems including: • $P(\alpha, t)$ is completely determined by expected returns whenever all possible returns in both F and G lie at or above the target t. • If two distributions have the same expected mean and one distribution is sure to provide a return at or above target while the other has a positive probability of generating a below-target return, then a risk-averse investor will prefer the sure thing. Gambling behavior is revealed whenever all returns from both distributions are at or below the target. However, for various values of the risk aversion parameter α, the below-target investor may either become conservative—lest a bad situation becomes worse--or become increasingly risk-seeking. "Depending on context and the circumstances of the decision maker or his firm, t might be formulated as a ruinous return, as the zero profit return, as the return available from an insured safe investment, or as a target which reflects a general attitude towards acceptable performance…." If an investor is primarily concerned about missing the target but the magnitude of the shortfall is not critical, the model will incorporate a smaller value for α than if both the shortfall probability and the magnitude of the shortfall are of importance. The value of α separates risk-seeking from risk-averse behaviors for below-target returns. Fishburn also explores how a mean-risk utility model can express a decision maker's preferences. Risk is defined as above, and for two distributions F and G, F is preferred to G if and only if: $U(\mu(F), \rho(F)) > U(\mu(G), \rho(G))$.	

		This results in the following form for the utility function: $U(x) = x$ for all $x \geq t$ $U(x) = x - k(t\text{-}x)^{\alpha}$ for all $x \leq t$. After reviewing a number of empirical surveys of how farmers, businessmen and investors view risk, Fishburn notes: "…most individuals in investment contexts do indeed exhibit a target return—which can be above, at, or below the point of no gain and no loss—at which there is a pronounced change in the shape of their utility functions. A relatively narrow range of utility functions (mostly linear) holds for above-target returns; a more wide range of α values are observed for below target returns. Generally, the α-t model suggests that investors will avoid distributions offering a probability of generating returns below the target even if such distributions have a greater mean. Additionally, "If the α-t utility model—which presumes the existence of a real valued function U in mean and risk which increases in mean, decreases in risk , and reflects the decision maker's preferences between distributions—is congruent with the von Neumann-Morgenstern expected utility model with utility function u, then u can be written as $u(x) = x$ for $x \geq t$, $u(x) = x - k(t\text{-}x)^{\alpha}$ for $x \leq t$, with $k > 0$."	
1981	"Uncertain Lifetime, Consumption, and Dissaving in Retirement," James B. Davies, <u>Journal of Political Economy</u> Vol. 89, No.3 (1981), pp. 561 – 577.	Older investors, lacking bequest motives, either continue to save in retirement or decumulate at rate slower than predicted under standard life-cycle models. Davies argues that lower-than-expected consumption is due to uncertainty in life span: "…in the absence of pensions uncertain lifetime will not only depress consumption at all ages but will also have an increasingly severe proportional impact beyond middle age….On conservative assumptions, uncertain lifetime may more than halve the mean rate of decumulation among the retired." The author cites previous research by Yaari [1965] and Levhari & Mirman [1977]. Yaari's model suggests that consumption, in the absence of insurance, under an uncertain lifespan grows more slowly than under	Yaari's model draws on the work of Irving Fisher [*The Theory of Interest*] published in 1930. Fisher contends that uncertainty in lifespan tends to increase 'impatience.' This, in turn, suggests that consumers prefer current consumption opportunities to future consumption opportunities simply because they may not be alive at a future date. Technically, an uncertain lifespan produces a higher subjective discount rate—future consumption is discounted at a rate greater than the prevailing return on savings.

		certainty. If insurance is available at an actuarially fair price, however, an investor lacking a bequest motive holds financial wealth in the form of annuities: "…actuarially fair insurance makes the price of future consumption P(τ|t)exp[-r(τ-t)], where r is the (constant) rate of interest, that is just low enough to offset completely the higher time preference." Under the Yaari model, with insurance, consumption increases at approximately the same rate under uncertainty as under certainty. When wealth consists exclusively of 'non-human' capital, Levhari & Mirman's model suggests that, under constant relative risk aversion (γ), consumption may be higher for low values of γ, and lower for higher values (where 1/γ is the elasticity of intertemporal substitution in consumption). Davies argues: 1. For values of gamma greater than 1, increased uncertainty of lifespan will lead to a decline in initial expected consumption provided that the savings rate (r) and the time preference rate (ρ) are not too high; and, 2. Models presented in previous studies are unrealistic because there is no actuarially fair insurance contract—"…insurance markets fail to provide annuities sufficiently attractive to outweigh the greater transactions costs and inconvenience of saving in this form." Davies' model holds expected lifetime constant and measures the difference in the shape and level of consumption as an individual investor's degree of uncertainty varies. Under a certain lifespan, consumption grows according to: $(r - \rho) / \gamma = g.$ Where the investor's "propensity to consume" is the inverse of discounted aggregate lifetime consumption divided by—i.e., normalized—initial consumption. Changes in lifespan alter the strength	Davies stresses that maximization of (discounted) consumption cannot exceed resources [R] where 'resources' include current wealth (both human and financial capital). Maximization, absent a bequest objective, is tantamount to a "spend-the-last-penny-at-the-last-breath" strategy. This concept finds mathematical expression in the terminal wealth condition: W[μ(t)] = 0. Absent both a bequest motive and ability to borrow—i.e., no negative net worth—the goal of utilizing all lifetime resources to finance consumption [terminal wealth = $0] is an optimal control problem. But this is an early expression of a 'feasibility condition' for retirement income planning. As time unfolds, academic literature seeks to understand the nature and scope of this condition in optimal control and boundary condition modelling; and, ultimately, in portfolio surveillance and monitoring protocols. Quantifying the feasibility condition and managing assets to maintain future goal feasibility become central to adaptive portfolio management. In the author's model, where the actual lifespan of the investor is fixed, and consumption growth exactly equals the savings rate (e.g., a form of constant-dollar consumption), normalized lifetime consumption equals remaining life expectancy. An increased probability of survival reduces the investor's propensity to consume. The Davies study finds further development in a number of studies included in this annotated bibliography. See, for example, "Estimating the True Cost of Retirement," David Blanchett, Presented at the Society of Actuaries 'Living to 100 Symposium' (January 8 – 10, 2014); "Optimal

		of subjective discounting. Under uncertainty (u), the rate of consumption (C-hat) changes according to: $$\widehat{C^u}(\tau) = \frac{1}{\gamma}[r - \rho + P(\dot{\tau})]$$ Or, in words, consumption growth over the uncertain planning horizon (τ) equals the elasticity of intertemporal substitution (an investor's tolerance for consumption variance) times the difference (r-ρ) between the available return on savings and current demand to consume (as expressed in a consumption time preference rate) plus delta in survival probability (which, when divided by gamma, is a risk-adjusted first derivative of mortality). Under this model: "wealth would not be run to zero as long as there was a nonzero probability of surviving any longer." The metric of interest to Davies is the ratio of Consumption under uncertainty [C^u] to Consumption under certainty [C^c]. The ratio's value is determined by the ratio of resources [R], and by the propensity to consume [PC] under uncertainty and certainty. PC, in turn, is a function of 'g' and 'r' and 'γ.' The author varies parameter values to determine the impact on PC. For example: "…raising γ above unity reduces 1/γ and tends to make PC^u fall. Clearly, with sufficiently large γ we obtain $PC^u/PC^c < 1$." Davies asserts that a gamma value of 4 is realistic. When this value combines with parameter values of r=.03 and ρ=.015, "…maximum negative impacts on consumption are 23.5 and 46.3 percent for the middle-aged and old, respectively" (without pensions). With pensions, "…uncertainty reduces consumption in middle age by up to 8.3 percent…[and] in retirement by up to 19.3 percent."	retirement consumption with a stochastic force of mortality," Huaxiong Huang, Moshe A. Milevsky and Thomas S. Salisbury, Insurance: Mathematics and Economics, Vol. 51 (2012), pp. 282 – 291; and, "Savings and Consumption with an Uncertain Horizon," David Levhari and Leonard J. Mirman, Journal of Political Economy Vol. 85, No. 2 (1977), pp. 265 - 281
1983	"Utility With Decreasing Risk Aversion," Gary G. Venter Proceedings of the Casualty Actuarial Society, Vol. 70. pp. 144 – 155.	Although this article is written for the Casualty Actuarial Society, it is a good introduction to utility with decreasing risk aversion. The article presents a fact pattern where the decision is whether to accept, simultaneously, an asset represented by a premium income stream and a liability represented by the possible loss from accepting the insured risk. To the extent that an insurer's surplus is under pressure, acceptance may or may not be attractive. Typically, an insurer demonstrates decreasing risk aversion as a function of surplus. One consequence of this type of	This is an early discussion of decision making where the investor does not exhibit CRRA. It is a critically important reminder that the CRRA assumption is valid only under a limited number of circumstances. Acceptance or rejection of a risk undertaking is not per se attractive; rather it depends on the "pressure" on current surplus. If surplus is

		utility function is as follows: "For many decision makers, the value of different potential levels of wealth is apparently not strictly proportional to the wealth level itself. A mathematical device to treat this is the utility function. Thus, a 50-50 chance at double or nothing on your wealth level may or may not be felt equivalent to maintaining your present level." The author details how the shape of the utility of wealth function determines whether a Loss function's risk is acceptable given the potential for gain. For an individual with a convex (steeply ascending) utility function the "value of potential wealth needed to risk losing everything on a 50-50 bet may be less than twice the current level." This is a gambler's utility of wealth function. By contrast, for an individual with a slowly rising utility of wealth function, the value of potential wealth required to accept the risk may be "…considerably greater than twice the current level…." Finally, if "the utility increases asymptotically to a value not greater than twice the utility of current wealth, such a bet would not be acceptable for any amount." Thus, instead of comparing diverse risk functions purely in monetary terms, "…the situation with the highest expected utility is preferred." The article reviews criteria for a valid absolute risk utility function: 1. $u(x)$ is increasing on $(0,\infty)$--more is always better or, marginal utility is always positive; 2. $u(x)$ is concave down. This is equivalent to the second derivative being less than zero. This is a measure of risk aversion because it implies that "the certainty of the expected value of the outcomes is preferred to an uncertain situation," or, the more wealth one has, the less utility value is generated by acquisition of an additional dollar. 3. Absolute risk aversion decreases as wealth increases. The measure of absolute risk aversion is: $ra(x) = -u''(x) / u'(x)$, where $ra(x)$ is the percentage change in marginal utility. "This property can be shown to equate to greater acceptance of risky situations with greater wealth."	"strained," risky propositions may be unwarranted. Monitoring surplus is, therefore, an important prerequisite to which asset management elections are acceptable to an investor. Increasing investment risk—or continuing risk by not exercising an option to annuitize, may result in a situation where the value of future gains is less than the utility of the "bet" under conditions of loss. That is to say, the potential improvement in the future budget constraint may not be sufficient to overcome current loss aversion. Yaari's model assumes constant relative risk aversion and concludes that an investor will annuitize all wealth given a complete market. Under the assumption of decreasing risk aversion, the portfolio management decision weighs the utility of exercising the option to annuitize against the utility of remaining invested. The Venter utility assumption does not automatically lead to annuitization. For individuals exhibiting decreasing risk aversion, the production of periodic income or terminal wealth is more important than the value of (1) performance ratios like the Sharpe Ratio, or (2) benchmark-relative performance comparisons. Note: Penalty Function metric parallels quadratic loss penalty metric—see: [2006] "The Management of Decumulation Risks in a Defined Contribution Pension Plan," Russell Gerrard, Steven Haberman and Elena Vigna.

		4. u(x) is bounded above—there is a number b such that u(x) is less than b no matter how large x is. [or, if you had all the wealth in the world, adding another dollar does not increase utility] 5. u'(x) = 0 for negative values of wealth. This reflects corporate bankruptcy laws. However, for individual investors, a wealth level below a minimum threshold may generate negative utility. The following paragraphs justify a penalty function approach to portfolio evaluation and asset management elections: "A utility function should capture the preferences of the decision maker, including the relationship of preferences to wealth….In other words, the utility function should be able to get at fundamental attitudes towards risk including how reactions will change with wealth. A key concept is that absolute risk aversion is equivalent to the percentage change in marginal utility as the level of wealth changes" "The marginal utility of wealth should decrease as wealth increases, but decreasing absolute risk aversion means that the percentage decrease in marginal utility should itself be declining. If a utility function does not reflect this decline, it is not properly valuing various wealth potentials." Thus a dollar increase in wealth has a declining marginal utility but the rate of decline slows down as the number of extra dollars piles up. The wealthier you are, the more shallow the slope of the second derivative—hence, the more risk tolerant you become: "…decreasing absolute risk aversion is not simply a matter of having different attitudes towards risk at different wealth levels. It is rather an aspect of the shape of the utility function at every point and reflects the relative desirability of the different levels of wealth themselves."	
1989	Martin L. Leibowitz and Roy D. Henriksson, "Portfolio Optimization with Shortfall Constraints:	This essay defines "shortfall constraint" in terms of a minimum return that an investor must equal or exceed with a given probability. The authors build their analysis on the normal distribution which is characterized completely by mean and standard deviation—the square root of variance. Assuming that portfolio returns are approximately normally distributed, the authors remind investors that such a portfolio	This is one of several articles authored or co-authored by Leibowitz and Henriksson in the late 1980s. These articles contributed to and motivated ongoing research efforts by many authors on the general topics of portfolio optimization with downside risk control for both institutional

	A Confidence-Limit Approach to Managing Downside Risk," Financial Analysts Journal (March/April, 1989), pp. 34 – 41.	with an expected return of R_p and a standard deviation of σ_p has only a 10% probability of producing a return less than 1.282 standard deviations below expected mean return. They make the analysis more concrete through a close examination of a portfolio offering an expected return of 8 percent with a standard deviation of 10 percent. The mathematics of the symmetric normal distribution suggest that the portfolio has a 78.81% probability of a return exceeding a return of zero; a 95% probability of exceeding a return of -8.45; and a 90% probability of exceeding a return of -4.82 [$R_p – 1.282\sigma_p = -4.82\%$] where returns and moments are calculated over a corresponding time period—presumably, annually. In point/slope format, the risk-return tradeoff is the line y = -4.82 + 1.282x [the Y axis is expected return; the X axis is standard deviation]. This expression facilitates a discussion of how various downside target returns result in different slope values in return/SD space: a 95% confidence limit corresponds to a return of -8.45% and a slope value of 1.645; a return of +8% (the expected mean return) exhibits a slope value of zero and, geometrically, is a horizontal line in the return/SD plane [$R_p – 0\sigma_p = +8\%$]. Increasing or decreasing expected return by 1%--holding SD constant—shifts the y axis intercept (downside risk) by 1%. However, holding expected return constant, a shift up or down in standard deviation results in a change in the intercept value of 1.645% (at the 95% confidence level) in a direction opposite to the SD shift. The authors argue that the mathematics of confidence limits gives investors the opportunity to constrain risk without a strategy of dynamic hedging to secure a portfolio's floor value. Dynamic hedging can fully guarantee success only in a market environment of frictionless and continuous trading. A confidence level approach provides a probabilistic assessment that downside risk will be limited. However, given the shape and level of the efficient frontier in the expected return / SD plane, the set of feasible downside limits is limited. The optimal portfolio is the tangent line from the Y intercept to the efficient frontier curve. For example, a portfolio wishing to exhibit a 97.5% confidence level for a return greater than -8.45% is infeasible because the tangent line's slope	investors (banks, insurance companies, pension funds) and, somewhat later, for individual investors. Optimization using downside risk [semi-variance], shortfall constraints, benchmark-relative targets, and so forth is a topic that, although generally outside of the bounds of this bibliography, is, nevertheless, conceptually important for a full understanding of efficient use of portfolio resources for retired investors. The issue of portfolio allocation by investors faced with liabilities is a branch of this general topic which receives attention because of its importance to retirees faced with managing portfolios to meet cash flow, gifting, and bequest "liabilities." Again, however, there is an extensive literature on fixed income cash matching and immunization theory and techniques which, for the most part, is beyond the discussions within this reference source. It is instructive to compare this 1989 article which offers a short-course on risk / return tradeoffs associated with a normal distribution--evaluated in discrete time--with: Evangelos Karagiannis, "Stochastic Investment Horizons in the Asset Allocation Decision and Liability-Driven Investing," CFA Institute (2014), pp. 1 – 4. www.cfainstitute.org. Karagiannis discusses the normal distribution in the context of continuous time mathematics developed, to a great extent, by Robert Merton in the 1970s. One notes that the Leibowitz / Henriksson approach parallels the "Chance-constrained Programming" methodology cited by Yaari ["Uncertain Lifetime, Life Insurance, and the Theory of the Consumer," Menahem E. Yaari The Review of Economic Studies

		places it above the efficient frontier. The article includes a short discussion of how the concept of a confidence limit applies when the investor's objective is to limit shortfall probabilities relative to a benchmark portfolio rather than a minimum rate of return. Briefly, the expected (mean) return required to avoid an unacceptable shortfall depends on the standard deviation of the portfolio and its correlation to the benchmark portfolio. As the value of the correlation statistic declines, the investor-owned portfolio must adjust the expected mean return upwards in order to compensate for the likelihood of high dispersion between portfolio returns and benchmark returns. The importance of correlation increases with the standard deviation of the benchmark portfolio: "Given portfolios with the same standard deviation and the same shortfall constraint, the portfolio with the lower correlation with the benchmark will require a higher expected return to overcome the greater dispersion of its return differences." Finally, the authors note that a shortfall-constrained optimal portfolio may lie below the unconstrained efficient frontier. An appendix details the mathematics underlying some of the authors' assertions.	Vol. 32, No.2. pp. 137 – 150. "The chance-constrained programming approach requires that the constraint (in this case the wealth constraint) be met with probability λ or more, where λ is some number fixed in advance, say .95….This approach is, of course very common in statistics where many of the standard tests are based on the idea of maximizing some criterion subject to the constraint that the probability of type I error be less than, say .05."
1990	William F. Sharpe and Lawrence G. Tint, "Liabilities—A new approach," The Journal of Portfolio Management, Vol. 16, No. 2 (Winter, 1990), pp. 5 – 10.	The article introduces the concept of the "liability hedging credit." A portfolio with a positive credit provides a utility benefit "…exactly analogous to, but in the opposite direction from, a risk penalty." It motivates plan sponsors—as well as other investors faced with a structure of liabilities—to form optimum portfolios by taking advantage of positive covariance between the assets and liabilities. The magnitude of the credit's benefit is a function of the investor's risk tolerance and the ratio of current assets to current liabilities. It leads to a process of portfolio surplus optimization: "Let L represent the value of the relevant liability concept (for example, an 'economic value' for a DB Plan's projected benefit obligation) and k the importance to be attached to it (e.g., 1.0 for a full surplus optimization). The relevant measure of surplus is S = A – kL, where A represents the value of the fund's assets and S is the surplus."	This article motivated further research studies in the area of Asset/Liability management for pension plans. Studies listed in this bibliography include: D. Don Ezra, "Asset Allocation by Surplus Optimization," Financial Analysts Journal (January-February, 1991), pp. 51 – 57. "Liability Investment with Downside Risk," Andrew Ang, Bingxu Chen & Suresh Sundaresan, National Bureau of Economic Research Working Paper 19030 http://www.nber.org/papers/w19030 (May 2013). "A liability-relative drawdown approach to pension asset liability management," Arjan Berkelaar & Roy Kouwenberg, Journal of Asset Management Vol. 11,

		The forthcoming year's return on assets is expressed as: 1 + $\widehat{R_A}$; and the forthcoming year's return on liabilities is 1 + $\widehat{R_L}$. If the value of surplus is designated as Z, the goal is to enhance the value of Z: $$Z \equiv R_A - k\frac{L_0}{A_0}R_L$$ Designating risk tolerance by 't,' the optimal asset mix maximizes utility: $$U \equiv Expected\ (z) - [Variance(\tfrac{Z}{t})]$$ Utilizing standard Modern Portfolio Theory formulae for expressing portfolio optimization, the objective becomes: $$Maximize\ Expected\ (R_A) - \frac{Variance(R_A)}{t} + 2\frac{k}{t}\frac{L_0}{A_0}Covariance\ (R_A, R_L)$$ Or, Utility = Expected Return – Risk Penalty + Liability Hedging Credit where the risk penalty is variance divided by the investor's risk tolerance 't.' Surplus optimization takes into account the covariance term. An investor can "…accept lower expected return and/or greater asset risk in order to increase the ability of an asset mix to hedge against increases in liability values." All else equal, an asset class return exhibiting high correlation with liability return provides a hedging—i.e. risk reduction to surplus—benefit.	Nos. 2-/3, (2010), pp.194-217. Yonggan Zhao, Ulrich Haussmann, and William T. Ziemba, "A Dynamic Investment Model with Control on the Portfolio's Worst Case Outcome," Mathematical Finance, Vol. 13, No. 4(October 2003), pp. 481 – 501. Dan diBartolomeo, "Asset/Liability Management for the Private Client," CFA Institute (March 2011), pp. 42 – 48. cfapubs.org Jarrod Wilcox, "Harry Markowitz & the Discretionary Wealth Hypothesis The Journal of Portfolio Management Vol. 29, No. 3 (Spring 2003), pp. 58 – 65. In many respects the concept of optimizing surplus within a corporate plan context also applies to monitoring a portfolio against shortfall risk within a retirement income planning context.
1991	"How Strong Are Bequest Motives? Evidence Based on Estimates of the Demand for Life Insurance and Annuities," B. Douglas Bernheim Journal of Political Economy, vol. 99,	The article contributes to the literature examining the motives for and strength of bequest objectives. One school of thought suggests that bequests are primarily accidental and arise from unspent wealth which is maintained because of incomplete insurance markets. Other schools of thought see bequests as motivated by altruism or by self-interested exchange with heirs (you take care of me and I'll take care of you). Bernheim argues that the data suggests a positive motive for bequests and that a bequest motive influences both savings and insurance decisions.	Bernheim's argument suggests that an annuity + life insurance approach where the annuity funds income to the investor and life insurance guarantees a minimum benefit to remaindermen, may be suboptimal. It is important to determine the extent of the "load" that the insurance company places on both sides of this type of transaction. The study by Patrick J. Collins and Huy Lam, "Asset

	no. 5, (October, 1991) pp. 899 – 927	In the classic Yaari model, an absence of a bequest motive should result in complete annuitization provided that the annuity pays a return higher than that available on conventional financial assets. Bernheim argues that changes (increases or decreases) in social security benefits will influence the extent to which households will want to buy term insurance (buy term = sell annuities) if they are over-annuitized, buy annuities (if they are under-annuitized) or do neither. The data is from the Longitudinal Retirement History Survey. Bernheim excludes households with more than \$500k in assets. Also, he defines annuities primarily as pension income received in the form of an annuity. This is not the population subgroup likely to hire investment advisors or establish private trusts. However, the article makes some interesting observations: Annuities are not actuarially fair because (1) adverse selection creates wedges between the buying price ($1/a_b$), the actuarially fair price (1/a) and the selling price ($1/a_s$) of an actuarial claim on one dollar one period in the future. This is because individuals with low survival probabilities tend to buy life insurance and those with high survival probabilities tend to buy annuities; and (2) annuities prices are adjusted for expenses, commissions, insurance profit objectives and other loads. In a life-cycle model, a couple cares about three distinct set of claims: joint, husband-only, and wife-only survival-contingent resources: "When the lone survival-contingent resources of either spouse are too low, it makes sense to purchase life insurance." However, the holding of life insurance among retired households may not be primarily a function of achieving an appropriate joint survival-contingent income stream: (1) until legally mandated, many households did not elect pension survivorship options, and (2) insurance coverage is primarily on husbands despite the fact that Social Security tends to treat spouses symmetrically. Maintenance of cash value policies may occur because of precautionary	Allocation, Human Capital, and the Demand to Hold Life Insurance in Retirement" Financial Services Review, vol. 20, no. 4 (Winter, 2011), pp. 303-325 provides a literature review on the demand for insurance and annuities. Furthermore, it details (1) the importance of including the right to receive life insurance payments when determining the sustainability of income throughout the joint life span of a retired couple; and, (2) the impact of surrendering life insurance policies and investing the net available cash value in the retirement portfolio.

		savings motives in the elderly population group. However, because such policies consist of cash value plus term insurance, Bernheim argues that the use of insurance for precautionary savings is rendered ineffective because of term insurance costs: "The term component of life insurance holdings avails consumers nothing during an emergency." Bernheim suspects that "some policies held after retirement may simply be the residue of efforts to insure human capital earlier in life." He concludes: "…approximately two-third of all life insurance policies owned by individuals in the LRHS sample functioned exclusively as savings accounts or were maintained because of inertia or irrationality." However, there were "powerful bequest motives for a large segment of the populations." Many people do not fully annuitize even the in presence of perfect insurance markets.	
1991	D. Don Ezra, "Asset Allocation by Surplus Optimization," <u>Financial Analysts Journal</u> (January-February, 1991), pp. 51 - 57	The author points out the value of a Defined Benefit [DB] Plan's liabilities become volatile whenever real interest rates are volatile. Maintaining a DB Plan surplus is an important corporate goal: "If fund assets exceed plan liabilities, future accruals of liabilities can be financed, at least partly, by drawing on the surplus. But if liabilities exceed assets, the asset shortfall must be made up by the sponsor; this constitutes a future drain on the sponsor's assets." The measure of plan surplus is the "funded ratio." However, "the ratio depends on how liabilities are defined and measured." Plans seeking to maintain and enhance their future surplus must employ suitable asset allocation policies. Ezra suggests that the following steps are "the natural approach to optimization": • "Analyze the composition of the plan's liabilities, taking into account (1) the liability valuation basis, (2) the extent to which liabilities are sensitive to elements in the valuation basis and (3) the expected values, standard deviations and correlations of the elements in the valuation basis." • Derive the distribution of liability returns • Estimate expected returns, standard deviations, and correlations for the investments in the asset portfolio, and estimate the asset portfolio's returns and the liability returns.	This essay illustrates how differing underlying risk model assumptions can lead to different outputs and conclusions. For example, when modeling a Pension Plan's liabilities under a static interest rate assumption, liabilities tend to behave like a short position in bonds; when modelling assuming volatile real rates, the liabilities tend to behave like a short position in stocks. Asset allocation recommendations will vary significantly depending on the risk model's structure and input assumptions.

		• Given the plan's current funded ratio, calculate the expected Surplus return, standard deviation, and inter-asset-class correlations. • Calculate the efficient frontier in "Surplus space." • Specify the sponsor's risk tolerance function and select the most appropriate asset allocation. The author discusses four cases and asserts that only case D provides a realistic scenario. Case A exhibits predictable liabilities; case B exhibits uncertain experience but uses fixed actuarial assumptions ("the use of constant actuarial assumptions for liabilities is very similar to the use of book values for assets."); case C assumes variable discount rates but, when liabilities are measured on an Accumulated Obligation Basis the income stream is fixed in nominal terms and the liability is like a short position in long-term bonds. Alternately, changes in the actuarial discount rate may simply reflect an assumption that constant dollar inflation remains steady (a fully indexed benefit is a predictable liability that acts like an inflation-adjusted annuity); finally, case D assumes that all assumptions are variable. This case leads to "substantially different conclusions about the composition of the efficient frontier in surplus space." Simulations based on Canadian stock and bond data reveal that, for the case D situation, the lowest risk portfolio "…contains a healthy mixture of all asset classes." The author deems higher allocations to stock as "eminently sensible" given that the volatility in liabilities is too great to permit immunization via a bond portfolio.	
1992	"Optimal investment strategies with investor liabilities," Edwin J. Elton and Martin J. Gruber, <u>Journal of Banking and Finance</u> Vol. 16 (1992), pp. 869 –	Most previous studies that explicitly incorporate liabilities into the asset allocation decision use an objective function focused on ending wealth rather than period-by-period change in net worth. This is true even for duration-matching ("immunization) models where the choice criterion is based on comparison of ending values. While immunization strategies are common in institutional money management: "…no theory exists on why investors should utilize them." Financial institutions offer their institutional clients seeking to discharge	Although Elton and Gruber focus primarily on institutional asset management where the investor is concerned with changes in net worth in the face of liabilities, they also point out that the "…analysis has implications for an individual investor where the analogy to the liability stream is anticipated consumption expenditures." Although seldom cited in recent studies, the

	890.	spending liabilities (e.g., defined benefit pension plan, an insurance company interested in reserve solvency conditions, or a bank interested in net-worth regulatory compliance) a choice among actively managed accounts, duration-matched accounts, and cash flow-matched accounts. Although treating pension liabilities as "negative assets" allows the investor to solve the asset allocation problem by quadratic programming, the "black box" approach obfuscates the "…questions as to why and when an investor should use various passive techniques versus active techniques to manage funds." The study provides a general framework that elucidates the relevant decision criteria. The study defines a riskless portfolio as one that matches liability outflows with default-free assets having the same maturity dates and amounts. It assumes that liabilities act like tradable assets capable of market value pricing. Under these conditions, a return on assets must match a return on liabilities, or $R_A = R_L$. This means that an exact cash-matched portfolio costs L dollars. The riskless strategy, assuming assets are greater than liabilities, is to place L dollars into default free, cash-matching assets with the excess (A – L) into a zero-risk Treasury instrument earning the one-period spot rate. The strategy guarantees that "…the one period spot rate and the risk of the portfolio is zero." Although a *cash-matched* portfolio is without risk, a *duration-matched* portfolio is not risk free. Any combination of the riskless strategy and "risky" portfolios traces out a capital market line in expected return / standard deviation space. In this case, however, the risky portfolio is not characterized by the variance of risky assets; rather, risk "…depends on the difference in return between the assets and the liabilities and the amount by which assets exceed liabilities." The efficient frontier exists in surplus return / surplus variance space. The asset allocation decision considers how much to weight the riskless strategy and how much to weight the risky strategy. Variance in the investor's net worth depends on the returns of both assets and liabilities as well as on the liability/asset [L/A] ratio. Although differences in net worth and liabilities mean that investors will want to occupy unique positions on the (surplus) efficient frontier, according to the basic tenets	Elton/Gruber study remains important in that it offers a theoretical basis for portfolio construction in the face of cash flow liabilities. Unlike many research studies that set up a model, calibrate it to appropriate empirical data series, and derive (and comment) on its output, the Elton/Gruber work analytically develops general rules using the equations that characterize modern portfolio theory's pricing of assets under conditions of equilibrium. The study's conclusions provide support for a portfolio that, in many ways, parallels the portfolios developed in behavioral finance literature—i.e., a pyramid structure where the base (Liabilities) are cash matched to a corresponding asset portfolio; and where the apex is a combination of an alpha generating portfolio and a duration-matched portfolio. Given that the objective function is to preserve surplus by eliminating all residual risk (guaranteed flooring), opportunity costs do not emerge as an important asset management consideration or decision-making criterion.

		of modern portfolio theory as expressed by the Capital Asset Pricing Model, all investors will want to construct a portfolio that combines some proportion of assets in the risk-free strategy with the remaining proportion in the optimal risky portfolio. "…combinations of [risky] duration matched portfolios and the riskless portfolio are still duration matched since each portfolio by itself is immunized." In the discussion of immunized portfolios, the authors point out that it is the existence of error terms [e_D and e_L] that creates risk. The impact of an immunized portfolio on net worth [N] depends on four elements: assets, return on assets, liabilities, and return on liabilities: $$R_N = \frac{AR_A - LR_L}{(A-L)}$$ But the return generating process for portfolio 'i' takes the form: $R_i = R_1 + D_iI + e_i$ Where: R_1 is the one period spot rate (risk free) D_i is the Duration of portfolio i relative to an appropriate risk-factor index and e_i is the error term. A duration matched portfolio is "…a portfolio of assets and liabilities which has zero sensitivity to the factor I. This condition requires that $$AD_A - LD_L = O,$$ and $$D_A = \frac{L}{A} D_L$$ Duration matching requires that duration must be adjusted by the ratio of liabilities to assets. The key point is that the risk of this portfolio is not zero because of the unsystematic variance of the asset error term and the liability error term: "…where $Var(e_A)$ and $Var(e_L)$ are the unsystematic variance of the assets and liabilities, respectively."	

		Where the risky asset portfolio is both duration matched and has the expectation of earning positive alpha, the preferred portfolio is the portfolio that maximizes the slope of the line emanating from the risk-free strategy in expected return/standard deviation space. If short sales are permitted, the investor should short the risk-free strategy to capture the expected alpha in the duration matched portfolio: "…if all assets are priced in equilibrium, no investor should hold any immunized (duration matched portfolios), except for the cash flow matched portfolio. On the other hand, if some assets are priced out of equilibrium it will almost always be advantageous to cash flow match some portion of liabilities while at the same time investing part of the funds in a portfolio which is duration matched but not cash flow matched." In the next part of the study, the authors extend the analysis to a situation where an investor can own a unique portfolio of assets with excess returns. This is a situation "…where a subset of bonds may be believed to offer a return above or below the equilibrium return." Equilibrium return is the return '…as implied by arbitrage pricing theory." If assets and liabilities are priced in equilibrium, then "…the optimum investment policy is to cash flow match the liabilities with the remainder invested in the one period government bill." However, if the alpha of the asset portfolio is greater than zero, the efficient frontier consists of all combinations of the *riskless portfolio* and the *risky portfolio*. "All portfolios that include the pure discount instrument in constructing the efficient frontier of <u>risky</u> assets are inefficient." Where 'X' is the proportion of assets invested in the risky portfolio, the preferred allocation is the one that maximizes the slope [θ] of the capital market line in the equation θ = [(surplus return – Risk Free Return) / Standard Deviation of Surplus]. Thus, the general rule is: (1) whenever all assets are priced in equilibrium: "…no investor should hold any immunized (duration matched portfolios), expect for the cash flow matched portfolio; and (2) "if some assets are priced out of equilibrium, it will almost always be advantageous to cash flow match some portion of liabilities while at the same time investing part of the funds in a portfolio	

which is duration matched but not cash flow matched."

The next section of the article expands the analysis by shifting the portfolio elements under consideration to include combinations of the riskless portfolio with three other portfolios:

1. A particular portfolio of efficiently priced assets;
2. A unique portfolio of assets with excess returns [α]; and,
3. The cash flow matched portfolio.

The preferred portfolio from the set of portfolios holding efficiently priced assets is the portfolio that minimizes residual (diversifiable) risk. This portfolio is at the risk-minimized point on the surplus efficient frontier; and, "With perfect factor replication it is the replicating portfolio." The cash flow matched portfolio eliminates "…the diversifiable risk associated with the investors' [sic] liabilities." The cash-flow matched portfolio may contain varying amounts of the one-period pure discount bond: "…the only difference between it and the riskless portfolio is that the riskless portfolio contains some investment in the one period pure discount bond." Finally, the investor may include a portfolio of assets with excess returns in the decision set because the "…returns…may more than compensate for the fact that it has residual risks."

The investor's task is now defined as maximizing the slope [θ] value by selecting an allocation that combines the Cash flow matched portfolio [C], the Factor portfolio [P] and the expected-alpha special portfolio [S]. 'C' assumes that the cash flow matched portfolio and the liabilities have the same duration. The authors calculate the first order conditions (delta θ = 0) with respect to C, P, and S to arrive at the optimal allocation [X] to each component.

When the factor portfolio can be sold short, the unconstrained optimum exhibits several properties:

- The allocation to X_C is to hold "…a cash flow matched portfolio exactly equal in size to the liabilities." This is the only way to eliminate all of the residual risk on the liabilities.
- The remainder assets (A – L) are allocated to the factor portfolio

		and the special alpha portfolio. The greater the reward-to-risk ratio for the special portfolio relative to the efficiently priced portfolio, the greater the allocation to the special portfolio. Additionally, the special portfolio "…could be constructed without consideration of the allocation between other portfolios or the characteristics of the factor portfolio." The authors prove: "…the composition of the special portfolio can be determined independently from the composition of the factor portfolio and depends only on characteristics of the special assets. In particular, note that characteristics of the liability structure such as its duration do not affect the composition of the special portfolio….it is independent of the factor portfolio, and it is independent of the liability structure." When short sales are not allowed, there is a significant change in the asset allocation decision. Specifically, the optimal solution is to invest less than L in the cash flow matched portfolio and more than (A – L) in the special portfolio: "…it is generally optimum to still engage in some cash flow matching but to only invest a fraction of the liabilities in this manner." The article explores a final variation where the portfolio of efficiently priced assets is not perfectly correlated with the risk-factor portfolio (i.e., contains some residual risk with respect to the liabilities). In this case, the asset allocation recommendation to invest an amount in the cash flow matched portfolio equal to the value of liabilities [L] is restored. Beyond this point, "…the minimum risk factor portfolio dominates the exact match portfolio." Beyond this amount (A – L) is split between the "minimum risk" factor portfolio and the special [α] portfolio. The combination that has the lowest residual risk is a combination of the cash flow matched portfolio and a 100% allocation to the minimum risk factor portfolio. However, a complete separation between the factor portfolio and the special portfolio no longer exists due to the fact that both exhibit residual risk. The investor is still capable of defining the composition of the special portfolio independently of the other portfolio choices; and, the allocation above (A – L) between the factor and special	

		portfolio still depends on the investor's expectation of the extra-reward-to-extra-risk ratio.	
1992	"The Buffer-Stock Theory of Saving: Some Macroeconomic Evidence," Christopher D. Carroll, Brookings Papers on Economic Activity, Vol. 23, No. 2 (1992), pp. 61 – 156.	In the Buffer-Stock Model "…consumers hold assets mainly so that they can shield their consumption against unpredictable fluctuations in income…." Buffer-Stock behavior emerges whenever the consumer, facing income uncertainty, is both impatient and prudent. An impatient consumer wishes to borrow against future income to finance current consumption; a prudent consumer values precautionary savings lest they suffer unacceptable economic consequences during bad economic times: "…impatience makes consumers want to spend down their assets, while prudence makes them reluctant to draw down assets too far." The result is a type of fear/greed dynamic: "…under plausible circumstances this tension will imply the existence of a target wealth stock. If wealth is below the target, fear (prudence) will dominate impatience and the consumer will try to save, while if wealth is above the target, impatience will be stronger than fear and consumers will plan to dissave." As the consumer becomes more pessimistic about his circumstances, the motivation to save increases. Carroll uses data from the University of Michigan's Panel Study of Income Dynamics to explore several aspects of consumer behavior in the face of possible fluctuations in household income. Both changes in expected level of future labor income and changes in the probability of "bad events" impact consumption and savings behaviors. Unlike most life-cycle models, the "interest elasticity" of savings is approximately zero. This condition arises due to the "target-saving" characteristics of the model. Carroll explores the distribution of shocks to both permanent and transitory components of labor income. Sample data, excluding self-employed individuals and those experiencing a change in marital-status, indicate that "…income typically recovers fully from near-zero events within three years, and mostly recovers within a year." Carroll's model assumes that shocks are serially uncorrelated, independent, lognormally distributed, and occur with a probability of 0.5 percent per period.	Although Carroll's model is primarily interested in exploring the consequences of--and investor reaction to--periods of income shocks due to unemployment, his insights are relevant within a retirement planning context wherein the investor is concerned about extraordinary, and perhaps unanticipated, demands on income (large expenses create a form of "zero-income event"). A drastic fluctuation in household income due to a period of unemployment may be analogous to a drastic increase in expenses during retirement—e.g., cost of medical/dental treatment—although care must be taken because the retirement decumulation period differs from the pre-retirement accumulation period in a life-cycle model context. Carroll cautions: "…whether buffer-stock savings behavior will occur depends on the expected future growth rate of income. If expected income growth is high early in life but lower (or negative) as retirement approaches, it is entirely possible that consumers will engage in buffer-stock saving when young but, after a certain age, will switch to a more traditional life-cycle saving behavior as their expected future income growth falls." Although the Carroll study focuses primarily on total wealth rather than on a cash reserve specially earmarked for emergencies, commonly recommended financial advice fits with the behaviors explained by a Buffer-Stock Model. Carroll quotes from a 1989 financial planning guide: "It is generally held that your liquid assets should roughly equal four to six months' employment

		Given a model that describes the evolution of wealth as a function of current wealth, labor income, and consumption (with corresponding growth factors and rates), and assuming a Constant Relative Risk Aversion utility function, "…the optimal consumption rule in a model with both transitory and permanent shocks can be written as a relationship between…the ratio of gross wealth X to permanent income, X/P, where gross wealth X is defined as assets plus current income…." Impatience limits the size of gross wealth that the consumer is willing to accumulate; the coefficient of risk aversion induces precautionary savings. Carroll estimates model parameters to fit realistic consumer behaviors as documented in the data sample. What is the target ratio value? As wealth approaches infinity, the consumption growth rate approaches the growth rate under income certainty: "Because everything in the model is continuous and monotonic, the expected consumption growth rate will cross the income growth rate curve at one point. The gross wealth ratio at this point will be called x*, the "target" gross wealth ratio." Spending/Savings are the control variables in this model: "…the gross wealth ratio x* is a target in the sense that, if actual gross wealth is below x*, the consumer will spend an amount small enough so that gross wealth will be expected to increase; however, if actual gross wealth is greater than x*, the consumer will spend enough so that expected gross wealth next period will decline….if gross wealth is at the target ratio, the expected growth rate of consumption is approximately equal to the growth rate of income." If the growth rate of future income increases, the target wealth stock will decrease because "…higher future income results in higher current consumption, hence lower saving and lower wealth." By contrast, increasing uncertainty increases the variance of consumption growth at any wealth level which will motivate increasing target wealth. However, increasing the coefficient of relative risk aversion has an uncertain effect. On the one hand, increased risk aversion tends to motivate more precautionary savings; on the other, it results in a lower intertemporal elasticity of substitution thus tending to reduce wealth as the investor maintains consumption levels. For example, given the	income. If you are in an unstable employment situation…the amount should probably be greater. [Your] need for liquidity is determined by the predictability of your cash income and expenditures, by your employment security, and by your investment strategy." It is a puzzle why the Buffer-Stock model is absent from the debate concerning the wisdom of creating an income reserve to cushion retirement investment portfolio drawdowns during bear markets. Both the Carroll model and the "Prudence" model (prudence = the third derivative of the investor's utility function; and, it motivates precautionary savings) developed by Miles Kimball ["Precautionary Saving in the Small and in the Large," Econometrica, Vol. 58, No. 1 (1990), pp. 53 - 73] (not discussed in this bibliography) offer insights and opinions important for this debate. For further discussion on the merits of an investment reserve strategy, see the bibliographical entries for: "Sustainable Withdrawal Rates: The Historical Evidence on Buffer Zone Strategies," Walter Woerheide and David Nanigian, Journal of Financial Planning (2012). [Available at SSRN: https://ssrn.com/abstract=1969021]. "The Benefits of a Cash Reserve Strategy in Retirement Distribution Planning," Shaun Pfeiffer, John Salter and Harold Evensky, Journal of Financial Planning vol. 26 no. 9 (September, 2013), pp. 49 – 55. One trend in retirement income planning is the rise of portfolio construction and asset management

		parameters of the model outlined above, a change in the probability of a zero-income event from 0.5 percent to 1 percent per year, increases the target net wealth to income ratio from 0.44 to 0.56. Optimal consumption under the new probability decreases as the consumer saves more. Carroll explores both short and long-term behavior in savings and consumption under various conditions including minimum income floors (e.g., safety net programs like unemployment insurance and expanded health insurance coverage, the rise of two-income households, expansion of credit by financial intermediaries, etc.), elimination of borrowing constraints on future income streams, various income growth rates, discount rates, interest rates, etc. Likewise, he compares and contrasts his Buffer-Stock model to other models aiming to explain consumer savings and consumption behaviors. This analysis lies beyond the scope of this bibliography. The Brookings Papers provide a discussion forum, at the end of the study, which offers other economists an opportunity to discuss Carroll's work. Forum participants point out that (1) the Buffer-Stock model is especially relevant to low income earners and that the model may not apply to wealthy investors; and (2) income is not expected to grow throughout retirement and investors must commit to major savings programs in order to finance retirement costs. This, in turn, introduces other savings motives and dynamics.	strategies that are "accounting based." Examples include the balance sheet approaches advocated by Wilcox and Fabozzi as well as the household balance sheet approach promulgated by the Retirement Income Industry Association [RIIA]. Carroll offers a type of "Income Statement" approach that may be an important complement for a comprehensive accounting-based retirement planning strategy. In this case, saving and consumption depend on net worth and on permanent income so that there is an optimal target ratio of net worth to permanent income.
1993	"Normal and Lognormal Shortfall-Risk," Peter Albrecht, <u>Proceedings of the International Actuarial Association</u>, Vol. 2 (Rome,1993), pp. 417 – 430.	Albrecht's paper reviews aspects of the normal and lognormal distributions over single and multiple periods in the context of shortfall risk. Shortfall risk is "…the probability that a special minimum return level (target return, benchmark return) will not be exceed…." In Albrecht's view, variance is a measure of financial asset volatility; while shortfall risk is "…the most elementary asymmetrical risk measure." The presentation distinguishes between the first two moments (mean and variance) of a normal vs. lognormal distribution. The normal distribution extends from negative to positive infinity; the lognormal distribution (absent investment leverage) cannot allow for returns less	This paper, presented at the 3rd annual AFIT Colloquium in 1993, lays the groundwork for a more developed presentation ("Shortfall Returns and Shortfall Risk") presented by Albrecht at the 4th AFIR Colloquium in 1994. Although the primary intended audience is insurance company actuaries charged with controlling investment shortfall risk over long planning horizons, this paper is also an important early contribution to the "surplus" management literature on the topic of liability-driven investing for individuals. It is possible to connect the

		than zero (the natural log is not defined at values ≤ 0). Whereas the mean of a lognormal distribution is the log of the return relative [1 + return where a 20% return = 1.20; and a 12% loss equals 0.82] is: $$m = \ln(1+\mu) - ½v^2$$ where nu is the substitute symbol for sigma (σ). Given the above expressions, the investor selects a rate of return which is the minimum desired return. The expression for shortfall risk [SR] in terms of the returns [R] of a lognormal distribution becomes: $$SR(M) = P(R \leq M).$$ where M is the minimum acceptable return. The probability bound, quantified in terms of the percentiles of the normal distribution for the minimum return target is $\mu \geq M + N_{\varepsilon}\sigma$, where Nε is the (1 – ε) quantile of a standard normal distribution. It is a “distance measure” or z-score: “The straight line $\mu = M + N_{\varepsilon}\sigma$ divides the (σ,μ)-plane into two separate sectors. The sector above the line (including the line itself) contains all (σ,μ)-positions with controlled shortfall-risk.” Note: for a comparable shortfall analysis using the normal distribution and the z-score distance measure, see for example, Martin L. Leibowitz and Roy D. Henriksson, “Portfolio Optimization with Shortfall Constraints: A Confidence-Limit Approach to Managing Downside Risk,” Financial Analysts Journal (March/April, 1989), pp. 34 – 41. Albrecht observes, however, that “The case of the lognormal distribution is much more complex.” The challenge is to identify the shortfall line separating failure from success in the mean/variance plane for lognormal distributions where the goal is to equal or exceed the minimum target	Albrecht papers to essays by authors such as Jerrold Wilcox [see, for example, Jarrod Wilcox, “Harry Markowitz & the Discretionary Wealth Hypothesis The Journal of Portfolio Management Vol. 29, No. 3 (Spring 2003), pp. 58 – 65]. The paper refers the reader to several textbooks for background information on statistical distributions. The lognormal distribution is the exponential of the normal distribution. The work assumes that the reader is familiar with the derivation of the moments of the lognormal distribution; and, the appendix provides mathematical expressions for the mean and variance. It is well known that the mathematical expression for the probability density function f(x) of a normal distribution is: $$\frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$ However, where $y = e^x$, and x is normally distributed, y is lognormally distributed: $\ln(e^x) = x$. This gives the equation $E[Y] = E[e^x]$. Designating $e^x = g(x)$ we have a kind a “double component distribution” for the log-normal PDF: a ‘$g = e^x$’ term and a ‘f(x)’ term for the normal distribution: $$\int_{-\infty}^{\infty} g(x)f(x)dx$$ Assuming a zero mean (μ = 0), plugging in the terms yields the following expression: $E[Y] = E[(g(x)] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{x-\frac{x^2}{2\sigma^2}}$ dx (by the law of

		return: $m \geq \ln(1+M) + N_\varepsilon \nu$. This is accomplished by a transform $G_\varepsilon(\nu)$ of the following form: $G_\varepsilon(\nu) = \frac{1-\exp[-N_\epsilon \nu - \frac{1}{2\nu^2}]}{[[\exp(\upsilon^2)-1]^{1/2}}$ A series of charts helps the reader visualize the difference between the slopes and intercepts of normal and lognormal ‘shortfall boundary lines’ in mean-variance space over a single-period planning horizon. In a multiperiod context, both arithmetic and geometric return series are monotonically decreasing with the length of the planning horizon. This suggests that procedures for controlling the arithmetic shortfall risk will also work for geometric returns. Given the multiperiod shortfall target [constraint]: $P(R_G(T) \leq M) \leq \varepsilon$ implies $\mu \geq M + G_\varepsilon\left(\nu/\sqrt{T}\right)\sigma/\sqrt{T}$. The implication is that over a long-horizon, a portfolio with suitable mean-variance parameters will achieve the desired return with “high ‘on the average’ probability.” To control risk during each subinterval, (1) the sub-periods over which a given shortfall control level operates; (2) the desired minimum returns; and (3) the desired control levels (distribution quantiles) must be specified. Note: compare to “The hurdle-race problem,” S. Vanduffel, J. Dhaene, M. Goovaerts, R. Kaas Insurance: Mathematics and Economics Vol. 33 No. 2 (October, 2003), pp. 405 – 413].	exponents) A common denominator reduces this to: $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma}} e^{\frac{2x\sigma^2 - x^2}{2\sigma^2}} dx;$ And, by completing the square for a quadratic in x: $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma}} e^{\frac{-(x^2-2x\sigma^2+\sigma^4)+\sigma^4}{2\sigma^2}} dx;$ Dividing the $2\sigma^2$ term in the exponent’s denominator into the σ^4 term in the numerator allows $e^{\frac{\sigma^2}{2}}$ to be brought out of the argument because it is a constant. This leaves the pdf of the normal distribution inside the integral (with the number σ taking the place of the number μ): $e^{\frac{\sigma^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma}} e^{\frac{-(x-\sigma)^2}{2\sigma^2}}$ But the pdf must, by definition equal 1 (100% probability); and the expression reduces to $e^{1/2\sigma^2}$ when μ=0. When μ ≠0, it is necessary to transform the variable (x-μ) into a new variable in terms of y. Following a comparable procedure, the expectation (mean) of the lognormal variable y becomes: $E(Y) = e^{\mu} e^{1/2\sigma^2}$; or, $e^{\mu+1/2\sigma^2}$ The variance term is 2ln [the arithmetic mean ÷ the

			geometric mean): Variance = $e^{(2\mu+\sigma^2)}[e^{(\sigma^2)}-1]$ The importance of distinguishing between normal and log-normal returns is highlighted in a 1974 essay by Elton and Gruber [Edwin J. Elton and Martin J. Gruber, "Portfolio Theory When Investment Returns Are Lognormally Distributed," The Journal of Finance, Vol. 29, No. 4 (September, 1974), pp. 1265 – 1273.] The Elton/Gruber article reviews the literature on the nature of the distribution of stock returns from the early econometric studies by Kendall, Osborne, and Moore, through the work of Fama and Mandelbrot in the 1960s. For a single period, Elton and Gruber restate the Markowitz efficient frontier theory when returns are lognormal. Part of their analysis relies on investigation of the distribution of z-scores for a lognormal distribution: Z = $\frac{\log r - m}{s}$, where 's' is the standard deviation of log r and 'm' is the mean. This gives log r = m + sz; and next period wealth [w_1] equals: $w_1 = w_0 r = w_0 e^{m+sz}$
1994	"Shortfall Returns and Shortfall Risk," Peter Albrecht, 4th AFIR Colloquium (1994), pp. 87 – 109.	Albrecht defines shortfall risk as "…the risk that a specified minimum return (target return, threshold return, minimum acceptable return) may not be earned by a financial investment." If return is designated by 'R' and the minimum return by 'm,' then R_m [returns from the "left-hand side" of the distribution target 'm'] characterizes the shortfall magnitude of return realizations below m; R_+m characterizes the magnitude of return realizations above m. R_m is "…the extent of the danger of not achieving the desired minimum return." The one-period probability law of shortfall returns is derived by	This article builds on Albrecht's 1993 presentation to the AFIR Colloquium. It is historically important in that it mathematically defines and characterizes shortfall risk metrics in both normal and lognormal distributions, and over a multiperiod planning horizon. It is concerned with return *rates* rather than dollar-value wealth *levels*. Albrecht offers a deliberate, carefully constructed analysis of shortfall risk, loss functions, and risk-

		integrating the density function of (R_): $$\int_0^{\infty} f_{\_(r)dr} = F(m)$$ Albrecht develops a "general conception of shortfall risk." He introduces a loss function [L] as a means of evaluating the consequences of different magnitudes of shortfall conditional on the occurrence of a shortfall in return. This leads to a general measure of shortfall risk [SR_m]: $$SR_m(R) = E[L(R_-)]$$ which can be expressed as: $$SR_m(R) = \int_{-\infty}^{m} L(m-r)f(r)dr.$$ When the power function $L(x) = x^n$ is selected as the Loss function, the resulting values for exponents from n = 0 through n =2 are the probability of loss (i.e., <u>shortfall probability</u>): $$SR_m = P(R \leq m) = F(m).$$ For normally distributed returns this is: $$SR_m = \Phi\left[\frac{m-\mu}{\sigma}\right]$$ and for lognormally distributed returns (where 1+R replaces R and 1+m replaces m), shortfall probability relative to 'm' is:	control criteria over both single and multi-period horizons. He concludes that shortfall risk constraints (probability / expected magnitude / semi-variance) should not be the exclusive investment decision-making criteria. Rather, the preferred approach is to analyze the complete "risk-value model" in terms of an appropriate loss-penalty function.

$$SP_m = P(1+R \leq q) \text{ where } q = 1+m.$$

In the case of n=1, the <u>expected value of the shortfall</u> is:

$$SE_m(R) = \int_{-\infty}^{m} (m-r)f(r)dr = mF(m) - E^m\,(R)$$

which, for a normally distributed random variable R is:

$$SE_m = (m - \mu)\Phi(m_N) + \sigma\phi(m_N); \text{ and,}$$

for a lognormally distributed random variable R is:

$$SE_m = q\ \Phi\ [q_{LN}] = -\,exp\left(\mu + \frac{\sigma^2}{2}\right)\Phi[q_{LN} - \sigma].$$

Where 'N' signifies a normal distribution and 'LN' signifies a lognormal distribution.

Finally, for n =2, we obtain the <u>shortfall semivariance</u>—"…a measure for the mean quadratic variation of the possible shortfalls."

$$SSV_m = \int_{-\infty}^{m} (m-r)^2\, f(r)dr = m^2F(m) - 2mE^m(R) + E^m(R^2)$$

Where $E^m(R)$ is the lower partial moment covering the area (-∞,b],

which, for a normally distributed random variable R is:

$$SSV_m = [(m - \mu)^2 + \sigma^2]\Phi(m) + \sigma(m - \mu)\phi(m_N)$$

And, for a lognormally distributed random variable R (with q = 1 = m),

		is: $SSV_m =$ $q^2\Phi[q_{LN}] - 2q\, exp\left(\mu + \frac{\sigma^2}{2}\right)\Phi[q_{LN} - \sigma] = \exp[2(\mu + \sigma^2)]\Phi[q_{LN} - 2\sigma].$ The most interesting portion of Albrecht's presentation is his discussion of multi-period shortfall risk. This discussion is part of the ongoing debate over whether time reduces risk. The answer to this question, in part, depends on the nature of the underlying distribution of returns, on the degree of mean-reversion in returns, and whether one defines risk in terms of *rates* or *levels*. Albrecht begins by pointing out that successive one-period returns can be arithmetically annualized: $RA = \frac{1}{T}(R_1 + \cdots + R_T)$ Or, geometrically annualized: $RG = \sqrt[T]{\prod_{t=1}^{T}(1 + R_t)} - 1$ Additionally, a shortfall risk metric relative to an end-of-horizon, total return target is possible by designating total return [RT] as: $RT = \prod_{t=1}^{T}(1 + R_t)$ Assuming stochastically independent returns, Albrecht provides analytical expressions for shortfall probability, shortfall expectation (first moment), and shortfall variance (second moment) for both arithmetic and geometric annualized returns. The reader is referred to the paper for the results. Given the analytic expressions for shortfall risk, Albrecht points out that the investor defines the relevant risk control criteria [C]	

		by applying an appropriate Loss function L(R): E[L(R)] ≤ C. When L(x) =1, shortfall risk is measured by Shortfall probability; and the appropriate control criterion is P(R≤m) ≤ ε. When L(x) = x, the shortfall risk metric is expected shortfall [SE_m(R)] and the appropriate control criterion is SE_m(R) ≤ c E(R). The interpretation of this expression is that "The shortfall expectation should not…exceed a certain percentage of the entire expected return." Albrecht devotes additional space to a discussion of "excess value" [R_+(m)] on the basis of a value function that is analogous to the loss function considered earlier. A full analysis of the possible return distribution employs "preference functionals" of the excess-value and shortfall risk types. Such functions are termed "risk-value models." The Markowitz portfolio section approach is an example of a risk-value model where risk is measured by return variance and value by expected value. The model "…quantifies the trade-off between risk and value." This approach can take a simple form in which the "optimal" portfolio minimize the Euclidian distance between value and risk; or, can incorporate a loss control function where the portfolio is optimized subject to E[L(X_)] ≤ C (where the random variable X is equivalent to portfolio return). In this case, shortfall probability, shortfall expectation, and shortfall semi-variance can be calculated relative to either the single period portfolio 'm' or the multi-period portfolio 'm' where $m_p = (m - \mu_p)/\sigma_p$). This approach is akin to maximizing the multi-period value of a "downside" Sharpe-like ratio. Although several studies have focused on downside risk under the assumption that it is the most appropriate investment decision criteria, Albrecht argues that "…excess value is superior to the expected value as a measure of value of a financial instrument."	
1994	"Asset Allocation, Life Expectancy and Shortfall" Kwok Ho, Moshe Milevsky and Chris Robinson	Issues: 1. Consumption vs. Running out of money before death 2. Impact of Inflation 3. Asset Allocation of Wealth	Lies at intersection of financial and actuarial research. An investor's aversion to shortfall is an implied utility function. The function is measured in terms of a portfolio's ability to provide a minimum return

	Financial Services Review vol. 3. no. 2 (1994), pp. 109 – 126.	Tests the advice of financial planners and others—e.g., Malkiel: retirees should own mostly bonds to produce retirement income (1990) Methodology: ➢ Rate of return required to minimize the probability of failing to meet the ROR necessary for consumption over the average of remaining lifetime. [Note: actual return may converge to average but actual wealth may fail to converge to expected wealth in the face of periodic consumption]. ➢ The implied utility function is the minimization of shortfall risk. The authors measure risk by the shortfall probability relative to a minimum <u>return</u> threshold. ➢ Authors assert that they require actual consumption from the portfolio so that they contribute to the time diversification debate—i.e., are equities risky over the long term? ➢ Authors do not consider uncertain inflation. ➢ Analysis is Pre-tax. Let d = 1+minimum rate of return necessary to support expected lifetime consumption given the average mortality rate applicable to the retiree. The Wealth/Consumption equation = $W = ({}_1P_n{*}C_1)/d + ({}_2P_n{*}C_2)/d^2 + \ldots + ({}_{T\text{-}n}P_n{*}C_{T\text{-}n})/d^{T\text{-}n}$ or, in constant dollars with d as the constant dollar or real discounting factor, $W/C = {}_1P_n/d + {}_2P_n/d^2 + \ldots + {}_{T\text{-}n}P_n/d^{T\text{-}n}$ The goal is to select the asset allocation best able to support a Wealth/Consumption ratio. The investor must solve for d periodically as age, health, etc. change. Asset allocation must be done in terms of the *remaining* lifetime not the expected lifetime at the time when the allocation was first determined. That is to say, allocation must by	over the planning horizon, or a threshold level of periodic income throughout the horizon. The authors' approach can be traced to Yaari's suggestions that the shortfall risk metric (the "chance constrained programming" approach) can substitute for a utility-based preferencing criteria. This article is significant because (1) it lays the foundation for using the Wealth/Consumption ratio as a performance monitoring benchmark; and (2) demonstrates why retirement income portfolio monitoring must be dynamic rather than static—i.e., the ratio's value changes over time. This article is the beginning of a series of papers, co-authored by Milevsky and others, on asset allocation and annuities. It argues that an annuity may be a feasible "solution" to sustaining consumption at a given Wealth/Consumption [W/C] Ratio. Consumption, returns, inflation, etc. are all either fixed or averaged in the article. Authors conclude that the greatest retirement risk is outliving capital and that 100% equity portfolios are essential for most retirees unless at a very advanced age or at modest W/C ratio. Note: High equity exposure as an optimal asset allocation is a common recommendation. See Bengen [1994], Bierwirth [1994] and Milevsky & Robertson [1996]. See also, [2002] Hughen, Laatsch & Klein.

		dynamic. Closed form solution = finding optimal T-Bill/Equity allocation given the expected returns and expected variance of the two asset classes—i.e., solve for optimal portfolio where return of portfolio $\geq$ d. Closed form solution requires inputs of <u>average</u> return and <u>average</u> variance. W/C ratio solution (i.e. solve for d) equates present value of consumption to current wealth. This is equivalent to a life annuity at the required rate of return: $W/C = \Sigma \text{Survival Probabilities}/d.$ When 'd' is the "real" as opposed to "nominal" discount rate, the required return above inflation is not large given the effect of the force of mortality over the planning horizon. On the other hand, T-Bills earn only a real rate close to zero. 'd' changes each year—each year can change consumption or rework the optimal allocation. Coping with shortfall risk may mean an entirely new portfolio allocation each year—i.e., solution is for a single period not a multiyear period. "A bad year may induce lower consumption, and this affects all future probabilities to earn enough."	
1994	"Investing For Retirement: Using the Past to Model the Future," Larry Bierwirth Journal of Financial Planning vol. 7 no. 1 (January, 1994), pp. 14 – 24.	The author asserts that the typical retirement "ledger" presentation contains unrealistic assumptions about inflation and investment returns. Usual ledger inputs are based on historical averages. The result of this assumption is that future projections unfold with "predictable regularity." In fact, there are substantial variations in inputs from one historical period to another. Bierwirth exploits the period-to-period differences in returns and volatility to illustrate the range of feasible spending rates over rolling 27-year periods (beginning 1926) for three asset allocations: Conservative (20% stock), Moderate (40% stock), and	A valuable and early discussion of sequence risk in portfolio sustainability. Return sequence rather than asset allocation is the key factor in sustainability. The greater the equity exposure, the greater the drawdown risk. However, continued exposure to equity produces the highest level of sustainable long-term spending. Bierwirth's examination of historical return

		Aggressive (60% stock). The portfolio under evaluation has an initial principal of $500 thousand, a goal of maintaining the nominal value of principal at the end of the planning period, and a goal of maintaining a constant, inflation-adjusted withdrawal amount. Bierwirth plugs in the yearly investment returns and inflation rates and solves for the amount of feasible (pre-tax) constant-dollar spending. The range of achievable spending is a function of the asset allocation and period-specific sequence of investment returns and inflation—while preserving the nominal terminal value of the portfolio. The author indicates that it is the sequence of returns and inflation that is of primary importance to determining the achievable spending rate. Although one 27-year period may have a higher average return / lower average inflation when compared to another, nevertheless the period with the less favorable averages may produce the higher achievable spending. A sequence of poor returns / high inflation early in the period has a deleterious impact from which the portfolio may be unable to recover. Bierwirth terms his historical back-testing methodology: "the dynamic ledger" to contrast it with the static assumption ledger. His dynamic ledger model covers all 27-year rolling periods from 1926 through 1992—a total of 42 periods. He notes several interesting model results: 1. "…the best time to retire…was the period from 1926 through 1951. This period produced the highest constant-dollar achievable yearly income. 2. A histogram of achievable income ranges strongly suggests that an investor should load for equity. The higher the equity position the more likely that the feasible income is right skewed. Additionally, over the entire period, the risk of high equity weighting is less than that of high fixed income weighting. 3. It took a high degree of risk tolerance to stay the course during the depression era years. This was the period in which the investor experienced the maximum portfolio drawdown; and,	sequences is a variation on the theme originally raised by Yaari: an "opportunity for an improved budget constraint" by continued ownership of stocks vs. the goal of meeting the budget constrain at a probability of λ or more. Bierwirth examines empirical evidence rather than assessing results of a mathematical model.

		the greater the equity exposure, the move severe the drawdown.	
1994	"Determining Withdrawal Rates Using Historical Data," William P. Bengen Journal of Financial Planning vol.7 no.4. (October, 1994), pp. 171 – 180.	The essay extends Bierwirth's work by testing the longevity of portfolios maintaining a constant asset allocation while distributing constant inflation-adjusted income stream. The author uses historical data for an annually rebalanced portfolio consisting of intermediate-term Treasury notes and common stock (1926 through 1994). However, all 50-year periods after 1944 require that missing years use historical averages as opposed to actual returns. The applicable planning horizon is a 50-year rolling period with a combined actual return series and an extrapolated average return series. The 50-year rolling period historical back testing leads Bengen to observe that a three-percent withdrawal rate was sustainable over all periods given a 50-50 asset allocation. A four-percent withdrawal rate was generally successful--the minimum portfolio survival time was 33 years. A five-percent rate generated a minimum survival time of 20 years. After setting up the base case, Bengen examines equity allocations of 0, 25, 50, 75, and 100 percent paired with withdrawal rates (percentage of initial portfolio value maintained over the planning horizon on a constant dollar basis) of 1, 2, 3, 4, 5, 6, 7, and 8 percent. The variable of interest is the minimum number of years that a portfolio will last given starting dates of 1926 through 1976. For example, at a one-percent withdrawal rate, all portfolio allocations last for 50 years. A two-percent withdrawal rate cannot be sustained over the 50-year period with a 100% Treasury note allocation. A three-percent withdrawal rate requires more than a twenty five -percent allocation to equities. Across all withdrawal rates, an allocation of 50 to 75 percent equities resulted in the highest survival percentages. These allocations also produced substantial ending portfolio wealth: "…I think it is appropriate to advise the client to accept a stock allocation as close to 75 percent as possible, and in no cases less than 50 percent."	Bergen's conclusions are U.S. market-centric. Compare to Nikkei 225—1989 = 38,915 / 2011 = c. 8,500. During the week of October 9, 2008, the index suffered a one-week loss of approximately 24%. Bengen's long-term stay-the-course results remind the reader of Keynes famous statement—In the long run we're all dead. For an age 65 investor beginning retirement in 1929, the 1992 portfolio value is available at the investor's age 134! Bengen's approach suggests that terminal wealth is a function of spending policy as much as asset allocation. He advocates a "stay-the-course" investment strategy. This is in contrast to, say, Ho, Milevsky & Robinson [1994] who advocate dynamic asset allocation. The stay-the-course approach assumes that markets will "cure" risk—perhaps by mean reversion. But see the Nikkei index 1989 through the present. The stay-the-course approach contrasts with approaches based on risk measurement and conditional probabilities. The practitioner community rapidly adopted the empirical approach to solving for optimal retirement portfolio design and strategy; the academic community continued to publish results from risk models incorporating a variety of statistical assumptions. This "split" became increasingly pronounced; and explains, in part, the heterogeneity in retirement planning

		Bengen argues that, under all circumstances, the client is best served simply by staying the course with respect to their initial asset allocation decision: "My research indicates strongly that as long as the client's goals remain the same, there is no need to change the initial asset allocation. It is likely to do more harm than good…." He provides an example of a hypothetical investor who retired in 1929 with a $500,000 portfolio allocated 75% to equity and operating under a 4% constant dollar withdrawal policy. The maximum drawdown exceeded $300,000 by the end of 1932. If the investor shifts to a 100% intermediate Treasury allocation, remaining funds are depleted after 17 more years; if the allocation is maintained, the 1992 portfolio value is $1.7 million. Similar patterns are evident for investors beginning retirement in 1937, 1946, 1969, 1973, and 1974. "This is a testament to the enormous recovery power of the stock market…." Finally, Bengen advises caution in the event that positive market returns realized early in retirement lead to a demand to increase distributions: "…excess returns earned today will probably be needed to offset losses in the future."	recommendations. Looking back historically, it appears, that 1994 us a watershed year because, for a period of time thereafter, the academic and practitioner communities employed different research methodologies and arrived at dissimilar conclusions.
1995	"The History of Risk 'Measurement'," Elton G. McGoun, Critical Perspectives on Accounting vol. 6 no. 6 (December, 1995), pp. 511 – 532.	Provides a history of risk measurement by economists. The Modern Portfolio Theory approach to risk is based largely on a probabilistic measure which reduces to relative frequency. The dispersion of the relative frequency distribution of past events (e.g., security returns) is a proxy measurement of risk. Expressing probabilities as a number between 0 and 1 [empty set and certainty] enables the distributions to be displayed as a curve. Although relative frequency probability is the only theory currently in general use in the field of economics, there are at least four probability theories: • Classical Probability: the notion of proportion or ratio of the number of ways an event may occur to the set of equally possible outcomes. Classical Probability theory is largely	Article distinguishes between *assessing* risk (relative frequency probability is one tool to accomplish this task) and *measuring* risk. A relative frequency distribution is not a measure of risk. It is merely a measure of historical realizations which may or may not be applicable to the current situation. This article is important because it elucidates potential weaknesses imbedded in both the empirical approach of practitioners and the statistical assumptions used in academic models: "The acceptance of a probabilistic measure of risk was an act of faith." McGoun's article is a strong criticism of the tendency to use empirical

		confined to games. • Relative Frequency Probability: limit as # of trials goes to infinity of the ratio of # of times an event occurred to the total number of trials. • Logical Probability: the degree of rational belief, relative to given information, in an event's occurrence. But the only information given is historical frequency, and so this method is not usable by economists. • Subjective Probability: the degree of personal belief in the occurrence of an event. Relative frequency probability, however, suffers from three problems: <u>The reference-class problem</u>: what elements of the past are common to the current situation and what elements of the current situation are so unique that past frequencies are inapplicable. Current events may be unprecedented. <u>The Law-of-Large-Numbers problem</u>: If the mean is the true long-term expected return, if the distribution is stable enough to use standard deviation as a risk measure, the investor is more and more certain to approach the mean as the investment "trial" is repeated period by period. In the long run, under this point of view, risk aversion should not matter. <u>The Estimation problem</u>: The dispersion of a real relative frequency distribution of historical events is used to estimate the dispersion of a hypothetical relative frequency distribution of possible future events. There is dispersion in the dispersion and hence the "measurement" of risk is itself risky.	distributions as valid proxies for risk. The Law of Large Numbers assumption is, perhaps, why some practitioners give short shrift to the concept of investor risk aversion—supposedly, it doesn't matter in the long run. Academic models use utility as a preferencing criteria; practitioner literature uses sustainability of spending and terminal wealth as preferencing measures. Although much modern economics follows the work of Kenneth Arrow, [relative frequency probability is a prerequisite to creating a "scientific" theory of choice], F. H. Knight's distinctions between risk and uncertainty have never been adequately addressed by economists. For many researchers, data is used to confirm *a priori* models and not as "a creative source of ideas for building new models."
1996	"Risk-Adjusted Retirement" Ho, Milevsky & Robinson, Canadian Investment Review, vol. 7 no. 3 (Spring 1996), pp. 19 – 27.	Essay discusses: 1. Tradeoff between minimizing shortfall risk and expected terminal wealth; 2. A Method of Optimization (minimize shortfall risk given the Wealth/Savings Ratio and given a target terminal wealth); 3. Results of random draws from specified distributions of T-Bills and Canadian Equity returns.	A short essay designed to acquaint readers with probabilistic vs. deterministic financial planning. Its main focus is on pre-retirement wealth accumulation rather than post-retirement wealth decumulation.

<table>
<tr><td></td><td></td><td>It concludes that higher long-term expected returns of equity facilitate the attainment of terminal wealth objectives and that investors should load portfolios towards a high equity allocation.</td><td></td></tr>
<tr><td>1997</td><td>“Asset Allocation Via The Conditional First Time Exit” Kwok Ho, Moshe Milevsky and Chris Robinson Review of Quantitative Finance and Accounting vol. 9 (1997) pp. 53 – 70.</td><td>This paper revisits the issues raised in the author’s 1994 paper (Asset Allocation, Life Expectancy and Shortfall”).

Modifications Include:
1. Utilize combined utility function and risk aversion parameter. Utility = desire to consume at an appropriate level; risk = probability of shortfall. This is a binary utility function comparable to immunization.
2. Asset allocation optimization cannot be expressed as a closed form solution—need numerical (i.e., Monte Carlo) methods.

Retirement Income Model:
➢ Consumption = liquidation of pro-rata share of funds from each asset class;
➢ Consumption is fixed—i.e. takes the form of an inflation-indexed annuity withdrawal;
➢ For K periods per year, Consumption of C/K occurs at the end of each period.

<u>Deterministic Case:</u> (W = Wealth)

$$W_1 = \max[0,(W_0 \times \sqrt[k]{1+r} - \frac{C}{K})]$$
and,
$$W_n = \max\left[0, W_0 \times (\sqrt[k]{(1+r)}^n - \frac{C}{k} \times \sum_{i=0}^{n-1} (\sqrt[k]{1+r})^i\right]$$

or,</td><td>Bengen [1994] and Bierwirth [1994] suggest an Assets Only [AO] approach to portfolio sustainability—i.e., stay the course and trust in the recovery power of equity markets. Here, the approach is more of an Asset/Liability Management [ALM] approach which is comparable to immunization of a bond portfolio. Optimization of asset allocation requires constant adjusting / rebalancing. This is like the adjustments required for immunizing a fixed income portfolio. This observation leads directly to a justification for dynamic asset allocation. The optimum allocation changes as a function of age and as a result of change in the wealth/consumption ratio.

It is a variation on the accounting decomposition of trust wealth: assets = current beneficiary interest + remainder interest + fees and expenses. Here, the decomposition is in terms of Wealth = compound growth of wealth – value of an annuity due. A 4% unitrust has a constant wealth consumption ratio of 100/4 = 25. As such, the article is an early expression of portfolio management in terms of a ‘free boundary’ problem.

The expression for investment risk includes both a term for utility value and a term for shortfall risk. This is an important contribution because it provides a theoretical bridge between the practitioner articles and academic articles.</td></tr>
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		W_n = max[0, (Compound growth of initial wealth – Accumulated Value of an Annuity Due)]. Because ‘r’ is deterministic, an investor’s wealth will equal the consumption demands placed upon it (i.e. PV of consumption = current value of wealth) until the following time N*: $W_0 \times (1+r)^{n/k} = \frac{C}{k} \times \frac{(1+r)^{n/k} - 1}{\sqrt[k]{1+r} - 1}$ or, solving for n = N* (the limit of time for which wealth is greater than consumption demands): $N^* = \frac{\ln[C] - \ln[C - W_0 K((1+r)^{1/k} - 1)]}{\frac{1}{k}\ln[1+r]}$ If an individual aged x lives beyond age x+N*, he or she will run out of money. The authors point out that life expectancy (i.e. the median value of an exponential distribution) is not an adequate measure of the probability of bankruptcy because the force of mortality has a different shape for men and women as well as for age (i.e., need at least a 2 parameter distribution). If ‘r’ is sufficiently large, N* goes to infinity—i.e., r ≥ $(1+\frac{C/W}{k})^k - 1$ or, under continuous compounding and withdrawals, r ≥ $e^{C/W} - 1$. <u>Stochastic Case</u> (r is uncertain): At retirement, investor selects fixed asset allocation vector $\underset{a}{\rightarrow}$. Return is a random variable ($\mathbf{R}^k$) which is an implicit function of the asset	If the portfolio’s continuously compounded rate of return is greater than $e^{C/W} - 1$, then consumption can continue infinitely. Thus, $e^{C/W} - 1$ is a shorthand approximation for required return for an endowment. [A no-free-boundary condition]. Equity allocation becomes a function of (wealth, consumption targets, gender, age, Δhealth, Δmarital status, bequest preference, etc.). Annuities destroy the time value of the option to change asset allocation. Retiree utility seems to involve a tradeoff between maximizing the probability of successful consumption (minimizing shortfall risk) and optimizing the consumption/bequest tradeoff. Note: paper suggests that a useful approach to dynamic asset allocation in retirement is use of the Hamilton-Jacobi-Bellman equations. These are equations to determine the “best path” for a non-linear process to take—like a curve expanding through space on a path that is perpendicular to its own surface. The rays of expansion are optimized using the HJB equations. Note: paper illustrates the need for a dynamic asset allocation policy in the face of a static spending policy. A fixed spending policy requires dynamic asset allocation especially where there is a lower bound designating an acceptable spending level. Later commentaries investigate the interactions between dynamic asset allocation and flexible

		allocation or return of the portfolio, where Return = $\mathbf{R}^k(\underset{a}{\rightarrow})$. The return of a portfolio therefore equals a fixed allocation vector x a distribution characterized by μ and a variance/covariance matrix Σ: $\mathbf{R}^k(\vec{a}) = \sum_{i=1}^{m} a_i \times \Lambda\left(I_i * \frac{1}{k}\mu,\ I_i * \frac{1}{k}\Sigma * I_i^t \right)$ Wealth at the end of one period: $W_1 \equiv \max [0, (W_0 * \mathbf{R}^k(\vec{a})_1 - C/k)]$, And, at the end of n periods, as: $W_n \equiv \max \left[W_0 \times \prod_{i=1}^{n} \mathrm{R}^k(\vec{\mathrm{a}})_{\mathrm{i}} - \frac{\mathrm{C}}{\mathrm{k}} \times \left(\sum_{\mathrm{i=z}}^{\mathrm{n}} \prod_{\mathrm{j=1}}^{n} R^k(\vec{a})_j \right) - \frac{\mathrm{C}}{\mathrm{k}} \right]$ This equation does not have a closed form solution for N* and the probability of bankruptcy [P_{starve}] equals the conditional probability of N* = I [i.e., time when wealth equals zero for each stochastic process vector] multiplied by the probability of surviving to x+i/k age. These conditional probabilities are then summed over all i. The model is designed to test various asset allocations with the goal of minimizing P_{starve}—the probability of outliving wealth. Conclusions: 1. For most retirees, shortfall risk is materially reduced by moving from 0% equity to 40-50% equity. The function is very flat in the vicinity of optimal allocation. 2. P_{starve} (Women) > P_{starve} (Men). 3. Small cap stocks materially reduce shortfall risk. 4. Slight increases in shortfall risk can be traded for large increases in the expected value of bequests (i.e. increases in equity allocation increase bequests) Note: calculations based on m = 2, or a 2 asset class portfolio. The	spending.

		conclusion appears to modify the earlier recommendation to maintain a substantial equity exposure in the retirement portfolio. Item #4 suggests that downside protection is expensive.	
1998	"Risks in Pensions and Annuities: Efficient Designs," Salvador Valdes-Prieto, World Bank Working Paper Number 20847 vol. 1 (1998).	The author states that pension risk can be divided into three parts: Mortality Risk—can be eliminated by the individual through purchase of an annuity. For the contract issuer/pension sponsor, however, there is a nondiversifiable component in this risk—1.e., average mortality turns out to be greater than expected at the time the contract was issued. The author calls this risk: "demographic risk." Investment Risk—the risk that the store of retirement wealth will buy less goods and services over time. Investment risk has three parts: 1. Capital risk: decline in asset values; 2. Reinvestment risk: necessity to reinvest at least some assets at future rates that are uncertain; and, 3. Inflation risk. Timing Risk—the risk of changing exposure to any previously-listed risk at an unfavorable time—e.g. buying a mortality guarantee just before expense charges fall. In a discussion of fixed annuities issued by insurance carriers, the author points out that annuity products can also accommodate bequest objectives. A retiree can purchase an immediate payout annuity with either an amount certain payout guarantee or a time certain payout guarantee. However, the price of the contracts will rise depending on the length and/or magnitude of the guarantee features. Much time is devoted to the retirement option known as the "programmed withdrawal" approach. Variations of this option are mandatory for several national pension programs. A general formula for an individual retiree (e.g., in England) is: Maximum monthly benefit in year t, or P_t =	Article points out that annuities obtained from commercial insurers are not risk-free products. Insurance solvency monitoring has become a complex activity. Even a 0.1% chance of insurer default compounds to a cumulative probability of 3% over 30 years. [For a discussion of issues in insurance company monitoring see the four-part article sequence: Patrick J. Collins, Kathryn A. Ballsun & Dieter Jurkat " Trustee Administration of Life Insurance," ACTEC Journal (Spring, Summer, Fall and Winter, 2006)]. Most asset management decisions are an exchange of risks. Is it prudent to exchange "capital risk" for "timing" and "inflation" risk? The article raises the possibility that, long term, the U.S. annuity marketplace may cease to be viable because of anti-selection. The "programmed withdrawal" approach which assumes that spending will be adjusted according to average life expectancy and a discount rate factor, mirrors the 2009 argument in "The Case for Flexible Retirement Planning" R. Gene Stout and John B. Mitchell; and, the 2011 argument in "Retirement withdrawals: Preventive reductions and risk management" John B. Mitchell. Valdes-Prieto's work makes it clear that a programmed withdrawal formula is optimal only under a limited number of conditions.

$$F_t\left(\sum_{x=t}^{110} q_x(1+r)^{-(x-t)}\right)^{-1} \div 12$$

Where F_t is the balance in the individual's account;
q_x is the probability of being alive at x given that the retiree is alive at t; and,
r = a real rate of return.
Note: F_t equals the Bequest amount if the individual dies.

The programmed withdrawal formula assumes that the retiree can spend an amount that assumes he lives for the average life expectancy and earns a deterministic real rate r on the account balance. At each (yearly) recalculation date, he is alive with probability 1 (i.e., he has not died during the year) and the pension must be adjusted downwards to reflect the recalculated mortality. [That is to say, the probability of a 70-year-old living to age 75 is greater than the probability of a 65-year-old living to age 75. Therefore, if the conditional survival probability goes up, the spending rate must be adjusted accordingly]. The formula does not insure/hedge individual mortality risk and is not designed to maximize individual utility.

If the individual's time discount preference for money equals the real rate of return, then the utility-maximizing programmed withdrawal for an individual retiree under the regime of no annuitization amounts to:

$P_t = F_t\left(\sum_{x=t}^{110} (q_x)^{\sigma} \cdot \prod_{z=t}^{z=x-1} (1+r_z)^{-1}\right)$; bequest = F_t if the individual dies.

Where σ = the inverse of the degree of relative risk aversion and/or the elasticity of intertemporal consumption.

The two equations will be equal (i.e. maximize utility) if σ = 1 and r is set using the market term structure of interest rates. This is the only time

		the programmed withdrawal is optimal for a retiree without the ability to annuitize and without a preference for making bequests. The author states that programmed withdrawals have several drawbacks: 1. No portion of wealth is annuitized and, therefore, the system is inefficient for all risk averse consumers. 2. Only a single bequest profile is allowed (the amount remaining in the fund) as opposed to annuity contracts that can build in various types of bequest guarantees. 3. It is impossible, under a programmed withdrawal regime, to reduce investment risk to zero. Even if a fixed income asset portfolio is immunized (duration matches planning horizon), a rise in interest rates can decrease the value of a bequest. [Cannot immunize the remainderman's interest] The author notes that reverse annuity mortgages are the extreme opposite of either immediate payout annuities or annuities that have bequest guarantee features. In the case of a reverse annuity mortgage, the annuitant receives a lifetime of payments in exchange for bequeathing his or her property to the insurance company. The author also discusses the problem of adverse selection in which retirees who think that they have high mortality risk will elect a programmed withdrawal. Therefore, the average mortality rate for remaining retirees becomes lower than the insurance company expects. This forces carriers to adjust their payout rates and generates another round of adverse selection: "In the final equilibrium the annuity market performs worse than optimally." The author points out that some commentators believe that this may be a source of serious market failure in the United States.	
1998	"Optimal Asset Allocation Towards the End of the Life	This paper reviews the "Annuity Puzzle." In 1963, Ando & Modigliani present a Life Cycle Hypothesis which suggests individuals will smooth lifetime consumption by annuitizing wealth. The puzzle exists because	The probability of consumption shortfall is the relevant measure of risk. Shortfall can be defined in several ways including: (1) probability of outliving

	Cycle: To Annuitize or Not to Annuitize?" M. Milevsky	the hypothesis has not proved to be true empirically. Possible answers to the lack of popularity of annuities include: 1. strong bequest motives 2. high fees, loads and expenses Thesis of Milevsky's paper: Retirees should defer annuitization until "mortality credits" overtake annuity costs [a 'do-it-yourself-and-then-switch" strategy]. The optimal time to annuitize is expressed in option valuation terms—not in utility-of-wealth terms. This paper, however, uses probability of consumption shortfall as the relevant measure of risk. The traditional finance approach entails utility maximization: Maximize E[Utility of Consumption + Utility of Bequest] Or, for a continuous function: Maximize E[$\int_0^T e^{-\rho t} U(C)dt + e^{-\Upsilon t} B(W_t)$] Where C = Consumption; B = Bequest p = personal consumption discount rate y=Bequest discount rate, T= stochastic date of death; W= wealth. <u>Discrete Time Model:</u> Basic Pricing of a \$1.00 Fixed Immediate Annuity: $a_x = (1+L_x)\left(\sum_{i=1}^{\infty} \frac{iPx}{(1+R)^i}\right)$ where a_x = Market Price of Annuity; L_x = load at age x, iPx = Probability of survival from age x to x+i; and R = Insurance Company annuity discount rate--cost of capital, corporate bond rate, etc. An individual is better off deferring purchase of an annuity if his "after consumption wealth" at the beginning of period x=1 is greater than the annuity's cost at the beginning of period x+1. Or, the condition for beating the ROR of the annuity is:	wealth, and (2) probability of a consumption level that is below a minimum acceptable threshold. The paper employs a top-down approach to annuitization in which the Wealth/Consumption ratio is reviewed after each period. If there is a "surplus"—if W > PV consumption, then it is beneficial to delay exercise of the annuitization option. This approach is conformable to a portfolio management system using a free boundary problem monitoring approach. The free boundary approach implies that Wealth must not be allowed to drop below the PV of consumption. However, the authors develop an optimal "stopping time," Although the approaches are not contradictory, care must be used when employing the optimal stopping time approach because the approach may ignore the level of wealth available for annuitization. Locking in a superior rate of return by annuitization is beneficial only if the annuity income is above the minimum needed. The optimal stopping time cannot only be a function of the portfolio's expected investment return. See the proviso in the last paragraph of this summary where the authors state that the annuity option is only appropriate for trusts with surplus. The stochastic investment returns model means that markets may not "bounce back" as assumed by Bierwirth [1994]. Annuity risks include (1) stochastic inflation—both a decrease in a nominal benefit's purchasing power

		ROR earned on investment portfolio (ROR = K) ≥ {[(1+R/discounted for probability of survival) * Delta Load] – (Year x+i load/Year x annuity cost) – Consumption of $1 end of year} or, K $\geq \frac{1+R}{1Px}\left(\frac{1+L_{x+1}}{1+L_x}\right)-\frac{L_{x+1}}{a_x}-1$ If K > RHS of equation, then delay. When L = 0, the equation reduces to K $\geq$ [(1+R/${}_1P_x$) – 1] where ${}_1P_x^{-1}$ equals the mortality credits that enhance the annuity's return. Note: there is a potential problem here because the "discount rate" is not the general market rate. Rather it is selected by the insurance carrier and is reflective of the insurer's cost of capital and profit targets. <u>Continuous Time Model</u>: (1) Deterministic Model—portfolio return is deterministic $a_s = (1+L_x)\int_0^{\infty} e^{-rt} tPxdt$ where the integral is solved to obtain a closed form solution (The Incomplete Gamma Function) for the price of the life annuity. The investor can buy an annuity or invest in a portfolio earning the continuously compounded deterministic return (δ) and consuming the annuity amount C_x^w. If $\delta \geq 1/a_x$ the investor can beat the annuity indefinitely; otherwise, *the unconditional probability of a consumption shortfall is the probability of surviving to the time of portfolio bankruptcy.* The optimal time to Annuitize (time = s) is an implicit function of the portfolio's investment return δ such that W_s / a $_{x+s} \geq C_x^w$. Since the mortality credit will eventually surpass δ, there will come a point when	*and* an increase in the cost of a future annuity income benefit, and (2) adverse mortality—the future annuitant mortality table reflects anti-selection. These factors may make a delay in annuitization more risky. Whereas annuitization is irreversible, the decision to annuitize involves a balance between two risky alternatives—a failure to generate sufficient lifetime consumption without annuitization, or elimination of liquidity and abandonment of potential future investment gains upon election of the annuitization option. This is an exchange of risks which must be prudently evaluated. Note: Mitchel, Poterba & Warshawsky [1999] use a utility maximizing "wealth equivalence" approach and conclude that for any reasonable utility function parameters, individuals are better off annuitizing even in the face of 30% loads. Note: It is interesting to compare the future annuity cost projections made in this paper to the [2008] Feng Li's Masters thesis on the distribution of the Future Value of an annuity.

the option to Annuitize may be beneficial.

(2) Continuous Time Stochastic Model:

The deterministic model is, in practice, subject to three sources of uncertainty:

1. stochastic investment return (risk of insufficient future wealth);
2. stochastic interest rates (annuity pricing risk); and
3. stochastic mortality (risk that adverse selection will force insurers to use less favorable mortality table / reserving requirements)

The stochastic model does not assume that δ is a constant. Rather it follows the classic stochastic differential equation [SDE] (with consumption):

$$d W_t = \left(\mu W_t - C_x^w\right)dt + \vartheta W_t dZ_t^k$$

where Z is a Brownian Motion process and the parameters μ and σ depend on the investor's portfolio.

Likewise, the stochastic model does not assume that R (the pure internal rate of return from the life annuity) is a constant. Rather R obeys a mean reverting SDE comparable to the interest rate behavior model developed by Cox, Ingersoll and Ross in 1985:

$$dr_t = \gamma(\bar{r} - r_t)dt + \sigma_r \sqrt{r_t}\, dZ_t^r .$$

Assuming zero correlation between interest rates and portfolio returns.

Mortality rates follow the current Society of Actuaries annuity mortality table using the Scale G improvement factor.

The Stochastic Model computes the probability that a decision to defer annuitization until time s will be successful. Success is defined as the probability that the investor will have, at time s, an amount sufficient to equal or exceed the income stream that could have been purchased at

		time x (x < s). A Monte Carlo simulation suggests that the probability of a successful deferral depends primarily on the current level of market interest rates relative to the risk premium (μ - r_0) and to the annuity load. Using a standard methodology for computing the Value per Premium Dollar (VPD) for annuity buyers, the load can be estimated (an actuarially fair annuity has a VPD equal to 1; a loaded annuity has a VPD less than 1). Discounting at the corporate bond rate, the implied average annuity load for Canadian annuities during the 1984-1996 period is approximately 12%. Whereas the purchase of an annuity is an irrevocable decision to pay a non-refundable sum to an insurance company, and because the decision eliminates liquidity and the ability to make bequests, the decision to annuitize should be deferred for as long as possible provided that the risk of failing to acquire an adequate lifetime consumption stream remains within tolerable levels.	
1999	"New Evidence on the Money's Worth of Individual Annuities" Mitchell, Olivia S., Poterba, James M., and Warshawsky, Mark J., American Economic Review vol.89 no. 5 (December 1999), pp. 1299 – 1318.	The authors' findings include: 1. Payout rates per equivalent premiums differ significantly among insurance companies; 2. The expected present discounted value (EPDV) of an annuity for an individual chosen from the general population is 80-85 cents on the dollar; and 90-94 cents on the dollar for an individual chosen from the pool of annuity purchasers; 3. Although the effective transaction costs are high, they have declined since the early 1980s; 4. Annuity taxation benefits do not materially alter the EPDV of the payouts; and, 5. The expected utility of an annuity is often positive even in the face of high transaction costs. The expected present discounted value [EPDV] is a function of (1)the amount of the annuity payout, (2) the rate used for discounting, and (3) the survival probability applicable to the annuitant. Most annuities provide fixed nominal cash flows. Discounting is done	This article is a contribution to the "annuity puzzle" research literature. This topic is important because many academic models—especially those assuming constant relative risk aversion—view annuitization as an optimal retirement income strategy. Annuities "guarantee" a consumption stream but investors should be aware that they are not risk free. Annuities have (1) default risk; (2) interest rate risk (reflected in different costs and different times for an equivalent income stream); and (3) purchasing power risk for fixed nominal payouts. Whereas the duration of an annuity is, in general, greater than the duration of an alternative bond portfolio, an investor who annuitizes by acquiring lifetime nominal periodic payments exchanges "systematic" longevity risk for "systematic" interest

either through application of the expectations theory of the treasury bond yield curve (interpolated spot rates) to future cash flows; or, by calculating the spread between treasuries and a 10 year BAA corporate bond and adding the constant spread to the first term structure calculation.

The authors calculate survival probabilities on a month-by-month basis with the assumption that a 65 year old will not live more than 600 months (115 years of age). The EPDV of an annuity purchased at age b is:

$$V_b(A) = \sum_{j=1}^{600} \frac{A * P_j}{\prod_{k=1}^{j}(1+i_k)}$$

Where P_j is the probability of survival.

For taxable investors with an exclusion ratio of λ for a period 'T' determined by IRS tables (with all income fully taxable after 'T' at a tax rate equal to (τ) the valuation equation becomes:

After tax value (A) =

$$\sum_{j=1}^{12*T'} \frac{(1-\lambda*\tau)*A*P_j}{\prod_{k=1}^{j}(1+(1-\tau)*i_k} + \sum_{j=12*T'=1}^{600} \frac{(1-\tau)*A*P_j}{\prod_{k=1}^{j}(1+(1-\tau)*i_k)}$$

In calculating the force of mortality, the authors use cohort tables rather than period tables. Period tables provide information on cross sectional past experience on a year-by-year or age-by-age basis. A cohort table follows the lifetime experience of a group of individuals born in a particular year. Assuming mortality improvement over time, the future mortality experience of the cohort may differ substantially from the past experience embodied in the period table. As the cohort moves through time, their actual experience means that the force of mortality is a

rate risk. It is important to understand the full dimension of the risk/reward tradeoffs in the decision to annuitize.

Annuity instruments are reserved by bond portfolios acquired in the capital markets. Whereas a profit-seeking insurer seeks to overcome its hurdle rate, the IRR of an annuity [PV of expected benefits – purchase price] may be lower than the IRR of investment alternatives. If interest rates are low and stable, however, the insurer might offer better *relative* pricing—i.e., reduce the gap between purchase price and expected future benefits. This reverses, to some extent, the conventional wisdom regarding annuity purchases in a low interest-rate environment.

Note: This argument is also advanced by Orszag in 2002 ["Ruin in Retirement: Running Out of Money in Drawdown Programs," Watson Wyatt Technical Paper 2002-RU06]

Note: Method One to compare annuities to other investments is similar to elements of the formula presented by Mitchell in 2011.

Note: The utility-based analysis parallels Don Ezra's [2009] argument in "Who Should Buy a Lifetime Income Annuity? And When?"

"moving target." The authors favor a projected cohort mortality table approach for both the general population and the annuity-purchasing population.

Assuming both a tax rate of 28% and 15%, the transaction costs of annuities (1-EPDV) are high under both the Treasury and Corporate Yield Curve discounting methods for the general population; and, for the annuitant population, implied transaction costs are still between approx. 7% and 16%. The effective duration (i.e. sensitivity to changes in the discount rate) of annuity payouts for the annuitant purchasing population is larger than that for the general population.

The authors use two methods to compare annuities to other fixed income investments:
1. Discount the annuity's EPDV using the return on the other asset as the applicable discount rate; or,
2. Compare the IRR of the annuity with the IRR of the alternative investment, where IRR is:

$$0 = -(\text{Annuity Purchase Price}) + \sum_{j=1}^{600} \frac{A * P_j}{(1+\rho *)^j}$$

where ρ is the pre-tax rate that brings the equation into equilibrium.

Note: see p.89 of the article for after-tax IRR formula.

The authors conclude that IRRs of annuities are substantially lower than IRRs of alternative investments. However, "when interest rates are low and stable, insurance companies may be able to price nonparticipating annuities more competitively with other fixed-income investments. In contrast, when interest rates are high and variable, insurance companies may be reluctant to assume that current yields will be maintained for the duration of annuities issued in that year, and therefore they act more conservatively and require larger contingency funds in their annuity

		pricing." Another way of comparing the value of an annuity to that of an alternative investment is to compare expected utility values. The authors employ the following utility function and maximize it using stochastic dynamic programming: $$U = \sum_{j=1}^{50} P_j^* \frac{\left[\left(\frac{C_j}{(1+\pi)^j}\right)^{1-B} - 1\right]}{(1-B)\times(1-\rho)^j}$$ where: π is the inflation rate and ρ is the time preference or consumption preference discount rate. [Note—this is a CRRA power-utility function with B equal to the risk aversion parameter]. The authors calculate the optimal consumption path for a person with assets of W_0 who does not have access to an annuity market and who wishes to maximize the above utility function over their lifetime. By comparison, the authors calculate the value of an actuarially fair annuity (NOT a commercial annuity subject to loads and expenses) where the payout from an actuarially fair annuity (θ) for a sixty five year old with a maximum life expectancy of 50 years is determined by: $$1 = \sum_{j=1}^{50} \frac{\theta * P_j}{[(1+r)(1+\pi)]^j}$$ The authors determine the fraction of annuitized wealth that produces the same utility value of consumption as determined in the absence of an annuity market. At risk aversion parameters of B=1 and B=2, they find that a reduction in age-65 wealth equal to between 30 and 38% produces comparable utility. That is to say, an individual may find it	

		advantageous to annuitize wealth in the face of transaction costs of 30% or more. By contrast, if half of an individual's wealth was already held in annuitized form (i.e. Social Security and/or Defined Benefit Pension entitlements), the equilibrium decrease in wealth is approximately 23 to 31%. The authors use a stylized model of uncertain inflation (inflation = one of four possible values with equal probability) and conclude that uncertain inflation does not substantially change parameter values. Therefore, under conditions of fixed or variable inflation, annuitization appears to provide high utility values. Thus high loads in conjunction with other factors such as Bequest preferences and liquidity needs are required to explain the Annuity Puzzle.	
1999	"The Behavior of Shortfall Functions in Asset Allocation," Paul Bouchey, David Carino, and Yuan-An Fan, <u>Russell Research Commentary</u> (August, 1999), pp. 1 - 16.	Given an asset allocation, the paper examines various types of shortfall penalties at the end of the planning horizon. A shortfall penalty can apply to a targeted wealth level, a rate of return, a peer group, a benchmark-relative goal, or a spending (cash flow) objective. Common shortfall risk measures include semi-variance or downside risk both of which are variations on a quadratic (mean/variance) function in which only the investor considers only the first two moments of the return distribution. However, a variety of shortfall penalty functions exist. These include: • Linear shortfall: Penalizes below-target results on a straight-line basis (e.g., every dollar below a target wealth level incurs a penalty 'unit'). • Quadratic shortfall: The further the shortfall relative to target, the greater the penalty weight. Optimization in the face of a shortfall penalty involves selecting an asset allocation that maximizes the value of the following expression: $$E\left[Wealth - \frac{Shortfall\ Penalty}{Risk\ Tolerance}\right]$$	The application of a shortfall penalty risk metric to asset optimization demonstrates that high expected return portfolios often fail to maximize risk-adjusted results for investors. Conversely, portfolios of "safe" assets also fail to achieve maximum risk-adjusted values. This is a variation on a number of themes within the retirement planning literature including: (1) the "cost" of "safety," and (2) the risk of shortfall when weighting a portfolio heavily towards equity vs. towards fixed income. The wealth-relative-to-target metric described in this article has some interesting similarities to an ALM approach in which wealth is optimized relative to the liabilities it is asked to discharge. "The goal of any decision model is to help decision makers gain insight into the problems they are faced with." This is a helpful statement--it presents investment modelling not as a solution tool but a tool for gaining insight into problems—a decision making tool.

		In general, higher expected return assets have a greater shortfall probability; and, therefore, are likely to incur greater shortfall penalties given the wider range of possible outcomes. The shortfall risk for these assets tends to depress the value of the objective function. Interestingly, however, a safe asset—i.e., one exhibiting a low dispersion in returns—also tends to produce a low value for the above mathematical expression. When running an optimization algorithm incorporating a shortfall penalty risk metric, the target level (amount of dollars, rate of return, cash flow, etc.) is of great significance. For low targets, it is unlikely that the investor will achieve below-target results; hence, the risk-metric is of little consequence. For high targets, it is possible that most every outcome may generate a shortfall penalty: "In this case, the target itself does not matter, but the basic *shape* of the shortfall function matters a great deal." Just as the curvature of an investor's utility function impacts his or her preference for one portfolio over another; so, also, the curvature of the penalty function greatly influences the preference ranking of optimization outcomes. Therefore, it is important for the investment advisor to assess accurately, the investor's risk attitudes. To this end, the authors provide a table of typical investor attitudes to both disappointing and satisfactory results. The responses characterize the investor's risk aversion as Decreasing Absolute Risk Aversion, Constant Risk Aversion, and Increasing risk aversion. However, "In a shortfall framework, the absolute level of wealth is not as important as the wealth relative to the target." The authors stress that use of a quadratic risk-penalty function "…will result in a more conservative asset mix as the target is lowered ("I do not want to erode my wealth") and a more aggressive mix as the target is increased ("the markets are my only chance"). This profile characterizes an investor with increasing risk aversion. The article presents a series of graphs to illustrate how, (1) using a quadratic shortfall function (a power function with a utility of wealth exponent equal to ½), (2) changing the target, and (3) the investor's risk	

		tolerance effects the optimal asset allocation—i.e., relative weighting of cash, stocks, and bonds within the portfolio. As the exponent value changes, the concavity of the utility function also changes with maximum curvature exhibited by ln(utility). The adjustments to utility (risk tolerance = the reciprocal of risk aversion), influences asset allocation significantly. At higher levels of curvature, significant shortfalls do not necessarily trigger a strategy of increasing equity exposure: "Even when almost all of the outcomes fall deep into shortfall, the increasing curvature prevents equity from becoming desirable at high target levels. This type of behavior is more consistent for investors who would prefer to become more conservative when faced with very low wealth relative to their target."	
1999	"The Risk and Rewards of Minimizing Shortfall Probability," Sid Browne The Journal of Portfolio Management vol. 25 no. 4 (Summer, 1999), pp. 76 -85.	Browne contrasts an investment strategy designed to maximize the utility of consumption and/or wealth with a strategy designed to minimize the probability of falling below a terminal wealth amount or a relative return target. Rather than optimizing in the Markowitz sense, one maximizes a success probability in the Roy sense where Roy's safety-first criterion was developed in a static, single-period setting. Browne draws on a continuous-time model [similar to Black-Scholes] and assumes an investment process consisting of a safe asset paying the risk free rate of return and a risky asset portfolio characterized by a lognormal distribution of asset prices. He considers two different return targets—beating an all cash portfolio (by 10%) and beating an all stock portfolio (by 10%) where both portfolios are simple buy-and-hold [B&H] portfolios. Initially, Browne compares the B&H portfolios to a constant mix asset allocation dynamically rebalanced in a continuous time context. He defines the optimal asset allocation for the dynamically rebalanced portfolio as the allocation that maximizes the value of the Merton Optimum. Assuming an iid distribution of prices, this is the strategy that also maximizes the value of a logarithmic utility function. The Merton Optimum is: $$\frac{\mu - r}{\sigma^2}$$	Important to define clearly the term "shortfall probability minimizing" strategy. Browne's use of this phrase occurs in the context of dynamic hedging. The goal is to assure a target terminal wealth level or a minimum relative target return. As such, it involves substantial risks and is contrasted with a utility maximizing strategy. Other interpretations of a "shortfall probability minimizing" strategy are akin to state preference criteria where the value of the portfolio is designed to be at or above a floor value during poor economic states. A direct comparison with a quote from study A with study B without understanding the context in which the phrase "shortfall probability" is used, may be misleading. Examples of a shortfall: 1. In terminal wealth level; 2. In target ROR; 3. Relative to acceptable standard of living; 4. Portfolio sustainability—portfolio depletion; 5. State preference payout; 6. Benchmark;

		The allocation is also known as the growth-optimal asset allocation, and is appropriate for CRRA investors holding portfolios that do not have interim cash flows. Assuming, for example, an expected return to stocks of 15%, a risk free rate of 7% and a standard deviation of stock price of 30%, the optimal asset allocation to equity is 8% ÷ 9% = 89%, and the optimal allocation to the risk free asset is 11% where leverage is not allowed in the portfolio. Under the model's assumptions, as the time period goes to infinity, the optimal allocation will beat any other allocation. For a given time period less than infinity, the lognormal asset price distribution assumption allows for the creation of a probability measure for the expected time to exceed the non-optimal allocation by a given "exceedence" level. The probability measure is simply the $1 - \alpha^{th}$ percentile of the standard normal distribution. Browne solves for the expected time for the optimal portfolio to exceed by a designated level the non-optimal portfolio by observing that the ratio of non-growth-optimal wealth to growth-optimal wealth is also a variable with a lognormal distribution. In Browne's examples, the optimal constant-mix portfolio may take an exceedingly long time to manifest its expected superiority at statistically and economically significant levels. Instead of a constant asset allocation, Browne suggests that investors consider dynamically changing the rebalancing constants to minimize the probability of a shortfall. When shortfall is defined in terms of a terminal wealth target, this strategy is equivalent to a Black-Scholes dynamic hedge strategy for a digital option—i.e., an option that pays a target benefit if it is in the money, or zero if it is not. This strategy, although minimizing the probability of a shortfall by dynamic hedging, carries with it substantial risks—the risk of a $0 payoff. Additionally, as the target payoff date approaches, if the portfolio is underperforming, the investor must increase risk through leverage in order to boost the chance that a sudden rise in stock price will save the day: "…the active probability-	7. Probability that initial withdrawal rate must be adjusted downwards [e.g., see, Pye—2000]. Note: The concepts discussed by Browne are important for investors grappling with the problem of wealth approaching a critical Wealth/Consumption Ratio value [see: Ho, Milevsky & Robinson 1994]. Continued investment may require dynamic allocation per the Merton Optimum. Such a strategy, however, may require substantial leverage if it is to have a reasonable chance of success. Risk is a function of both probability of a shortfall plus the potential magnitude of the shortfall. This article contains significant insights into the concept of risk that increases at an increasing rate. Simply "curing" an emerging shortfall by increasing equity exposure is a strategy fraught with peril.

		maximizing strategy gives results that are orders of magnitude better than the comparative results for the constant allocation (optimal-growth) strategy analyzed. The downside of course, is that under this strategy, the terminal value of the portfolio at time T has positive probability of being 0, as it is essentially an options strategy." "The major problem with probability maximization is that the payoff function is binary valued (1 at the investment goal and 0 elsewhere). Therefore, if there is a finite deadline, significant risk-taking occurs near the deadline if wealth is far from the investment goal."	
1999	"Using Asset Allocation to Protect Spending," Philip H. Dybvig Financial Analysts Journal vol. 55 no. 1 (January/February 1999), pp. 49 – 62.	The author claims that asset allocation decisions and spending decisions must be linked. The article is intended primarily for managers of endowment funds (perpetual-life portfolios). Current practice is to link spending to long-term expected return. This linkage is static, however, and does not provide any dynamic feedback. Endowments traditionally have dealt with perturbations from long-term expected returns by adopting a smoothing rule for spending—spend x% of the average value over the previous y months. Dybvig claims that such rules are *ad hoc* and have little basis in theory: "Averaging reduces the typical magnitude of year-to-year changes in spending, but it does not change the overall shape of expenditures over time. Indeed, in principle, if a more persistent decline in the stock market should occur than has occurred in this post-World War II sample, a much larger dip could occur under averaging than under the traditional rule." Note: This is a criticism of an autopilot unitrust distribution formula often recommended for family trusts. Endowments wishing to achieve a level of "protected spending" on a perpetual basis, can only spend at the rate of W*r [r = the risk free rate]. This will result in a substantially lower spending level than is currently found in most endowment programs. Conceptually, the author suggests that an endowment split into a "safe" portfolio (fixed income investments delivering immunized cash flows to cover future projected	An endowment's budget constraint [Current wealth must equal or exceed PV current projected expenditures] is also a retiree's budget constraint. Dybvig points out that the US stock market is atypical of other world stock markets because of its survival and prosperity. In the Dybvig formula for perpetual protected spending, the "Floor" is determined not by an arbitrary lower bound wealth amount but by the Present Value of Minimum Spending Needs. Because the risk free rate constantly changes, the PV of the spending need is stochastic. This means that the "cushion" is also a random variable. A built-in decline in spending [i.e., a "front-loaded" retirement] makes it possible to support a higher level of current spending: $\alpha = K[(w_t - s_t)/(r + d)]$ where d is the rate of decline. Dybvig advocates a sequential (year-by-year) decision making process in which asset allocation changes dynamically to reflect changes in value of

		expenditures) and into a "performance" portfolio (stocks seeking returns in excess of the risk free rate). To the extent that the fund's current wealth is in excess of the PV of projected expenditures, it has a "cushion" that can be invested in risky assets. The PV of the fund's projected expenditures must equal current wealth—this is the budget constraint. It is only a short distance from this concept to a floor + multiplier asset management strategy. The proposed strategy is as follows: The amount to invest in the risky asset portfolio [α] is: $$\alpha = K\left(w_t - \frac{s_t}{r}\right)$$ Where K is a weighting factor (the "Multiplier"), w is wealth, s is the dollar spending target, and r is the riskless rate. The term $\left(w_t - \frac{s_t}{r}\right)$ is the "Cushion" or current wealth less the amount that would have to be invested to generate interest sufficient to meet the spending target. If fund value decreases to the point where it equals the PV of spending, all assets are in the immunized portfolio—thus minimum spending is protected in a down market: "If s_t/w_t equals r, then the portfolio is fully invested in bonds."	wealth, endowment fund donations, interest rates, spending objectives, etc. Changes are made to protect spending [Standard of Living] *not to reflect market forecasts*. Dynamic asset allocation does not equal market timing. Note: This article is an important extension of Tobin's two-fund solution: a "safe" or immunized portfolio + a "performance" portfolio for surplus wealth. However, in a low interest rate environment, as 'r' approaches 0, the s_t/r term goes to infinity—the cushion evaporates and the endowment fund is "stopped out" of the potential for future portfolio growth. Note: The traditional view of Investment Policy is that spending depends on investment policy. Dybvig turns this upside down and makes dynamic asset allocation a function of target spending. If asset allocation and spending decisions are linked, a credible portfolio monitoring is key.
2000	"Annuity Markets in Comparative Perspective," Estelle James and Dimitri Vittas, Policy Research Working Paper 2493, The World Bank (November 1999).	The authors examine the annuity markets in several countries [Canada, UK, Switzerland, Australia, Israel, Chile and Singapore] in order to ascertain if they offer a favorable alternative for producing retirement income when measured by the money's worth ratio [MWR]: "…the present value of the expected stream of benefits divided by its initial cost." In many countries, when discounting by the risk free rate, the ratio approaches one. This result is unexpected because "…the annuity company is incurring some administrative expenses and is providing investment and longevity insurance, which are not cost free." Higher	

		discount rates, of course, reduce the ratio's value; and, inflation-adjusted annuities have ratio values that are 7 percent to 9 percent lower than comparable nominal annuities. One possible reason for the low ratio value on inflation-adjusted annuities is greater adverse selection because the benefits are back loaded. However, the authors believe that the best explanation lies in the fact that the issuing insurers cannot engage in effective risk intermediation because the market for inflation-adjusted bonds does not provide suitable long-term securities: "If they try to avoid inflation risk, they get lower yields. If they invest in higher yields, they face inflation risk. Even if they choose the lower yield route, indexed annuities expose them to higher reinvestment risk and consequently to higher reserve requirements." Evidence indicates that "…voluntary annuities are a 'luxury' good with a high income elasticity of demand so wealthy people, who have greater longevity, are disproportionately buyers." For insurance companies the costs of issuing an annuity are as follows: in Canada total expenses are 5.5% of premiums, including 3% sales commission; in Singapore they are 4% including a 1% commission. Commissions in Australia and Chile are as high as four and five percent respectively. In the authors' opinion, the costs are "…covered by the spread between the risk free rate on which the high MWR was based and the riskier portfolios in which they invest. Moreover, insurance companies can invest their reserves in long term assets which may earn a higher return, while individuals still get some of their returns in short term payouts." Annuities convert risky assets into safe payout streams "…by investment diversification, by sharing risk across several different product lines including life insurance and annuities whose risk is negatively correlated, and by paying a premium to stockholders whose profits (positive or negative) act as a buffer between unexpected events and their insured customers." However, for any block of policies, "It will	

		be many years—perhaps 20 or more—before the company knows for sure whether it has made a profit or a loss….”	
2000	“Optimal Annuitization Policies: Analysis of the Options” M. Milevsky, North American Actuarial Journal vol.5 no. 1 (2001), pp. 57 – 69.	Paper recaps and summarizes several earlier papers by Milevsky. It recognizes that bequest objectives and high annuity loads may act as deterrents to annuitization; but, submits that the impact of mortality credits at older ages is sufficiently strong to make annuitization attractive for many retirees in the face of all but the strongest bequest preferences. Prior to reaching advanced age, retirees should consider self-annuitization because, given the loads on annuity products, a portfolio of risky assets provides a high probability of allowing the investor to make disbursements equal to the annuitized periodic income stream, while, at the end of a specified planning horizon, retaining sufficient wealth to purchase an equal or greater income stream via a commercial annuity. Milevsky addresses some potentially unresolved issues. For example, he states that the propensity of a portfolio of risky assets to beat the returns available from annuities purchased around normal retirement age 65, is not simply a function of expected outperformance of equities over fixed income over five to ten year periods. Milevsky notes that the risky-asset portfolio must also beat the mortality credits that unfold for survivors in the annuity pool. He contends that this is difficult for a proxy one or two asset class portfolio to outperform annuities purchased on or after ages 75-80. In addition to the list of factors (loss of liquidity, loss of ability to make bequests, high loads and expenses) that researchers have traditionally advanced to explain the “annuity puzzle,” Milevsky cites several additional items: (1) the adverse selection factor inherent in the annuitant mortality table; (2) the absence of a real or inflation-protected annuity; and (3) non-rational behavioral justifications (control issues, etc.). However, when interest rates are extraordinarily high, annuities may	Creating a synthetic annuity is, in some respects, what a trustee does for a current income beneficiary. The question becomes, when, if ever, would a trustee be justified in using some or all of the trust assets to purchase a commercial annuity for the benefit of the current beneficiary; and, when, if ever, would a trustee be justified in using either some or all of the remaining assets, or using a portion of the annuity income stream, to purchase a life insurance policy for the benefit of the remainder beneficiary(s). See [2000] article by Rubin: “Advantages of Annuitization versus Systematic Withdrawals” Note: the annuity table issue is important. Using an annuitant table or an annuitant table with a projected mortality improvement is, in effect, using a loaded table because it is the table upon which the insurance industry bases it statutorily-required reserves. The table itself is loaded and profitability for the insurance industry emerges over time as the longevity of the actual annuitant pool diverges from the projected annuitant pool. This is why the SOA RP 2000 table is a good one—it is adjusted for high-income white-collar retirees; and, there is evidence that higher income translates into longer life expectancy. The essay’s “top-down” annuitization strategy differs from the earlier “optimal stopping time” [PV Consumption = Current Wealth] strategy advocated by Milevsky in earlier papers. Compare with [1997]

		become sufficiently cheap to appeal to retirees of all ages. The analysis is very sensitive to the spread between the interest rate credited to the annuity (the insurance company discounting rate = cost of capital or other benchmark) and the rate or return available in the investment markets. As the spread approaches zero, it is increasingly difficult to beat an annuity. Milevsky creates a two-period utility of consumption “microeconomic consumer choice model.” Assuming that $1 of wealth must be consumed over two periods (i.e. no bequest) and assuming logarithmic preferences—i.e., log utility— [where p = probability of survival; and ρ = consumption preference discount rate], the utility maximization model is: Max E[U] = $\frac{p_1}{1+\rho}\ln[C_1]+\frac{p_2}{(1+\rho)^2}\ln[C_2]$ Subject to: 1= $\frac{C_1}{1+R}+\frac{C_2}{(1+R)^2}$ Milevsky forms the Lagrangian (lamda times the constraint equation) and sets the first derivatives to zero so that he arrives at three equations in three unknowns: $\frac{\delta L}{\delta C_1}=\frac{p_1}{(1+\rho)C_1}-\frac{\lambda}{(1+R)}=0$ $\frac{\delta L}{\delta C_2}=\frac{p_2}{C_2(1+\rho)^2}-\frac{\lambda}{(1+R)^2}$ =0	Ho, Milevsky and Robinson paper. Note: the implication of a forecasted low future equity risk premium may have important implications. If interest rates move higher, the annuity in the future becomes relatively cheaper. Compare this argument with [1999] Mitchell, Portoba, Warshawsky article: “New Evidence on the Money’s Worth of Individual Annuities.” If interest rate environment is historically low, it may be unwise to lock in an annuity payout rate.

		$$\frac{\delta L}{\delta \lambda} = -\frac{C_1}{(1+R)} - \frac{C_2}{(1+R)^2} + 1 = 0$$ Thus, the optimal consumption, in the absence of annuities, is: $$C_1^* = \frac{p_1(\rho R + R + \rho + 1)}{p_2 + p_1\rho + p_1}$$ and $$C_2^* = \frac{p_2(1+2R+R^2)}{p_2 + p_1\rho + p_1}$$ In the presence of an annuity market, however, the budget constraint equation changes (to reflect the effect of mortality credits): $$1 = \frac{p_1 C_1}{1+R} + \frac{p_2 C_2}{(1+R)^2}$$ This changes the lagrangian, and the optimal consumption in the presence of annuities equals: $$C_{1annuity}^* = \frac{\rho R + R + \rho + 1}{p_2 + p_1\rho + p_1}$$ and $$C_{2annuity}^* = \frac{1+2R+R^2}{p_2 + p_1\rho + p_1}$$ Where $\rho = R$ and (by definition) $p_1 > p_2$. With a large spread between the values (i.e. high mortality credit), the utility of optimal consumption with an annuity will be higher than the utility of optimal consumption in the no annuity case.	

		The remainder of the essay follows past work. Milevsky calculates the possible Wealth / Consumption ratios that exist in the marketplace given the pricing of annuities. [Consumption = (Wealth ÷ cost of annuity)]. If the risky asset portfolio earns specific deterministic rates, then it will either last into infinity or decline to zero in a specified number of years. The probability of bankruptcy is conditioned on the probability of survival at or longer than the year of bankruptcy. To model non-deterministic earnings rates, Milevsky employs Monte Carlo simulation analysis comparable to that used in his previous essays.	
2000	"Sustainable Investment Withdrawals," Gordon B. Pye The Journal of Portfolio Management vol. 26 no. 4 (Summer 2000), pp. 73 – 83.	The traditional approach of economists to optimizing withdrawals from a risky portfolio has been to maximize the discounted expected value of a utility function for withdrawals. This approach has met with several problems: 1. It is difficult to specify the utility function, 2. There is usually the assumption that the utility of a withdrawal is independent of previous withdrawals—i.e., no habit formation. 3. Practitioners assume that clients wish to achieve a "sustainable standard of living" which is often comparable to a pre-retirement standard. Risk, therefore, is the likelihood that the portfolio will be unable to sustain the targeted initial (real) withdrawal. "Beneficiaries must decide on the extent to which they are willing to sacrifice standard of living for sustainability and future value." Note: Risk is here defined as the probability that the initial withdrawal amount will have to be adjusted downwards in the future. Pye defines "sustainability" as not being required to ever reduce the initial constant dollar withdrawal amount while still preserving the portfolio's targeted ending value. Sustainability does not equal portfolio depletion. "The key is when and how to reduce the withdrawal when there has been a shortfall. This has an important effect on sustainability." Conceptually, if there is a limit on the extent to which withdrawals are	The traditional utility-based approach demands an explicit form of the utility function. However,, in this approach, "beneficiaries exercise their preferences after the results have been obtained, making only implicit reference to a utility function to do so." Pye's model places a limit on the withdrawal amount. The limit is calculated [and, periodically recalculated] as the largest stream of constant dollar withdrawals that can be made over the remaining planning horizon so that the required ending target amount is preserved. This is one possible model for a trustee seeking to balance the interests of competing beneficiary classes. Returns are considered to be independent and are modeled as a lognormally distributed time series. "Assuming the annual real return on stocks has an expected mean of 8% and standard deviation of 18%, a 4% initial withdrawal is highly sustainable over a thirty-five-year horizon, given the other base case assumptions."

		allowed to decline across the entire planning horizon, then long-term sustainability risk decreases but the short term risk of having to adjust the withdrawal downward increases. Pye tests the sensitivity of sustainability to changes in model inputs. One important lever to enhance the probability of success is to reduce the initial target withdrawal amount so that more money remains in the portfolio during its initial years of operation. Pye uses the term reserve for this "excess" money. Under his model, cutting the initial withdrawal amount by 25% increases the probability of portfolio sustainability—i.e., no decline in the withdrawal amount-- throughout the entire withdrawal period from 15% to 45%. The risk/reward tradeoff is deciding "how much standard of living he is willing to sacrifice to improve sustainability and future values." Note: this is a variation on a "back-loaded" distribution strategy.	Pye continues the line of investigation opened in this article, and more fully develops it, in his 2012 book, <u>The Retrenchment Rule</u>.
2000	"Self-Annuitization and Ruin in Retirement," Moshe A. Milevsky and Chris Robinson, North American Actuarial Journal vol 4. No. 4, (2000) pp. 112-124	This paper is an early version of "Is Your Standard Of Living Sustainable during Retirement? Ruin Probabilities, Asian Options, and Life Annuities," published in 2001. The authors review their model and point out that it provides an approximation both to the distribution of an Asian call option as well as to the distribution of ruin probabilities. The paper highlights the fact that a critical variable in the sustainability of fixed real consumption during retirement is the Wealth/Consumption ratio selected by the retiree. The consumption decision must be appropriate in terms of the annuity cost per unit of wealth. Self-annuitization at real rates carries a high risk of ruin even for diversified portfolios. The paper is followed by comments and criticisms of several commentators in a Discussions Appendix to the article. A response by Dhaene, Goovaerts & Kass, points out that the approximation is equivalent to the "Callogero model of mathematical physics, which was solved in the early 1970s." The commentators also provide alternative formulae and representations of stochastic annuities that provide upper and lower bounds for present value functions.	A good discussion of the methodological differences between the "risk of ruin" preferencing metric and the "utility maximization" preferencing metric. The shortfall probability metric may be measuring the likelihood of an event that no rational investor would permit. The Reciprocal Gamma Approximation suggests that, for a male age 65, the allocation offering the lowest probability of ruin is 60% Canadian equity/ 40% Canadian bond with a failure rate of 17%. The consumption target is a wealth to consumption ratio of 14 [7.14% of initial capital] For a female age 65, the optimum allocation is 80% equity / 20% bond with a failure rate of 26.7%. Historically, articles have tried to identify the critical factors in determining portfolio sustainability. These include: 1. Asset Allocation; 2. Spending Policy;

		J.R. Brown comments that the probability of ruin statistic is useful in that it provides, in a single number, helpful financial planning information—i.e., a risk metric—to the retiree. However, Brown contends that from the perspective of economic theory, the value of the probability of ruin approach, as modeled in the paper, is limited. The model does not follow the standard model of consumer utility maximization. The optimal consumption rule generated by the life-cycle model is not a constant real amount consumed in each period during retirement. Indeed, any consumption path that allows the possibility of generating zero consumption during any time period would not be followed by a rational consumer (zero consumption = infinitely negative utility). Under classic economic theory, Brown points out, the individual maximizes the following utility function: Max $\sum_{t=0}^{T} \frac{\prod_{j=0}^{t}(1-q_j)*U(C_t)}{(1+\rho)^t}$ Where T is the max life span, ρ is the time preference discount rate and q is the mortality hazard rate vector. In the absence of annuities, the present value of consumption is constrained because, discounted at the riskless real interest rate 'r', it cannot exceed the value of an individual's initial wealth. [Note: this argument is echoed in "Developments in Decumulation…." by Olivia Mitchell in 2002. The argument appears in a number of future papers by Mitchell and others; and, to a certain extent, recalls Dybig's arguments in his 1999 article "Using Asset Allocation to Protect Spending."] What is the optimal consumption path? Brown defines a one-period utility function exhibiting constant relative risk aversion as:	3. W/C Ratio. Compare Brown's comments to Albrecht, Maurer & Ruckpaul [2001] "Shortfall-Risks of Stocks in the Long Run." Note: Compare with the distribution of the PV of annuity in Feng Li [2008] Ruin Problem in Retirement Under Stochastic Return Rate and Mortality Rate and its Applications.

$$U(C_t) = \frac{C_t^{1-B}}{1-B}$$

where B is the Arrow-Pratt coefficient of relative risk aversion, and 1/B is the elasticity of intertemporal substitution in consumption. By substituting U(C) into the utility maximization function, one can solve for the optimal consumption path. This path must satisfy the following relationship:

$$C_{t+1} = C_t * \left[\frac{(1-q_{t=1})(1+r)}{(1+\rho)} \right]^{1/B}$$

A constant real income stream is optimal *only* when the above equation is equal to 1. But even if it is equal to 1 during any period, the constant change in the mortality vector would change this condition over an individual's lifetime unless B approaches infinity. But this implies that the individual is infinitely risk averse and has an intertemporal elasticity of substitution approaching zero. Further, most empirical evidence suggests that individuals have a value of B close to one, which corresponds to log utility.

Brown concludes that Milevsky and Robinson illustrate the dangers of self-annuitization but end up measuring the probability of an event that no rational investor will allow to occur.

Finally, Mark Warshawsky comments that individual preferences are important in the decision to annuitize wealth. He refers to the preference for liquidity, charitable giving, and testamentary bequests. These factors may be more important to some individuals than the uncertainty surrounding date of death or investment returns (especially for high income individuals).

Warshawsky also points out that the authors illustrate high consumption/wealth ratios; and, therefore, it is not surprising that the

		ruin probabilities are also high.	
2000	"Optimum Withdrawals from An Asset Pool," Jaye C. Jarrett and Tom Stringfellow. Journal of Financial Planning vol. 13 no. 1 pp. 80 – 93.	Tests four spending policies [Constant nominal, Increase by set factor, Increase by inflation factor and Withdraw a fixed percentage of market value] across three portfolio objectives: (1) Final Portfolio = zero, (2) nominal principal is preserved, and (3) inflation-adjusted principal is preserved. All tests are historical back-tests over rolling periods of various length. Portfolios are rebalanced annually. Sample period is 1926 through 1998. Note: this article argues that it is more prudent to increase bond exposure rather than equity exposure.	Sample test conclusion: "…a portfolio of 100 percent large cap stocks would allow you to withdraw 3.92 percent a year over a 20-year time frame…..if the fund contained 25 percent large cap and 75 percent intermediate government bonds, the withdrawals can increase to 6.7 percent…."
2000	"Financial Consequences of Distribution Elections From Total Return Trusts," Patrick J. Collins, Sam L. Savage and Josh Stampfli, Real Property, Probate and Trust Journal vol. 35 no. 2 (Summer 2000), pp. 243 – 304	The article initially presents a base case analysis for a trust-owned portfolio with an initial $1 million value. The portfolio's allocation is 60% to the S&P 500 stock index and 40% to the general US Government/Corporate bond market. The trust instrument directs the trustee to preserve the value of the initial corpus while providing a lifetime income to the trust's income beneficiary. The base case assumes that the trust employs a unitrust distribution formula of 4% of yearly trust value. The base case analysis utilizes a simulation model to examine the range of probable inflation-adjusted income distributed to the lifetime income beneficiary, as well as the range of terminal wealth distributions upon the death of the income beneficiary. The analysis focuses on the amount and sustainability of income and the amount remaining for final distribution. As the analysis moves beyond the base case, it considers a variety of unitrust distribution percentages as well as a variety of asset allocations for the two-asset class portfolio. The base case, across a spectrum of asset allocations, illustrates important interrelationships between the variables of interest. Specifically, it depicts the tradeoffs between higher distribution percentages and total inflation-adjusted income streams; and, between higher distribution percentages and terminal wealth. The following tables summarize model results:	Many irrevocable family trusts contain provisions directing the trustee to provide lifetime income to the current beneficiary and to distribute terminal wealth to remainder beneficiaries. A trust creates a dual set of claims against its assets whenever the trust instrument directs the trustee to preserve either the nominal value or the inflation-adjusted value of initial assets for the remaindermen. One claim is the claim against trust income held by the current beneficiary; the other is the claim against terminal wealth held by the remaindermen. In such cases, trustees must invest trust assets prudently and, in their trust administration, must impartially balance the competing claims of each beneficiary class. A failure to fulfill trustee duties may lead to allegations of a fiduciary breach which, if upheld by a court, often result in the trustee's personal liability to pay economic damages. The stakes can be high in fiduciary breach litigation; and, therefore, the trust and estate section of the bar began extensive discussion of investment issues and trust distribution strategies following the 1994 adoption of the Uniform Prudent Investor Act by

Remainderman's Interest: 80/20 vs. 60/40 allocation					
Distribution Formula	30 Yr. Expected Value	Standard Deviation from Expected Value	Minimum 30 Year Portfolio Value	Maximum 30 Year Portfolio Value	Failure Rate
80/20 Unitrust 4% of value per year	$4,603,049	$6,029,717	$439,455	$14,061,508	c. 14%
60/40 Unitrust 4% of value per year	$3,782,785	$4,636,270	$457,106	$11,205,796	c.18%
80/20 Unitrust 5% of value per year	$3,288,200	$4,255,121	$335,539	$10,758,923	c. 24%
60/40 Unitrust 5% of value per year	$2,808,599	$3,446.862	$339,463	$8,301,355	c. 26%

Current Beneficiary's Interest: 80/20 vs. 60/40 allocation				
Distribution Formula	30 Yr. Expected Value	Standard Deviation from Expected Value	Minimum 30 Year Portfolio Value	Maximum 30 Year Portfolio Value

the National Conference of Commissioners on Uniform State Laws. In many cases, this discussion has been confined to legal journals and monographs most of which fail to appear as bibliographical references for articles regarding portfolio sustainability and wealth preservation in investment-oriented journals. This is the case despite the fact that many of the issues addressed by financial economists and investment advisors first appear in legal journals.

The authors are among the first to provide an integrated and coherent discussion of the effects of the following factors on prudent portfolio management:

- Role of inflation paths as a key determinate of portfolio success
- Impact of diversification on portfolio sustainability
- Sequence risk—the role of variance on compound return.
- Examination of the fallacy of income smoothing techniques as tools for enhancing sustainability
- Model risk and the impact of jump discontinuities in asset price trajectories.

They are also among the first to develop and implement risk modeling techniques designed to facilitate portfolio monitoring.

80/20 Unitrust 4% of value per year	$4,603,049	$6,029,717	$439,455	$14,061,508
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The article recalculates the risk model's outputs by expanding the asset allocation from a simple two-asset class portfolio to a globally diversified thirteen-asset class model. Outputs indicate a substantial reduction in the failure rate for every measured category within the spectrum of equity weightings.

Note: This is an early—if not the first—exploration of the benefits of diversification with respect to portfolio sustainability.

The analysis then turns to a consideration of fixed constant-dollar trust distribution formulae—i.e. formula more commonly termed the 4% and 5% spending rules in the financial planning literature. Histograms compare the risks and benefits of adopting either an inflation-adjusted formula or a percentage of value formula for both current and remainder beneficiaries in selected years—e.g. monthly income in year five under each type of distribution strategy.

Part two of the article presents a lengthy discussion about why algebraic

		formulae derived from back testing historical returns cannot produce safe and sustainable withdrawal formulae. Historical back testing and closed-form analytical solutions are inherently flawed methodologies. The argument has two parts: 1. Any irrevocable distribution formula (autopilot withdrawal rule operating under all economic conditions) is both prone to failure and is not in the best interest of either beneficiary class; and, 2. It is the mathematics of stochastic calculus that underlies the manufacture and sale of derivative financial instruments that provides the tools to elucidate the underlying risks and returns faced by managers of investment portfolios. Algebraic calculations are irrelevant in a stochastic environment; calculations based on expected value give way to calculations based on the magnitude and direction actual price movements. Much of the argument is couched in terms of the exercise of the duty of trustee discretion. However, it forcefully underscores the importance of flexibility of portfolio management with respect to distribution and investment policy.	
2000	Annuity and Insurance Products for Impaired Lives, Ross Ainslie The Staple Inn Actuarial Society (May 2000)	A detailed discussion of the impaired lives market from the prospective of risks faced by insurance companies in the development and marketing of appropriate products. Much of the article focuses on factors that separate a standard annuity from an impaired-life annuity. The definition of an Impaired Annuity is: '…the client is offered a higher annuity income than that available to a 'standard' annuity purchasers." The author describes three market segments for impaired annuity products: (1) structured settlements, (2) retirement income enhancement, and (3) nursing care annuities. Among the interesting observations made by the author is: "If these 'unhealthy' lives were to buy enhanced rather than standard annuities, then the average mortality experienced by the standard group would be lower. This could result in lower profitability for this business."	
2000	"Wealth Planning	An introductory level article directed at financial planners who are	Relatively safe withdrawal rate is three percent.

	Under Uncertainty," Bernard J. McCabe & Charles P. Boinske Journal of Financial Planning vol. 13 no. 3 (March 2000), pp. 84 – 96.	unfamiliar with simulation. The authors argue that Monte Carlo simulation of normal or log-normal distributions is superior to a bootstrap process. A bootstrap assumes "…that the same kinds of returns that occurred during the past 30, 60 or 100 years will happen again, but permuted and shuffled around randomly." A Monte Carlo simulation, by contrast, enables the model user to redo the experiment with parameter changes in order to determine parameter sensitivities. They note some rules of thumb: "If you expect to have 25 years of living off your fortune and you start out spending six percent or more of the total, you're in the low-end category [at risk for portfolio depletion]; if everything else is the same but spending is three percent, high-end risk [risk of unspent funds] should be the focus. Between three and six percent means both should be addressed." Note: Monte Carlo simulations assume constant 2.2% inflation.	Above six percent is risky.
2000	"Risk tolerance and asset allocation for investors nearing retirement," Govind Hariharan, Kenneth S. Chapman and Dale L. Domian Financial Services Review vol. 9 no. 2 (Summer, 2000) pp. 159 – 170.	The authors point out that, in general, financial planners recommend that risk averse individuals hold a lower percentage of high-risk assets (i.e., equity) in their investment portfolios in favor of owning lower-risk bonds. Additionally, as risk tolerance allegedly decreases as retirement age draws nearer, planners recommend a decrease in the portion of equity held within the investment portfolio. Such advice, although intuitively appealing, conflicts with the behavior of rational expected-utility-maximizing investors as predicted by the CAPM. According to Tobin's 'separation theorem,' if risk aversion increases, the investor will own more of the risk free asset (CDs, Treasuries. Government savings bonds, etc.), but will not change the stock-to- bond ratio within the risky asset portfolio. The authors seek to test whether retirees follow the CAPM decision making process by looking at data from the 1992 Health and Retirement Survey. This is an ongoing longitudinal study that collects data on risk preferences, age, health, bequest objectives, asset ownership, education, etc. The first part of their study is a brief review of "investor	In general, if human capital constitutes a high proportion of total wealth, this asset "crowds out" T-Bills. One in a series of articles that assert that an individual's risk tolerance "…can be inferred from the asset allocation decision." See Sid Browne's 1999 article on using the Merton Optimum to derive the optimal asset allocation.

		risk aversion" literature. In 1975, Friend and Blume infer the individual's risk tolerance by calculating the percentage of assets held in risky securities. They trace this approach through a number of authors up to the 1999 article of Bajtelsmit, Bernasek and Jianakoplos ("Gender difference in defined contribution pension decisions") where, according to CAPM, an individual determines the proportion of wealth devoted to risky assets (a_i) as: $$a_i = \left(\frac{E(r_m - r_f)}{\sigma_m^2}\right)\left(\frac{1}{C_i}\right)\left(\frac{1}{1-h_i}\right)$$ where: The initial term on the right is the "Merton Optimum," C_i is the relative risk aversion of person i h_i is the ratio of human wealth to net worth. All else equal, investors with a high percentage of human capital hold larger proportions of wealth in risky assets. The authors take the survey data and consider the factors determining a participant's propensity to hold a portfolio of risky assets (stocks and bonds) and a portfolio of risk-free assets (CDs, T-Bills). They specify a regression equation with the dependent variable defined as the risk-free asset portfolio. After controlling for a variety of independent variables (inflation expectations, depression expectations, age, gender, bequest preferences, financial net worth, etc.) the authors calculate the coefficients on RISK (quantified by a series of preference questions posed to survey participants) and EDUCATION (# of years completed). In general, the results of the regression are consistent with CAPM (although R^2 values are very low). "Our regression evidence suggests that risk-tolerant individuals invest lesser amounts in Treasury Bills…. In addition, we found that the division of individual portfolios between	

		stocks and bonds was not systematically related to our measure of risk tolerance." Other findings include that men are, in general, more risk tolerant than women; and, as a consequence, may accumulate more retirement wealth per dollar of investment; and, that individuals with higher bequest preferences tend to own higher proportions of equity.	
2000	"Advantages of Annuitization versus Systematic Withdrawals," Larry H. Rubin Retirement Needs Framework, Chapter 14 Society of Actuaries (2000) pp. 153 – 158.	A Chapter from the Retirement Needs Framework SOA monograph: "…when compared to annuitization, systematic withdrawals result in higher risk with a suboptimal standard of living." The case for annuitization is based on its ability to control longevity risk: "Data from the Social Security Administration show significant improvement in mortality….If we were to apply these mortality improvement factors to the Annuity 2000 table, we find that for a couple both age 65 the probability that one will be alive past age 94 is over 50%." The author points out that TIAA, in 1984, introduced a graded-benefit payment annuity with an assumed interest rate of 4%. Earnings above 4% are used to purchase additional annuity income. The fixed annuity thus has some capacity to protect the purchasing power of the periodic benefit during inflationary periods. Note: Compare to [1999] Jeffrey Brown article maintaining it is irrational to buy an annuity for a price above actuarially fair value and to sell simultaneously an annuity (i.e., buy an insurance policy) for less than actuarially fair value.	This is one of the few "non-sales-agenda" articles that suggests that the investor can combine annuities and life insurance contracts: "If there is a desire to bequest some of the assets built up in a defined contribution plan to an heir, then the retiree can use a portion of their annuity income to purchase a life insurance policy with benefits free from state and federal income tax and through proper planning can be made free from estate taxes….A whole life insurance policy with a $60,000 face amount purchased with a single life annuity results in a slight initial decrease in income in the early years compared to systematic withdrawals crossing over in approximately ten years….The after-tax death benefit for the life policy is higher beginning in the 24th year. This assumes no state income taxes or estate taxes which would only serve to make the life insurance policy perform relatively better."
2000	"Decision analysis using targets instead of utility functions," Robert Bordley & Marco LiCalzi Decisions in Economics and Finance (May 2000), Vol. 23, No. 1. Pp, 53 – 74.	The study seeks to outline conditions under which target-based decision making is conformable with the precepts of rational, utility-optimizing decision making as expressed in the work of von Neumann and Morgenstern and L.J. Savage. In a world of perfect certainty each feasible action 'd' will lead to a known outcome 'x_d.' The rational decision maker will select the action that maximizes—i.e., optimizes—a value function 'υ (d).' In contrast to the optimization approach to decision making, the 'satisficing' approach suggests that the difficulty of evaluating all possible feasible actions leads the decision maker to: • Pick an acceptable target which he hopes to meet or exceed;	The article represents an interesting extension of research that explores interrelations between shortfall probability risk metrics and utility-based metrics. The field of decision analysis contains a plethora of such research studies. Generally, one finds studies in support of a 'rational' school, a 'behaviorist, school, or a 'third alternative' / 'synthesis' school. Knowledge of a broad range of decision analysis

		and, • Select the first action that will be likely to meet the target. However, such a 'target-based' model is incomplete because there is often uncertainty about the target itself. When a concrete, well-defined target 't,' is replaced by a set of (acceptable) random consequences 'T,' the model suggests that the decision maker should opt for an action 'd' which maximizes the probability of meeting the uncertain target: $v(d)$ = P($X_d \succcurlyeq$ t), where $\succcurlyeq$ signifies "succeeds in equaling or exceeding the target." There is no way of telling if a decision maker is optimizing expected utility or maximizing the probability of meeting an uncertain target. However, by interpreting U(x) as equal to P(x $\succcurlyeq$ T)—i.e., the utility of an action leading to the outcome which is consequence 'x'—as the probability that x will meet the target, the decision models become equivalent. The authors continue by describing the target-based model. Given an decision 'd,' there is a set of associated random consequences 'X_d' associated with 'd' and a probability distribution 'P_d' for the random consequences. The classical expected utility model expresses the value function as follows: $$v(d) = EU(X_d) = \sum_x U(x) P_d\,(x)$$ However, the target-based model expresses the value function as follows: $$v(d) = P(X_d) \succcurlyeq T) = \sum_x P(x \succcurlyeq T) P_d(x)$$ where $P(x \succcurlyeq T)$ is the cumulative distribution function of the uncertain target and T is stochastically independent of X_d. This is the summation of the probability that the realized outcome is equal to or better than the set of acceptable consequences multiplied by the probability that action 'd' will lead to an acceptable outcome. This turns the classic von	articles is important for several reasons: 1. They provide an intellectual underpinning for structures and risk metrics found in retirement income risk models; 2. They offer a perspective that facilitates new insights; and, 3. They can promote use of helpful vocabulary for discussing difficult-to-express concepts. The article presents two probability densities for consideration. The first is a decreasing probability density (e.g., a negative exponential) suggesting that the most likely "left-hand" targets to meet are the worst outcomes. The better-outcomes on the right side are in a lower-probability region. The second is a unimodal and symmetric probability density (e.g., normal distribution) with the mode centered on the status quo values. When the two densities are integrated, the cumulative distribution functions look very different: density one takes the form of a typical concave utility function because a high likelihood of unfavorable outcomes contributes to risk aversion. By contrast, the cumulative distribution function of the symmetric unimodal distribution assumes an 'S' shape as the agent moves away from the status quo value. Moving to the left, the agent tends to be risk seeking over a loss region; moving to the right the agent manifests risk aversion over the region of gains. "In the target-based language, a unimodal probability density for the target with fatter right tails implies a behavior that is less risk averse over gains than it is risk seeking over losses."

		Neumann Morgenstern utility function into a probability distribution and makes the two models formally identical. One advantage is: "…the target-based model can avoid the notion of a cardinal utility function U(x) over consequences….For target-based reasoning, the agent must only be able to handle probability judgements." This advantage does not hold, however, for state-dependent preferences. If an agent seeks to maximize expected utility, the decision making process is one that maximizes the probability of meeting an uncertain target [T] with a cumulative density function [U(x)] provided "…that the target is stochastically independent of the random consequences to be evaluated." Given independence, the value function is $$\sum_x P(x \geqslant T) P_d(x)$$ Which is an expected utility model since U(x) = P(x $\geqslant T$). However, there are often situations where the preference weighting given to a target depends on the specific state. For example, a nominal return target may be evaluated differently when earned in a high inflation versus low inflation state—i.e., the evaluation of 'x' is state-dependent. But this returns the process into a comparison of cardinal utility across differing states; and, according to the authors: "there is no easy general recipe to convert a target-based model into a utility-based model" under such circumstances. "It is possible that target-based reasoning may offer a more convenient approach:" 1. The agent identifies relevant states and assesses the probability for each state; 2. For each state, the agent estimates his uncertain target and the conditional distribution [X_d] for that state; 3. The agent computes the probability that X_d will meet the state-dependent target; and,	

		4. The agent aggregates the assessments. [Note: this is a rationale for a multi-regime risk model.] The authors discuss several variations of individual decision making for cases lacking stochastic independence in a target-based model. These include cases in which the target in various states exhibits an underlying correlation structure—e.g., a rate of return target is a function of the underlying random variable of inflation. In other situations, the decision maker may "adapt" the target to the circumstances with which he is presented: "The agent who has access to some very promising random consequence X raises his expectations. The agent who is offered a poor prospect lowers his target." In an interesting observation, the authors state that some situations are better modeled by converting the target-based model back to a standard expected utility model. The article continues with a discussion of how well a current utility function—a reflection of the uncertainty about the current target—predicts a future utility function: "…today's estimate of the utility function in the future might differ systematically from today's utility function." For example, uncertainty regarding today's target may be expressed as uncertainty regarding the variance around a distribution's mean. Future uncertainty may be expressed as uncertainty regarding how the mean itself may change. "This has two implications: on the one hand, the change in U_t is likely to result in temporal inconsistency; on the other hand, the agent is now given a way to take into account how the future arrival of information will impact his willingness to implement today's choices and to predict how his preferences may change." The influence of uncertainty over long planning horizons has an uncertain effect on the value of options—i.e. the agent may be inclined to be more or less risky given uncertainty regarding prospects. The article concludes with a brief discussion of multiattribute decision making. Under certain conditions, the authors state "…that the problem	

		of maximizing a multiattribute utility function can be mapped to the problem of minimizing the probability of failure in a fault tree (more generally, an event tree)." Normally, evaluation of a multiattribute utility function requires stochastic independence of attributes. For example, "The assumption of utility independence is necessary to quantify how much of the overall utility of a house depends on the single-attribute utilities of price, location and facilities. It postulates that each single attribute must contribute to the overall utility independently from the other attributes. This strong assumption is often implausible." Reliability theory uses event trees to describe how higher-level objectives are influenced by combinations of lower-level objectives: "taking expectations over this tree defines the probability of achieving the overall target as a function of various combinations of price, location and facilities." Extending this concept, the authors point out that "…we can develop an alternative way of deriving the multiattribute utility function by identifying the combinations of lower-level targets whose attainment is necessary for the achievement of the overall target." The authors invoke the concept of a pyramid along with the concept of a reliability tree: "Thinking of a pyramid with the overall target T at the top, there are second-level targets T_i supporting T; and then third-level targets supporting the second-level targets T_i, etc. For each higher-level target, specify how its achievement is related to the attainment of its (possibly uncertain) lower-level targets: whether we need to meet all of them; or if it suffices to meet at least one; or if we have to meet at least a majority of the supporting lower-level targets; etc. Finally, we would compute the probability of achieving the higher-level targets as the probability of attaining the right combination of lower-level targets. Since this enables us to translate the fundamental objectives hierarchy into the standard event tree of reliability analysis, we are able to exploit the hierarchical structure of the event tree to compute these probabilities. This makes it possible to use a 'bottom-up' approach to the computation of the probabilities of meeting the higher-level target."	

		Note: this approach into the problem of multiattribute targets might be a way to reconcile the Maslow pyramid approach with the Markowitz MPT approach to portfolio construction. The reliability tree restores the concept of correlation among nodes or areas of the pyramid back into the discussion.	
2000	"Differential Mortality and Wealth Accumulation," Orazio P. Attanasio & Hilary Williamson Hoynes, The Journal of Human Resources Vol. 35, No. 1 (Winter 2000), pp. 1 – 29.	The authors assert that there is no conclusive evidence that "…people actually decumulate assets in the last part of the life cycle…." Two methodological concerns present themselves: (1) a cross-section age/wealth profile in any cohort does not necessarily translate into a profile for any individual; and (2) "If poorer individuals have higher mortality risk, one will overestimate the last part of the wealth age profile when using cross-sectional data because means (or other measures of location) are taken over a population which becomes richer as it ages." The main purpose of the study is to "correct" the wealth-age profile to evaluate the relationship between wealth and mortality. The article tabulates large amounts of data from the Survey of Income and Program Participation [SIPP]. A weighting methodology is employed to correct for a potential tendency of poorer individuals to die younger: "It is straightforward though, to deal with the sample selection by constructing weights equal to the inverse of the probability of survival for each individual in the data. The weights can be easily constructed from estimates of an equation linking mortality to wealth." The model uses a measure of relative wealth rather than an absolute level of wealth; and, it assumes that mortality depends on relative wealth with the dependence relationship effective beginning at age 50. Results indicate that "…most of the variation in mortality rates is concentrated in the lowest wealth percentiles." "…most of the differences in mortality rates are between the lowest 20 percent of the wealth distribution and the rest of the population…." Additionally, the evidence suggests that "…there is no strong evidence of wealth decumulation in the data." The study concludes that there is a strong differential mortality as a function of wealth.	
2000	"Can Annuities Pass Muster?" Bests	Short Article on Class Action suit against several insurance companies for	Information important re: trustee duty to avoid unjustified / unwarranted costs.

	Review vol. 101 no. 3 (July 2000) pp. 103 – 108.	selling variable annuities in pension plans. "Fees are exorbitant for services provided" Fees can deplete up to one/third of account value over time. Variable Annuity, M&E charges and Sub-Account management expenses can total up to 250 basis points per year.	
2000	"Analysis of Financial Needs in Retirement: A Multistate Approach," Bruce L. Jones in Retirement Needs Framework SOA Monograph M-RS00-1 (January 2000), pp. 81 -86.	The author presents a stochastic multistate model of a Markov transition process wherein the investor's spending needs are, in part, a function of the retiree's health state. In states of good health, the need to spend heavily on medical care is low; in poorer health states, the need for cash increases. For example, a change in health may necessitate a change in residence. Jones states: "A retirement income determined so that the actuarial present value of the income equals the actuarial present value of the costs will likely not provide the retiree with adequate security, since there may be a high probability that the income will be inadequate. To help determine a more appropriate income, it is useful to know the distribution of the 'adequate income amount,' that is, the income that, if paid through the lifetime of the retiree, has the same present value as the costs." To determine the distribution, Jones simulates a large number possible health state transition processes, calculates the amount of 'adequate income' for each trial, and rank orders the outcomes. Depending on the number, magnitude, length and sequence of less-than-optimal health states—as well as transitions back to improved health states—Jones can generate histograms of retirement cash needs. The model incorporates four states: active, frail, disabled, and dead—with a cost structure associated with each state. Only certain transition sequences are permitted—it is rare to find a transition from state four. Many investors die without experiencing significant expenses for frailty, sickness or disability. However, the distribution of costs exhibits a strong right skew which quantifies the financial risk faced by retirees for health care expenses. The pattern of income needs faced by retirees varies	This article provides a useful extension of model of retirement spending needs. It links spending to the state of health and stochastically models changes in the health state throughout the retirement period. This is in contrast to most studies which assume that spending is either a constant percentage of the portfolio's initial value, or a more well-behaved function that is tied to a state variable like inflation. Stochastic spending is also examined in Robinson and Tahani [2010]. The retiree's health state is incorporated into the study by Kim Balls [2006] which also models health transitions via a Markov transition matrix. The utility of consumption is a function of life expectancy in Balls' model. See also Turra and Mitchell [2008] "The Impact of Health Status and Out-of-Pocket Medical Expenditures on Annuity Valuation." Calls into question the use of an annuity as a valid benchmark for calculating the total cost of retirement. An annuity is a mortality-adjusted present value cost for pre-determined periodic cash flows. But many retirement income needs cannot be pre-determined—i.e., they are stochastic. When costs are stochastic, a level income may be suboptimal.

		considerably: "The level income that has the same actuarial present value as the costs may not provide adequate security. It is therefore of interest to examine the distribution of the adequate level income. This is the level income that has the same present value as the costs given the outcome of the multistate process."	
2001	"Why Stock Return Volatility Really Matters" G. Andrew Karolyi Working Paper http://bryongaskin.net/education/MBA%20track/Current/Mba611/Assignments/Project/WhyVolatilityMatters.pdf	Survey article examines econometric characteristics of volatility: mean, standard deviation, skewness, kurtosis and autocorrelations. Significant values for autocorrelation coefficients indicate that volatility remains volatile—"volatility clustering." The author measures annual volatility not as the sum of squared deviations from the 12-month mean return, but as the sum of squared deviations plus twice the cross-covariance in monthly returns to reflect autocorrelation. The summary of research reveals five "stylized facts" about stock market volatility: 1. Macroeconomic factors cannot explain return volatility. Volatility does not stem from innovations in dividends or discount rates. Interest rates, term yield spreads or default yield spreads have only slight forecasting power. 2. There is a relationship between trading volume and return volatility (arrival of new information motivates trades). Although the association can be demonstrated, there is no evidence that trading drives volatility or visa-versa. They may both be generated by an unknown underlying process. 3. Volatility in stock prices increases more following bad news than good news. A traditional explanation is the "leverage effect" (equity becomes more risky as stock prices decline). Other explanations include time-variation in the risk premium [higher required risk premium leads to lower current stock prices]. More recent studies suggest that there is a positive relation between volatility today and returns today, but a negative relation between volatility tomorrow and returns today. There is no consensus explanation, however. 4. Volatility may "spill over" into international markets. However, it is difficult to find and fundamental or macroeconomic news that could explain the volatility spillovers.	Conventional Monte Carlo simulation analysis does not consider time varying volatility or risk premiums. Volatility innovations are not random draws from a normal distribution. Such a model may significantly distort "true" probabilities. Any argument advancing the proposition that Prudence = Probability demands that you have the correct probabilities. Compare the leverage effect in stock risk with the "leverage" in the remainderman's position as the wealth/consumption ratio changes.

		5. Derivatives do not exacerbate volatility.	
2001	"Sustainable Withdrawal Rates From Your Retirement Portfolio," Philip L. Cooley, Carl M. Hubbard & Daniel T. Walz Financial Counseling and Planning vol. 10 no. 1 (2001), pp. 39 – 47.	Whereas previous studies considered retirement portfolio sustainability given a fixed withdrawal rate or amount, this study seeks to determine the percentage withdrawal amount that will result in a terminal portfolio value of at least $0.00 at the end of a pre-specified planning horizon (15, 20, 25, and 30 years). The authors develop a probability of success measure by back-testing historical portfolios using monthly returns on the S&P 500, the Salomon Brothers Long-Term High-Grade Bond Index / S&P monthly high-grade corporate composite, and the 30-day return on U.S. T-Bills over the period January 1926 through December 1997: "In that 72-year period there are 58 overlapping 15-year payout periods, 53 overlapping 20-year payout periods, 48 overlapping 25-year payout periods, and 43 overlapping 30-year payout periods." Withdrawal amounts are adjusted annually for CPI, and portfolios are rebalanced monthly. For withdrawal rates ranging from 3% of initial portfolio value through 12% of portfolio value, each historical period return sequence either succeeded or failed to provide portfolio sustainability. After solving for the success rates for each withdrawal percentage over each period-specific planning horizon, the authors present a table of success rates for a portfolio allocated 60% stock / 30% bonds, and 10% T-Bills. The study provides coefficients of success [percent of all past payout periods supported by the portfolio] for both nominal withdrawals and inflation-adjusted withdrawals. Additionally, solving for the withdrawal rates from a 60-30-10 allocation assuming a terminal value of $0.00, the authors provide a statistical table of means, standard deviations, medians, minimums and maximums of sustainable withdrawal rates (% of initial value) on both a nominal and constant-dollar basis for each payout period. The data indicate that long-term sustainability at nominal withdrawal rates higher than 6 to 7 percent requires at least a 50% equity weighting for payout periods longer than 25 years. High success rates for inflation-	The article explores the topic of portfolio sustainability for various withdrawal rates and planning horizons. The authors note: "Since the choice of a withdrawal rate involves individual preference for current consumption, uncertainty of life expectancy, and variable financial needs, there is no single globally optimal withdrawal rate." Somehow this statement was conveniently lost on many future practitioners advocating a 4% withdrawal rate rule. The essay represents a high-water mark in the use of historical, back-testing methodology. However, the limitations of this approach are apparent. Other authors prefer bootstrapping and Monte Carlo simulation methods, or prefer to build closed-form analytical models. However, a new form of back-testing / historical methods will soon appear with respect to "withdrawal rules" designed to improve portfolio sustainability. The authors' comment on the sub optimality of autopilot distributions points to this development. Unfortunately, the quest to find such rules often approaches an elaborate exercise in data mining.

		adjusted withdrawals require limiting the withdrawal rate to 5% or less while maintaining an equity weighting of at least 50%. The authors note: "…the lower withdrawal rates of 3% and 4% recommended by some analysts appear to be excessively conservative for portfolios with at least 50% stock, unless the investor wishes to leave a substantial portion of the initial retirement portfolio to his/her heirs.	
2001	"Adjusting Withdrawal Rates for Taxes and Expenses," Gordon B. Pye Journal of Financial Planning vol. 14 no. 4 (April, 2001), pp. 126 – 135.	The author executes a Monte Carlo simulation of a portfolio (normally distributed returns) with a mean of 8% and a standard deviation of 18%. Assuming that an initial withdrawal rate of 4% is sustainable, a 1% expense ratio requires a reduction in the withdrawal rate of approximately 50 basis points in order to maintain a comparable probability of long-term sustainability. An algorithm adjusts the *initial* withdrawal by a fraction equal to the annual expense divided by the portfolio's expected mean return. The article considers the effect of taxes in the case of a 100% taxable portfolio and a 100% IRA portfolio. In the taxable portfolio, non-dividend returns are taxed at the long-term capital gain rate. This adjustment is adequate assuming that the portfolio basis is close to market value, the dividend yield is low, and turnover is moderate. For the IRA account, withdrawals are reduced by the average applicable tax rate whenever the withdrawal is characterized as ordinary income. The model indicates, for an IRA account with an initial withdrawal rate of 4%—inflation adjusted—the portfolio has the same sustainability rate as the taxable portfolio when (1) the withdrawal rate is multiplied by (1 – tax rate), (2) when the applicable tax rate is flat rather than progressive, and (3) when the RMD requirements do not exceed the withdrawal target. "Withdrawals must be made to obtain funds for both consumption and taxes. The objective no longer is sustainable withdrawals, but a sustainable stream of consumption in real terms." Pye notes that the progressive federal income tax schedule is adjusted for inflation. Although the tax rate is a function of the size of the withdrawal, the average applicable rate for the initial withdrawal should remain constant because the withdrawals themselves are inflation	This is an important contribution to the sustainable withdrawal research because it accounts of the economic consequences of taxes and fees.

		indexed. For low levels of withdrawals the investor's standard deduction and tax exemptions mean that the withdrawal is essentially tax free. This suggests that the long-term sustainability of the portfolio does not change significantly in the presence of taxes assuming that all other assumptions continue to hold. An important factor for taxable portfolios is the tax basis. A large basis "…tends to improve sustainability by making a portion of the withdrawals tax free. On the other hand, appreciable dividends tend to reduce sustainability because of the higher tax. When both of these factors are included, their effects tend to offset." Likewise, Pye asserts that turnover also has only a marginal effect. Although high turnover accelerates taxes, the tax payments increase the basis of the remaining portfolio. In addition to determining the annual inflation-adjusted withdrawals, the basic assumptions of the model's tax algorithm assumes: • A fixed portion of the portfolio is sold each year to maintain diversification and to upgrade investment performance; • The cost basis of the sold securities is the same portion of the total basis as the portion of securities sold; • Losses are carried forward to offset future gains; • Dividend yields follow a relation fit to historical data for the S&P 500. Annual turnover is 40%, initial dividend yield is 1.5%, ordinary income is taxed at 30% and capital gains are taxed at 20%. Bottom line: key variables such as turnover, dividend yield, and cost basis—reduced from 100% to 60% of market value—do not have significant effect on portfolio sustainability as long as the initial distribution rate is reduced by the applicable rate on capital gains. This observation is also generally true for IRA portfolio as long as the initial distribution rate is reduced by the applicable rate on ordinary income.	
2001	"Is Your Standard Of Living Sustainable	The authors compute the conditional and unconditional probabilities of ruin for an individual who consumes a fixed real amount from a portfolio	An important synthesis of the concepts of the stochastic present value function, asset allocation,

	during Retirement? Ruin Probabilities, Asian Options, and Life Annuities," Milevsky, Moshe and Robinson, Chris Retirement Needs Framework, Chapter 9 Society of Actuaries (2000) pp. 87 – 97.	of risky assets—i.e. assets earning a stochastic rate of return. Using insights from option pricing theory, they conclude that the probability of ruin corresponds to the probability that a suitably parameterized Asian call option will expire in the money. An Asian call option is a "path dependent contingent claim whose payoff at maturity is based on the average price observed over the life of the option." Payoff = max $\frac{1}{n}\sum_{i=1}^{n} S_i - K, 0$ Where S_i is the price of the security on each of the n measurement dates and K equals the strike price of the option. Purchasing an Asian call option is akin to betting that the weighted average return from the underlying asset, over the specified life of the option, will exceed a predetermined threshold delineated by the exercise price. But this is analogous to calculating the discounted average consumption from a portfolio over the lifetime of a retiree. If average discounted consumption is greater than initial wealth, the individual will be ruined. In a stochastic return environment, the issue becomes the probability for ruin. The authors conclude that the probability of ruin is minimized with high allocations to equity for most all of a retiree's life span. Method: <u>The conditional probability of ruin</u> = probability of being alive given the probability that current wealth equals zero. <u>The unconditional probability of ruin</u> = probability of wealth ever equal to zero; or, the sum needed to fund a perpetuity. The unconditional probability is of interest to investors with strong bequest motives or to endowment plans. The model assumes fixed *real* consumption and an uncertain life span and investment return. Mathematically, the probability of ruin equals the probability that the present value of lifetime consumption (in real dollars) is greater than the initial wealth available to support consumption plus the stochastic return	and initial wealth. The paper lays the groundwork for an ongoing monitoring system that tests the Wealth-to-Consumption ratio throughout retirement. The paper presents the concept that retirement consumption "costs" are properly measured by an actuarial calculation.

		thereon. In a deterministic world with fixed return and a fixed date of death: $$PV_T(c) = c\int_0^T e^{-rt}dt = \frac{c(1-e^{-rt})}{r}$$ The stochastic analogue [$SPV_T(c)$] adds two sources of randomness: T and r. When T and r are stochastic, the right hand side equals the present value of a lifetime annuity under stochastic discounting. This equals, also, the scaled payoff from an Asian put option. The authors' model assumes that investment returns follow a Brownian motion process in which the return of the portfolio assets is a function of asset weighting (allocation vector) plus the stochastic differential equation governing change of wealth – period-by-period consumption under conditions of uncertainty. The stochastic process is expressed explicitly as: $$W_t = H_t\left[w - c\int_0^t (H_s)^{-1}ds\right]$$ where $H_s = \exp\left[\left(\mu_\rho - \frac{1}{2}\sigma_\rho^2\right)s + \sigma_\rho B_s\right]$ The authors maintain that H_s obeys a reciprocal Gamma distribution; and, because it is the stochastic present value of a lifetime annuity, the probability of ruin can be approximated by the reciprocal Gamma distribution.	

		The model employs a Mortality Function that is a "Gompertz fit" to a mortality table. When consumption is fixed in real terms, the probability calculation becomes the probability that the stochastic present value of lifetime consumption of one real dollar is greater than the initial wealth to consumption ratio: w/c. Finally, the article demonstrates how to calculate the first two moments (i.e. mean and variance)of the stochastic present value of consumption. These two moments are then input into the reciprocal Gamma function to calculate ruin probabilities. The authors proceed to determine the asset allocation that, for any sustainable wealth/consumption ratio, yields the lowest probability of ruin. Note: the authors suggest an alternative approach to the problem: for any given confidence interval for the probability of ruin, calculate the maximum lifetime consumption that is sustainable at a given asset allocation weighting. The paper concludes that the conditional probability of ruin is minimized by a high allocation to equity.	
2001	"Are The Elderly Really Over-Annuitized?" Jeffrey R. Brown Themes in the Economics of Aging ed. David A. Wise (University of Chicago Press, 2001) pp. 91 – 126.	Immediate annuities represent a lump sum payment in exchange for periodic lifetime income. Term insurance is the payment of periodic sums in exchange for a lump sum payable at the end of life. Therefore, if a household considers itself "over-annuitized," it may undo the excess annuitization by holding a term insurance policy. By contrast, the author considers holding of cash value insurance to be either an artifact of a decision made many years in the past, or to be a form of tax-preferenced precautionary savings: "the cash value of a policy is not a negative annuity, but rather represents a non-annuitized financial asset, much like a savings account." The paper examines the issue of whether households are seeking to undo Social Security for bequest reasons. Buying a life insurance contract is analogous to selling an annuity. Classic	Note: the author does not consider "refund" or "period certain" annuities the pricing of which combines the elements of a pure "life-only" annuity with an installment payout life insurance settlement option on a decreasing term policy. For example, insurance companies sell annuities with a payout to the annuitant for life; and, if the annuitant does not survive for at least x years, to a designated beneficiary for the balance of time between death and year x. Compare Brown's thesis regarding motive to hold insurance to Bernheim's 1991 article.

		financial economics suggests that a retiree should, absent a preference for bequest, use wealth to purchase annuities rather than selling annuities by purchasing life insurance. In fact, because insurance policies are not priced on an actuarially fair basis, even retirees with a desire to make bequests should avoid purchase of life insurance because risk-free bonds would strictly dominate life insurance as a form of wealth transfer. Empirical evidence indicates that few US households hold private annuities and that ownership of life insurance is widespread. In households aged 70+, approximately 8% hold private annuities while, in 70+ *couples* households, approximately 78% own a life insurance policy on at least one spouse. The existing literature suggests two hypotheses: 1. Elderly married couples use life insurance to reallocate annuity streams across survival states; and, 2. Life insurance is held to offset an excessive level of mandated annuitization in the form of Social Security benefits. Specifically, term insurance is used to "sell" annuities in order to leave bequests. The author suggests four tests of the annuity offset hypothesis: 1. No household would hold both private annuities and term insurance ('no one would rationally purchase annuities above the actuarial cost only to sell them back below the actuarial cost"); 2. An increase in the level of mandated annuity (Social Security) benefits should increase the demand for term life insurance; 3. Term life should behave as an inferior good because it is a negative annuity and annuities are normal goods ("an increase in the demand for annuities corresponds to a decrease in the demand for term life insurance."); and, 4. Individuals holding term life policies must have Social Security	Brown asserts that low wealth levels decrease the demand to annuitize due to the potential liquidity needs in the face of unexpected medical costs—use of wealth to purchase an annuitized income stream increases vulnerability to economic shocks. This argument becomes increasingly important in the retirement income literature. The liquidity risks of annuitization may require a reassessment of their utility value. For trustees administering trust distributions, if the ascertainable standard encompasses health, education, maintenance and support, the trustee may be enhancing "maintenance and support" to the detriment of "heath" needs. Note: A trustee should also evaluate whether an annuity purchase constrains the optimal consumption path. Such a result might adversely impact the interests of both the current and the remainder beneficiaries. That is to say, an annuity solution might distort the Settlor's intentions.

		benefits in excess of the desired retirement consumption. Alternatively, if the desired level of consumption is higher than the Social Security benefit, the retiree would want to supplement income through the purchase of additional private annuities as opposed to making payments to an insurance company for term coverage. The author points out that individuals might feel over-annuitized for reasons other than bequest objectives. An annuity income stream might exogenously constrain an individual's optimal consumption path (individual reallocates consumption across time rather than across generations); an annuity might imperil a retiree subject to unforeseen economic shocks (nursing home care). Such an individual may wish to hold marketable wealth as a hedge against financial catastrophe (consumption in the face of economic shock vs. bequest motives). The author examines data from the Asset and Health Dynamics of the Oldest Old (AHEAD) survey. One advantage of this data source is that it distinguishes between term and cash value permanent insurance assets. The general econometric model developed by Brown tests for the effect of Social Security Benefits and total economic resources on holdings of life insurance: $LI_i = \text{Max}\{0, \beta_0+\beta_1 SSB_i+\beta_2 LR_i+\beta_3 X_i+\varepsilon_i\}$ Where: LI is life insurance holdings, SSB is Social Security Benefits LR is a vector of other lifetime economic resources including the present value of pension benefits , and X is a vector of demographic characteristics such as race, gender, marital status, etc. The model is calculated from a Tobit specification as well as from a	

		simple Ordinary Least Squares method. The results clearly reject the hypothesis that households would not simultaneously hold private annuities and term insurance coverage. Of all married households 50% own both; and, among widows and widowers, 21% own both. With respect to the assumed positive correlation between the level of Social Security benefits and term insurance holdings, the Tobit results report a coefficient of 0.04 which is not statistically different from zero. OLS results are similar. After an exhaustive demographic decomposition of the results, the author concludes that there is no evidence to suggest a positive correlation between term insurance and benefit levels among the retired elderly. The author tests the third hypothesis and, while the sign of term life is, in some specifications, slightly negative relative to economic resources, total life insurance holdings exhibit a sign that is strongly positive. On the whole, there is no evidence that term insurance is behaving like an inferior good. Finally, the author tests the relationship between desired consumption, benefit levels and term life holdings. If the annuity offset hypothesis is correct, households with term insurance should be also saving some fraction of their Social Security income. However, the fraction of households holding term insurance and spending down assets does not vary greatly from the fraction that spend down assets and do not hold term insurance. Thus the author concludes that the annuity-offset model does not explain why elderly households hold life insurance programs. The author considers four alternative hypotheses to explain holding life insurance during retirement: <u>Couple Protection</u>—life insurance is used to insure against a loss of	

		pension benefits or Social Security income upon the death of the first spouse. Thus, insurance is purchased in order to re-allocate life-contingent income. Evidence for this hypothesis is mixed. Most owners of term life in married couple households name their spouse as beneficiary. This might be considered evidence against the intergenerational bequest motive. However, 62% of widowers and 49% of widows hold life insurance policies. <u>Inertia</u>—Policies are a "residue" from the attempts made earlier in life to insure human capital. Most life insurance policies are quite old (median age = 42 years). However, a significant number of elderly purchased a policy since the age of 65. However, only 8% of the post 65 policies were term insurance. <u>Estate Tax Planning</u>—While high net worth households with illiquid assets (farms, businesses, etc.) may wish to hold insurance to provide heirs with liquidity sufficient to pay estate transfer expenses, fewer than 5% of households are above the threshold for estate tax liability. This is much lower than the percentage of AHEAD households owning life insurance; and, therefore, cannot be a comprehensive explanation. <u>Funeral Expenses</u>—This is the most likely explanation for the widespread holding of life insurance among the elderly. Policy proceeds are not held up in probate and are directly available for payment of final expenses. For example, one study reported that 83% of widows holding life insurance indicated that the purpose was to pay final expenses.	
2001	"The Lifecycle Model of Consumption and Saving," Martin Browning & Thomas F. Crossley Journal of Economic Perspectives, vol. 15 no. 3 (Summer 2001)	The authors review the history of the life-cycle model which "…is the standard way that economists think about the intertemporal allocation of time, effort and money." There is no single lifecycle model; rather, there is a range of models each of which has strengths and weaknesses. A life-cycle model "…asserts that agents make sequential decisions to achieve a coherent (and 'stable') goal using currently available information as best they can."	Compare the economic definition of Prudence to a trustee's duty to exercise "care, skill and caution." The economic concept of Prudence suggests the appropriateness of maintaining a retirement income reserve. This element is sometimes missing from papers that focus on the benefits of maintaining such a reserve—e.g., Woerheide and Nanigian [2011]

	pp. 3 – 22.	A life-cycle model can include models that do not incorporate expected utility: "agents may have a preference for the early or late resolution of uncertainty even when this does not confer any planning advantages." This is a characteristic of Epstein / Zin utility. [Note: this preference may lead retirees to prefer "safe," fixed-income investments despite their sub-optimal characteristics]. What a life-cycle model does rule out, however, is "rule of thumb" behaviors in which households simply spend a fixed fraction of current income without regard for the amount and character of future income or potential income shocks. Within a life-cycle model, households strive for consumption smoothing. This does not mean keeping expenditures constant. Rather it refers to the attempt to keep the marginal utility of money constant over time. This, in fact, may lead to significant variations in expenditures. Smoothed consumption is not constant consumption. The life-cycle model thus assumes that younger consumers will borrow, mid-life consumers will accumulate, and retirees will prefer dissaving. One of the many life-cycle theories mentioned in the article is the 1972 paper by K. Nagatiani ["Life Cycle Saving: Theory and Fact", American Economic Review] suggesting that households are not so much liquidity constrained as they are "prudent": "Prudence leads households to treat future uncertain income cautiously and not to spend as much currently as they would if future income were certain (with a value equal to its mathematical mean). Thus, prudence is the precautionary motive for saving." The authors distinguish between a utility function—having a negative second derivative—and prudence—having a positive third derivative. The classic Markowitz optimization, for example, assumes that risk aversion reflects only quadratic preferences but not prudence. They state: "…the role of prudence in consumption and saving decisions has been one of the central themes in the literature for the last 15 years…."	The life cycle approach illustrates important differences between sequential decision making and autopilot formulae. Annuitization is akin to an autopilot distribution policy. Article makes the important observation that investors may prefer a "resolution of uncertainty" even if an early resolution is suboptimal. Such a preference may be contrasted with retirement income strategies—i.e. annuitization—based on expected utility or option valuation.
2001	"Conserving Client Portfolios During	This study compares alternative withdrawal strategies from retirement income portfolios. Bengen developed the concept of a MAXSAFE	Note: Bengen does not advocate a stay-the-course strategy for any withdrawal strategy. Rather he

	Retirement, Part IV" William P. Bengen Journal of Financial Planning vol. 14 no.5 (May, 2001), pp. 110 – 119.	withdrawal rate based on the historical sequence of returns to various stock/bond asset allocations. His initial work concludes that for reasonable allocations to equity, a retiree faced with a 30-year planning horizon may safely withdraw 4.14 percent of the portfolio's starting value. Annual withdrawals are adjusted for inflation. Following the decomposition of retirement spending phases outlined in the book by Michael K. Stein (*The Prosperous Retirement*) , Bengen assumes a three-stage retirement withdrawal strategy: 1. Phase One (through age 75)—Withdrawals increase annually by the actual inflation rate. 2. Phase Two (through age 85)—Withdrawals increase annually by the actual inflation rate less 4 percent. 3. Phase Three (beyond age 85)—Withdrawals increase annually by the actual inflation rate less 2 percent. Using the sequence of historical returns and inflation, Bengen concludes that a retiree following the modified three-stage withdrawal strategy is able to achieve a MAXSAFE—zero percent failure rate— withdrawal rate of 4.8 percent of initial portfolio value. Bengen also explores a performance-adjusted withdrawal strategy. The initial step is to specify the withdrawal percentage that will be withdrawn based on the portfolio's value at the beginning of each year. Although a fixed-percentage withdrawal formula prohibits the portfolio from running out of money, Bengen concludes that the historical drop in the dollar value of withdrawals for portfolios launched during a bear market become unacceptably low for most investors. To correct for this "flaw," Bengen tests a distribution strategy that allows for (1) withdrawal increases during a bull market of up to 25% above the inflation-adjusted value of the first year's withdrawal, and (2) withdrawal declines during a bear market period of up to ten percent below the real value of the first year's withdrawal. For this strategy the MAXSAFE rate for the initial year's withdrawal is 4.58 percent. Bengen observes that clients are free to change withdrawal strategies to	reminds readers that strategies can and should be periodically reconsidered in the light of the retiree's current circumstances. However, follow-on commentary by other authors establishes elaborate sets of rules from which the investor must never deviate lest he or she fall victim to financial ruin.

		reflect the economic impact of unfolding events: "Clients are not married to any one withdrawal model for their whole retirement. They can adjust their approach based on their experience early in retirement."	
2001	"Retirement Investing: A New Approach" Zvi Bodie Boston University School of Management Working Paper No. 2001-03.	This is a short essay that distinguishes between diversification (to balance defensively by dividing investments among various industries and securities), insuring (payment of a premium to avoid downside risk while retaining upside potential) and hedging (eliminating the risk of loss by sacrificing the potential for gain). Bodie implies that most investors follow Markowitz portfolio optimization algorithms that lead them toward diversification. They should, however, consider hedging and insuring as well; and, he argues that hedging can be as important as diversifying with respect to demand to hold certain types of assets. Critical (i.e. minimum acceptable level of retirement consumption) can be hedged (purchase of inflation-protected bonds) or insured (purchase of inflation-adjusted annuities). Furthermore, using the classic put/call parity relationship (put + stock = call + T-Bill), Bodie suggests that investors put the bulk of their financial wealth into inflation-protected fixed income securities to "immunize" the minimum retirement consumption objectives and buy a staged series of long-term call options with the remainder in order to provide growth. Note: This suggests a "bottom up" portfolio strategy.	The payoff diagram to this suggestion is comparable to Sharpe's Insured Portfolio "floor multiplier" payoff which uses T-Bills plus a multiplier for the stock position. The payoff diagram has some resemblance to the annuity/life insurance package available to trustees to provide income to the current beneficiary and to provide a guaranteed remainder interest. See Milevsky [2000] and Rubin [2000].
2001	"The Role of Real Annuities and Indexed Bonds in an Individual Accounts Retirement Program," Jeffrey R. Brown, Olivia Mitchell & James Poterba Chapter 9, Risk Aspects of Investment-Based	The authors point out that retirees face two inflation-related risks: 1. Over time, purchasers of nominal annuities suffer a decline in the purchasing power of the income stream in the face of a positive rate of inflation; and. 2. Unexpected inflation has an additional adverse impact on the real value of the right to receive income. The article explores several annuity options, including: ➢ A constant dollar annuity offered by Irish Life Insurance Co of North America (ILONA);	Authors point out that annuities provide constant income—but constant income may not equal optimal income because of changes in the preference (discount) rate due to changes in age, health, etc. Note: The authors mention, but do not consider, graded payout annuities. Thus, with some torturing of language, one could say (because nominal annuities provide a higher initial dollar payout than real annuities) that:

	Social Security Reform (eds.) John Y. Campbell and Martin Feldstein (University of Chicago Press, 2001), pp. 321 – 370.	➢ A variable annuity offered by TIAA-CREF linked to the payout of a TIPS index; and, ➢ A variable annuity linked to the payout of a stock index. Specifically, the authors calculate the expected utility of annuitizing wealth under the above-listed elections. Depending on the inflation process (iid random draws vs. persistent inflation) and depending on the degree of investor risk aversion, options manifest a different rank order with respect to their utility. A review of annuities in the United Kingdom indicates that the initial payout from real annuities is approximately 25 to 30 percent lower than for nominal annuities. The authors calculate the expected present discounted value (EPDV) of UK annuities (nominal annuities discounted by the nominal rate of interest and constant dollar annuities discounted by the real rate of interest). However, they use general population mortality tables rather than annuitant mortality tables, and, they point out that the PV of late-in-life increases in real payouts from constant dollar annuities are not fully reflected in their calculated values. Subtraction of the EPDV from the cost of the annuity yields a measure of the annuity's "moneys worth." The EPDV of a nominal UK annuity is approximately 90% of premium cost, the EPDV of a real UK annuity is approximately 85% of cost; and, therefore, the cost of inflation protection is about 5% of the premium. Turning to the US, the ILONA real annuity offered an initial payout approximately 30% less than the company's nominal annuity. Discounting via the implied structure of the corporate bond yield curve (nominal) or the term structure of real interest rates derived from TIPS (real annuity) indicated that the EPDV is approximately 86% v. 70% with inflation-protection costing approximately 15% of the premium. The TIAA-CREF variable annuity offers an inflation-linked bond account (ILBA) investment option (although, by prospectus, the fund can include	1. nominal annuities "front-load" retirement consumption; 2. real annuities offer smoothed consumption; and, 3. graded benefit annuities "back-load" retirement consumption. Proposition three is, of course, correct only if the annual increase in graded benefit annuities is greater than the annual increase in inflation. Also, smoothed consumption is the ability to keep the marginal utility of consumption steady—not the ability to keep the dollar value of expenditures steady. Note that the AEW is recalculated based on updated data in "Longevity-Insured Retirement Distributions: Basic Theories and Institutions," Mark J. Warshawsky and Jeffrey R. Brown Chapter 3 in <u>Retirement Income: Risks and Strategies</u> Massachusetts Institute of Technology (2012), pp. 57 – 84.

		fixed income securities other than TIPS—e.g. up to 25% foreign bonds). The authors point out that, irrespective of increases or decreases in inflation, an increase in the real rate of interest will cause the bonds to decline in value—thus, the option is not true inflation protection. Additionally, the initial payout is based on the same assumed interest rate (AIR) of 4% that underlies the payouts from all other investment options (to promote ease of changes in internal asset allocation). Although high AIRs will front-load the payout, future investment results that fall below the AIR will result in decreased payouts. Given the costs of inflation protection (ILONA) or the uncertainty of variable annuity ILBA options matching actual inflation, the authors conclude "there is currently no market for genuine real annuities in the United States." Finally, the authors explore investment in variable annuity equity sub accounts as a method to achieve a measure of inflation protection. They conclude that, although equities have a higher expected payout, they expose owners to market volatility and cannot guarantee a fixed real return. Their investigation methodology employs a series of four-order lagged regressions (dependent variable equals current inflation and independent variables are four quarters of lagged inflation plus four quarters of t-bill returns. The residual term represents unexplained or unexpected inflation. Security returns are then regressed against unexpected inflation and, not surprisingly, exhibit a statistically positive negative correlation. Thus, only a short position in equities can provide a hedge against unexpected inflation. The article then takes up the issue of measuring increased utility of annuity options through calculation of 'annuity equivalent wealth." After determining an actuarially fair payout, the expected lifetime utility of consumption generated by an annuity payout equals:	

		$$U = \sum_{j=1}^{50} P_j \frac{\left(\frac{C_j}{(1+\pi)^j}\right)^{1-\beta} - 1}{(1-\beta)(1+\rho)^j}$$ Where P is a survival probability, π is the rate of inflation, β is the coefficient of relative risk aversion (or $1/\beta$ = the degree of intertemporal substitution in consumption), and ρ is the subjective preference rate for discounting. The authors define annuity equivalent wealth (AEW) as "the amount of wealth that a retiree needs—if he does not have access to an annuity market—to achieve the lifetime utility level that he can attain with access to an annuity market." They are interested in determining the extent to which increased AEW can "overcome" an EPDV of < 1. AEW is calculated with deterministic returns and stochastic returns under conditions of uncertainty regarding mortality. Following the literature (most households have risk aversion close to log utility--β = 1), the risk aversion parameter values range from 1 through 10. Modified "triangle" distributions are used for inflation and risky asset return modeling. Results and Conclusions: For individuals with no preexisting annuity wealth (e.g. social security not considered), the annuity equivalent wealth represented by ownership of a real annuity ranges from 1.502 (coefficient of relative risk aversion of 1) to 2.004 (coefficient of 10). This range of AEW values dominates those for ownership of a nominal annuity either in the face of iid inflation (1.451 to 1.592) or persistent inflation (1.424 to 1.346). Interestingly, the monotonic increase in AEW relative to risk aversion does not hold for nominal annuities at high risk aversion coefficient values. Nominal annuities generate positive utility because they eliminate the risk of outliving wealth; they generate negative utility in the presence of future inflation. The effects differ over the range of risk aversion. A similar pattern of relative AEW values holds across values of risk aversion when half of an individual's wealth is tied to a preexisting	

		real annuity, although, under this fact pattern, utility of nominal annuities in the face of inflation does increase monotonically in the fact of uncertain inflation. Across all risk aversion parameter values and all annuity (real or nominal) options and inflation conditions, annuities produced AEW values greater than 1. Purchase of variable annuities may or may not produce AEW values > 1. For individuals with log utility and no preexisting annuity income, an equity-linked variable annuity under a model assuming 6% risk premium produces AEW of 1.623; and, under a 9% risk premium produces AEW of 2.024. However AEW *decreases* monotonically as the risk aversion parameter increases. In the case of log utility, "the individual always prefers an equity-linked variable-annuity product. At higher risk-aversion levels, however, the fixed real annuity usually dominates." [Note: this parallels Milevsky's conclusions that retirees are often better off with high allocations to equities]. However, the authors caution that equity-linked annuities are attractive because their model assumes that investors can access equities only through a variable annuity instrument.	
2001	Reinventing Annuities, Mike Wadsworth, Alec Findlater, and Tom Boardman The Staple Inn Actuarial Society (January 2001)	A research study from The Staple Inn Actuarial Society. Primarily concerned with rethinking UK annuity market structure. However, makes a series of interesting observations: "…writing annuities carries very great risk as it involves estimating future improvements in mortality over long periods in circumstances in which advances in scientific and medical knowledge appear likely to have a substantial but unpredictable impact." "…for those whose primary requirement from their assets is income generation the retention of assets in non-annuitized form is costly." "…annuitized funds have a wider potential role to play in the extraction of income from assets…" The authors emphasize that retired individuals face two risks: (1) the risk of outliving their assets (poverty risk), and (2) the risk of restricting their standard of living needlessly (excessive assets at death).	Charts provided by the authors illustrate the "force of mortality"—the difference in distribution of male and female deaths at ages 60 through 90. Both age and gender differences change the shape of the distribution significantly. A static asset allocation / spending policy solution derived at age X may not be appropriate at age Y.

		Note: Asset management requires a trustee to exchange—balance—the two risks.	
2001	"Private Pensions, Mortality Risk, and the Decision to Annuitize" J.R. Brown, Journal of Public Economics vol. 82 (October 2001), pp. 29 – 62.	Extends the concept of Annuity Equivalent Wealth (AEW). The Utility of additional annuitization of wealth varies from investor to investor. Utility is a function of mortality, risk aversion, marital status, social security entitlement, etc. For constant relative risk aversion, the utility measure is invariant to the level of wealth. Thus, AEW is a proxy measure for investor utility. The author seeks to determine the extent to which AEW correlates with the propensity for retirees to annuitize wealth. Justification for the study is, at least, two-fold: 1. Annuitization has important economic consequences for the economic well being of the elderly; and, 2. Annuitization decreases intergenerational wealth transfers and may have public policy implications (e.g. amount of revenue generated by estate taxes). The author utilizes survey data from a U. of Michigan study (Health and Retirement Survey); and, the paper concludes that "…the utility-based measure of annuity equivalent wealth that is constructed from a stochastic life-cycle model is significantly positively correlated with the ex ante probability of actually annuitizing DC [Defined Contribution Pension Plan] balances. Specifically, a one-percentage point increase in the calculated annuity equivalent wealth is associated with nearly a one percentage-point increase in the probability that an individual before retirement reports that they will annuitize." The author points out that his conclusion fits with the classic finance model known as the "life-cycle model;" rather than with other models such as behaviorist finance models. Although the life-cycle model is a good first approximation for predicting investor behavior, it leaves a significant amount of variation unexplained. A subset of the population exhibits myopic behavior (planning limited to a single period); and, in contradistinction to the model, the stated preferences for bequests seem not to influence the intention to annuitize.	Net income trusts allocate income to the current beneficiary and principal to the remainderman; total return trusts often have indexed annuity or unitrust percentage distribution formulae that can either bankrupt the trust or make the portfolio become vanishingly small. Annuities, however, are the ultimate in autopilot distribution formulae. However, the "actuarial argument" is that annuities present an opportunity to place the current beneficiary on the optimal consumption path; and, that life insurance presents the opportunity to guarantee the interests of the remaindermen. There is the additional issue of how a trustee's election to annuitize the trust portfolio may run counter to the intent of the grantor to grant discretionary distributions for the benefit of trust beneficiaries. Trustee must evaluate based on both a monetary metric and a utility metric. Milevsky's articles point out that "set in stone" asset allocation may also be suboptimal; and, although he recommends annuities for retirees of advanced age with only modest bequest preferences, he, nevertheless, points out that optimal allocation is a function of many constantly changing variables such as delta (health, wealth (bad or good investment results), age / force of mortality across remainder of life, marital and family status, etc.). Brown develops a dynamic programming approach

		The author provides a summary of research on "the annuity puzzle." The classic statement of the life-cycle hypothesis [Yaari] is: "…when the consumer has no bequest motive but he is constrained to meet the requirement that his transferable assets at time of death should be non-negative with probability one…the consumer's assets will always be held in actuarial notes rather than in regular notes." Brown summarizes and restates the hypothesis as follows: "A consumer can choose to hold assets either as ordinary notes which pay the market rate of interest 'r,' or as actuarial notes, which pay the market interest rate 'r' plus a 'mortality premium' equal to the mortality hazard rate. The 'cost' of this higher return is that the actuarial notes, or annuities, are cancelled upon his death. In the case where the life-cycle consumer places no value on wealth after death, i.e., he has no bequest motives, then the 'cost' of annuities is zero, and the individual will always prefer to invest in the higher yielding actuarial notes." Previous studies using the AEW methodology suggest that utility gains of annuitization are equal to an approximately 50% increase in financial wealth for a single 65 year old male retiree with log utility; and an approximately 25% increase for married individuals. Although Yaari's hypothesis has been criticized for comparing annuities to riskless bonds, as long as the annuity pays a premium above any underlying investment portfolio, absent a bequest objective, the full-annuitization option dominates "in an Arrow-Debreu setting." [Note: this parallels Milevsky's argument that the retiree should annuitize at the point where the mortality credits plus interest credits of the annuity equal or exceed the expected return of an investment portfolio]. However, relatively few retirees annuitize wealth. Explanations range from high costs of commercial annuities, strong preference for bequests, precautionary savings motives in the face of uncertain medical/assisted living expenses, substitution of multiple family incomes for the annuity	where the optimal solution is a function of AEW, utility, and the optimal consumption path. The optimal consumption path may not result in the spending of all period annuity income. The article manifests a top-down approach where high amounts of wealth will, all else equal, decrease the demand to annuitize. Thus, the article falls into the "annuity-as-safety-net" school of thought. Calculation of utility over the optimal consumption path relies on incorporation of constant relative risk aversion [CRRA] into the dynamic program life-cycle model. Future academic studies will point out that such an assumption is dubious in that there is scant empirical evidence that most retirees exhibit CRRA.

		market, etc. The author points out that loads arise from two sources: (1) expenses off the top; and (2) annuitant mortality tables. It is uncertain whether there is a self-selection process involving asymmetric information between individuals (who have private information suggesting an expectation for a long life span) and insurance companies (who must load the tables to protect themselves against adverse selection); or, whether annuitization is an option most often chosen by wealthy individuals who may have more favorable mortality expectations. [Note: justification for using SOA RP 2000 table adjusted for wealthy retirees when modelling longevity risk]. <u>Methodology</u>: The simple Yaari life-cycle model with actuarially fair annuities, no utility of bequests, no non-mortality sources of uncertainty, suggests that all investors will have a positive gain from annuitization; although the magnitude of the gain will vary from investor to investor because of differing consumption preferences (elasticity of consumption); marital status, amount of existing wealth annuitized under social security and pension entitlements, and so forth. Brown takes these variable and develops a dynamic programming algorithm to construct an AEW measure consistent with underlying utility theory. Brown follows the standard AEW calculation formula in that he assumes an individual with zero preference for bequest and with access to an actuarially fair annuity market. The individual fully annuitizes initial wealth and selects a consumption path to maximize utility (i.e. the periodic annuity payments need not be fully consumed in each period. A portion of the payment may be saved, at the riskless rate, to provide funds for future consumption). The author takes away the annuity market and calculates the amount of additional initial wealth that would be needed to provide the equivalent measure of utility (i.e. place the	

investor on the same utility curve).

Where ρ = the utility discount rate and T = maximum life span (age 115), the consumer's objective is:

$$\text{Max } E_t\left[\sum_{t=1}^{T-age+1}\frac{U(C_t)}{(1+\rho)^t}\right]$$

Where the expectation is taken over states of survival, and subject to the following constraints:

- W_0 given;
- $Wt \geq 0$ at all times t; and,
- $W_{t+1} = (W_t - C_t + S_t + A_t)(1+r)$

Where W_t is non-annuitized wealth, C is consumption, S is annuity payment from Social Security and pensions, and A is an actuarially fair annuity payment.

When an individual fully annuitizes all wealth, then $W_0 = 0$ and A_t is the expected discounted value of the initial premium (W*):

$$W^* = \sum_{t=1}^{T-age+1}\frac{A_t\prod_{j=1}^{t}(1-q_t)}{(1+r)^t(1+\prod)^t}$$

Where q is the probability of death before period t=1 conditional on surviving to period t; and Π equals the inflation rate.

[Note: purchase of an actuarially fair life insurance policy is the equivalent of selling short an annuity].

The author states: "The value function at time t is the present discounted value of expected utility evaluated along the optimal path. The value function satisfies the following recursive Bellman equation:

		$$\text{Max } V_t (W_t) = \text{Max } U (C_t) + \frac{(1-q_{t+1})}{(1+\rho)} V_{t+1}(W_{t+1})$$ Note that the expectation operator has been dropped, as we are now accounting explicitly for the survival probabilities, and there are no other sources of uncertainty in this problem. The Bellman equation reduces the full maximization problem to a series of two-period problems, which can be solved numerically by solving back from the final period." The Author solves the problem again assuming that annuities are not available—i.e. A_t = 0 for all periods. The difference is the measure of AEW. This difference is also the upper bound on loads that the consumer would be willing to pay to access the annuity market. <u>Calibration of the AEW Measure</u>: The author states: "to implement the dynamic program to solve for the annuity equivalent wealth, one must put some structure on the utility function. I invoke the standard assumption that individuals exhibit constant relative risk aversion: $$U (C_t) = \frac{C_t^{1-B}}{1-B}$$ Where B is the coefficient of relative risk aversion. Importantly, 1/B also measures the elasticity of substitution between consumption at two points in time, which is an important parameter in valuing annuities. The author calibrates risk aversion (the value of Beta) according to an ordinal scale based on the survey responses to a "lottery-type" question regarding income. Additionally, the author uses a cohort mortality table, an assumption that the utility discount rate and the market rate of interest are both equal to 0.03, that inflation is fixed at 3.2%, and that utility is concave in consumption (first units of longevity insurance are	

		the most valuable). An OLS regression in which AEW is the dependent variable shows that AEW increases with risk aversion, decreases for married couples, decreases with the fraction of wealth already annuitized, and increases with age. Interestingly, bequest motives and subjective assessment of health were not significant factors in the decision to annuitize except for those who listed their health as very poor. Also, wealthier individuals are less likely to annuitize. This might be explained by: 1. Less likely to exhaust financial resources; 2. Have more pre-existing annuity wealth; 3. More likely to try to earn high investment returns; or, 4. More likely to have a bequest motive. Bequest motives had little influence on the decision to annuitize retirement wealth ("bequest motives are not a significant determinant of marginal annuitization behavior") Author concludes that the paper provides strong evidence in support of the usefulness of the traditional life-cycle hypothesis as a predictor of empirical behavior.	
2001	"Annuity Markets and Retirement Security," James M. Poterba [Third Annual Conference of the Retirement Research Consortium ("Making Hard Choices About Retirement") May 17-18, 2001 Washington DC]. Text available at	The focus of the paper is on the role of adverse selection in annuity markets. The author describes annuities as "insurance policies that pay their beneficiaries for as long as these beneficiaries are alive." After describing the bump up in yield due to mortality credits, the author continues to explore the relationship between annuities and life insurance: "Annuities are sometimes referred to as "reverse life insurance." A life insurance policyholder pays the insurer each year until he or she dies. When the insured individual dies, the insurance company pays a lump sum to the beneficiaries of the life insurance policy. With annuities, the annuitant makes a lump-sum payment to the insurance company before the annuity payout begins. In return, the insurance company makes payments to the annuitant until the annuitant's death."	The paper contributes to the discussion on the "Annuity Puzzle" topic. It focuses on adverse selection as an important reason for failure to annuitize wealth. Poterba's thesis regarding the "packaging" of annuities and insurance should be contrasted to the argument that a rational consumer will not want to purchase an insurance contract and an annuity contract concurrently. This argument states that the consumer is buying an annuity for more than actuarial fair value and selling an annuity (i.e. buying insurance) for less than actuarial fair value.

	Center for Retirement Research at Boston College, Working Paper #2001-10.	Annuities and life insurance are combined in certain guaranteed payout annuity contracts. These contracts promise to pay for the life of the annuitant or x years, whichever is greater. Thus, if the annuitant experiences an early death, a bequest may still be forthcoming: "payment certain contracts may satisfy the annuitant's desire to purchase insurance against outliving his or her resources, while at the same time delivering some potential benefits to children or others to whom the annuitant might wish to leave a bequest." There is also a running discussion of the theme of "time diversification" of annuity purchases. This is based on the fact that variation in bond returns generates a substantial variation in annuity payout rates over time; and, therefore, annuitizing all wealth at a single moment may not be optimal. The main theme presents the argument that adverse selection in annuity markets is an important factor in dampening the consumer's enthusiasm for voluntary purchases of private annuities. The author acknowledges that bequest motives and precautionary savings motives also are important factors in the "annuity puzzle" explanation. A general characteristic of the annuity market, however, is the assumption that those who purchase annuities have a life expectancy greater than the average population. Dynamic adverse selection effects can ultimately lead to the disappearance of the annuity market. One way to quantify the effects of adverse selection is to employ the concept of the "Moneysworth" of annuity products. Moneysworth is the ratio of the expected present discounted value of payments over an uncertain future time period, to the purchase price of the annuity. Moneysworth can be calculated in nominal terms (using a fixed interest rate or the term structure of interest rates) or in real terms (real rate or term structure of real rates) by dividing the sum of payments over the annuity premium. Previous studies suggest that consumers should be	Poterba notes that annuitizing all wealth at a single moment ("an optimal stopping time") may not be optimal. It is a type of 'annuity market timing.' Given the anti-selection bias, annuitization may not be a suitable strategy for the general population. Indeed, the future market for annuities may cease to be viable. This argument parallels the position advanced y Salvador Valdes-Prieto [1998]. The argument that consumers who prefer a 'back loaded' retirement income may do better by purchasing nominal annuities and saving a portion of the periodic benefit parallels Brown [2001].

		willing to pay a substantial premium for the increased utility of consumption enhancement and smoothing. However, Poterba notes that Moneysworth calculations differ depending on whether you assume the carrier's perspective or the consumer's perspective. An actuarially fair annuity has a Moneysworth value equal to 1 (PV of costs = PV of expected benefits). However, such an annuity cannot exist because it allows for no profit, cost of capital, and expense factors. A randomly selected consumer considers the Moneysworth calculation as it is derived from a general population mortality table. This table reflects a force of mortality that, all else equal, diminishes the value of the aggregate payout stream. From the insurance company's standpoint, however, the discounted present value of the payout stream is much higher than Moneysworth perspective of the consumer. The insurance carrier prices the annuity from a mortality table with a much lower force of mortality. The differential between the consumer perspective and the insurance company perspective is one measure of the disutility of adverse selection for average consumers contemplating an annuity purchase. The author extends the concept of adverse selection to explain why the Moneysworth of fixed nominal annuities seems to be higher than the Moneysworth of escalating or inflation-adjusted annuities. The latter have "back loaded" payouts that would, all else equal, be favored by purchasers expecting to have a long life. Insurance carriers, recognizing the potential for anti-selection, will add a pricing factor to reflect the probability of having to provide longer-term payout streams. Consumers who prefer back loaded retirement incomes may do better by purchasing nominal annuities and saving a portion of the early payments in order to finance consumption at a later date. The author stresses the need to explore further the question of the value of options to purchase annuities over time.	
2001	"Shortfall-Risks of	The article contributes to the time-diversification controversy. Empirical	Risk models inputting distributions with differing

	Stocks in the Long Run," Peter Albrecht, Raimond Maurer & Ulla Ruckpaul. Financial Markets and Portfolio Management vol. 15 no. 4 (2001), pp. 481 – 499.	evidence may suggest that stocks always beat bonds in the long run and, therefore are the true riskless investment. According to this argument, in the long run, bonds are a redundant asset class. Advocates of this position usually take a purely historical approach. However, when the approach utilizes 10-, 15- or 20-year overlapping periods, the resulting returns have a high degree of correlation. "This results in a serious estimation bias. A high degree of statistical significance requires independent returns based on non-overlapping …. periods." A second approach to the time diversification issue analyzes empirical data on the basis of a stochastic model for stock evolutions. Commonly used models include a version of the random walk—e.g., Brownian motion—or other mean-reversionary model. Yet another common split to assessing stock risk contrasts an evaluation metric based on an assumed investor utility function with a "preference free" metric such as option pricing theory or shortfall risk. However, shortfall risk implies the selection of a target benchmark or portfolio floor value, and, therefore is not entirely preference free. The authors distinguish among three measures of risk: <u>Shortfall Probability</u>: SP(z) = P(R < z). This measures the probability of a shortfall relative to a stochastic or deterministic target (z). It does not measure the magnitude of a shortfall. <u>Shortfall Expectation</u>: SE(z) = E[max(z-R,0)]. This considers the extent of a shortfall as well as the probability of shortfall with respect to the target return. <u>Mean Excess Loss</u>: MEL(z) = E[z – R | R < z]. This considers the consequence of the mean shortfall-level assuming that a shortfall actually happens. It is a conditional shortfall measure. The relation among the measures is: SE = SP * MEL. Using inflation-adjusted data from the German stock and bond markets over the periods 1970 – 2000 and 1980 – 2000, the authors develop a model that assumes a lognormally distributed price change process	assumptions, operating under different stochastic processes, may produce significantly different results. The authors use a standard Brownian motion model for the stochastic process. The distributional assumptions are that asset returns are iid. The authors point out that such risk models are purely probabilistic—that is to say, their results are independent of economic fundamentals. A utility based risk metric can be reconciled with a shortfall based risk metric. The authors offer the following preference function as an example: $\Phi(V) = E(V) – [k*SP(V)^{\alpha}*MEL(V)^{\beta}$. Such a function allows the investor to model the tradeoff between SP and MEL. See, also, Brown's comments in Milevsky and Robinson [2001] "Self Annuitization and Ruin in Retirement" Makes important observations on the use of a shortfall metric to evaluate portfolio risk. It also prompts a reassessment of the prudence of a stay-the-course asset management strategy.

		evolving according to standard geometric Brownian motion. Evaluation of shortfall risk is not strictly based on historical results, but rather on a representative distribution that is consistent with empirical data. The outputs suggest that shortfall probability is monotonically decreasing with time. However, "…the shortfall probability of a stock investment does not converge rapidly towards zero depending on time – as purely historical studies suggest – but rather that this convergence is rather slow." This characteristic is termed "persistence of risk." Shortfall expectation, by contrast, initially increases but eventually monotonically decreases. It, too, evidences persistence. However, Mean Excess Loss (independent of both the benchmark and the return chosen as the target) reveals "…that the risk of a stock investment increases and thus shows the true risk of a stock investment." "Taking a shortfall relative to a benchmark of 0% return in real terms, the level of mean excess loss after 30 years is – depending on the supposed distribution of stock returns – in average about 35%-44% of the corresponding value. This is a substantially high shortfall level." The authors conclude: "…these results make clear that the use of the shortfall probability alone is insufficient for the assessment of the risk of stock investments in the long run." The authors explain: "the probability of a loss or a shortfall decreases with the length of the time horizon. However, the average level of the loss or the shortfall respectively, given a loss or a shortfall has occurred, increases."	
2002	Mortality Risk, Inflation Risk, and Annuity Products Brown, Mitchell & Poterba Innovations in Retirement Financing (eds.) Mitchell, O., Bodie, Z., Hammond, B., and Zeldes, S. (University of	Retirees should be interested in annuities because they "provide households with a structured way to draw down the assets that they have accumulated during their working lives." However, there is some evidence suggesting that retirees with large accumulations have a lower demand for annuities. This study updates earlier work on the "money's worth" of annuities. A review of payouts per $100,000 of annuity premium shows steady declines from 1995 through 1999 (Male 65 average payout = $794.12 in 1995 vs. $734.77 in 1999). However, carriers vary widely in their periodic payout amounts.	Primarily an update on the Moneysworth of annuities by examining the market place circa 1999.

	Pennsylvania Press, 2002), pp. 175 – 197.	The "Money's Worth" calculation uses the standard method: The Expected Present Discounted Value (EPDV) is calculated by summing the probability-adjusted payout (payout x probability of surviving) divided by the sum yearly discount rates (term structure of risk free, corporates, etc.). Real annuities are discounted by the real rate of interest. The money's worth is defined as the ratio of EPDV over the purchase price. In general, the EPDV of a US annuity is between 80 to 90 cents per dollar of purchase price for members of the general population; but between 90 and 100 cents per dollar for members of the annuitant-mortality table population. An examination of the only inflation-adjusted payout annuity (ILONA) indicates that the inflation protection adds approximately 15% to the cost of the annuity; and, the money's worth of the inflation-protected annuity was less than that of nominal annuities. The money's worth of nominal annuities also is greater than real annuities for non-US markets. CREF offers a Variable Annuity with an Inflation-linked Bond Account option. However, if the required rate of return on fixed income securities rose, there is no guarantee that the account will track the CPI.	
2002	"Self-Annuitization, Ruin Risk in Retirement and Asset Allocation: The Annuity Benchmark" Peter Albrecht & Raimond Maurer, Journal of Pension Economics and Finance vol. 1 no. 3 (2002), pp. 269 – 288.	Authors use data from German capital markets and insurance companies. They first develop a matrix of annuity payouts based on purchase age and interest rate environment. They proceed to use the payout of each matrix cell to establish a comparable self-annuitization program with stock, bond & real estate funds. For each age/interest rate annuity payout, the authors create an optimal asset allocation via simulation. Finally, they calculate the probability of outliving the optimally diversified investment fund. They conclude that annuities purchased in low interest rate environments produce modest payouts that, in most cases, can be achieved through self-annuitization. At more advanced ages and in higher interest rate environments, the mortality credits and annuity pricing factors make the risk of ruin through self-annuitization much	German annuity costs are estimated at 40bp of initial premium for acquisition costs, 125bp for commission costs and 150bp of payout for ongoing administrative expenses. The annuity "yield premium" is calculated as (annuity payout yield – risk-free rate). The premium is due to the ultimate "default" of principal plus the "mortality credit" that results from risk pooling. Note: The article develops the Annuity Timing / Interest Rate environment theme—best time to annuitize is when interest rates are high. See, for example, Milevsky [1998], Mitchell, Poterba, and Warshawsky [1999] and Milevsky [2000] articles.

		higher. Bequest motives are not considered.	This article makes the case that it is as important to monitor changes in the interest rate environment for investors considering annuitization as it is to monitor changes in investor circumstances and preferences.
2002	"An Analysis of Investment Advice to Retirement Plan Participants" Zvi Bodie Pension Research Council Symposium	The paper demonstrates that consumption demands create path dependencies; the average return is not a valid predictor of terminal portfolio wealth because low early returns may exhaust wealth despite higher than average late returns. In the long run, according to Bodie, the only safe investments are inflation-protected bonds and real annuities. Traditional US annuities, however, are not inflation-protected. Likewise, variable annuity benefits can increase or decrease with the fortunes of the stock market. An annuity escalating with the cost of living guarantees a flow of lifetime consumption. The author suggests that a prudent retirement asset management strategy would be to invest a portion of wealth in an inflation-adjusted annuity which, when combined with Social Security, would finance a minimum consumption floor. The remainder of wealth would be invested in a risky asset portfolio. A fraction of this portfolio would, each year, be invested in further units of the inflation-adjusted annuity to purchase additional real income. Call options with multi-year maturity dates may be an ideal way to invest risky assets—in a year when they expire in the money, additional income can be purchased.	A two-fund, bottom-up approach to portfolio management.
2002	"Discrete-time Drawdown Analytics: Annuities and Drawdown in a Retirement Income Model," Mike Orszag Watson Wyatt Technical Report 2002-RU04 (April, 2002).	The author compares consumption financed through income drawdown strategies in a discrete time setting with consumption financed through annuitization. The most common drawdown strategy is: $c = F/a_x$ where a_x is an annuity factor calculated at an assumed interest rate: r, and F = the Fund of Wealth. If the assumed interest rate is zero, then the annuity factor equals the IRS minimum distribution table for IRA's, which is based *solely* on a life	The author takes a valuation approach in that the optimal time to annuitize is the point at which the annuity produces an income stream higher than a portfolio drawdown program. Thus it is the ratio of consumption from the portfolio to consumption provided by the annuity that is determinative of the optimal time for annuitization. Whereas annuity mortality credits become greater at older ages, the author generally recommends a top-down approach in which annuitization is

		expectancy table. Roughly, a male age 65 takes out 1/18th of the fund; increasing to 1/17th at age 66; 1/16th at 67; 1/15th at 68, and so on. A drawdown strategy results in a consumption profile relative to annuities of: $$\frac{c_x^{(d)}}{c_x^{(a)}} = \varpi \prod_{y=x_0}^{x-1} \frac{(1+r_y)}{(1+\tilde{r}_y)(1+q_y)}$$ where ϖ = ratio of drawdown annuity factor to an actuarial annuity factor (usually close to one); r_y = rate of actual investment return at age y; $\tilde{r}$ y = assumed or projected return at age y; and, q_y is the mortality rate at age y. As long as actual (realized) investment returns equal assumed returns and there is no difference in annuity factors, a drawdown program never produces a level of income higher than annuities. However, if realized returns exceed assumptions (or if the annuity factors differ) then, over some period of time, a drawdown strategy will produce a higher income stream than annuities. The author examines both fund (F) growth and annuity factors. If consumption is c = F/a_x, then the fund's growth follows the following dynamics: $$F_{x+1} = [1 + r_x]F_x – c_{x+1}$$ Or, $$F_{x+1} = [1 + r_x]F_x – (F_{x+1}/a_{x+1})$$ Or, factoring F_x from the right hand term:	delayed until an optimal stopping time. However, annuities lock in the budget constraint. This means that there is a risk of losing potentially higher income at later ages if annuitization occurs early in retirement. This is a variation on the 'investing-as-an exchange-of-risks' theme.

		$F_{x+1} = \frac{(1+r_x)}{(1+\frac{1}{a_{x+1}})} F_x$ If $a_x \leq 1/r$, the fund will decrease. However, if $1/a_{x+1}$ is small relative to r_x, then the fund will increase. For individuals with a long life expectancy (i.e. long planning horizon), the expected return on investments may be larger. When the threshold interest rate is greater than the reciprocal of the annuity factor, annuitization is sub-optimal. However, annuity payout rates increase quickly at older ages because the annuity factors decrease with age. The equilibrium threshold interest rate (or "indifference interest rate') gets higher and higher. The author proceeds to model consumption growth *rate* which follows the following evolution: $g = \frac{\frac{F_{x+1}}{F_x}}{\frac{a_{x+1}}{a_x}} -1$ but the numerator equals $\frac{(1+r_x)}{1+\frac{1}{a_{x+1}}}$ and $1 + 1/a_{x+1}$ equals $\frac{a_x(1+r)}{a_x(1+r)-(1-q_x)}$ with the result that consumption growth depends only on rates of return, assumed rates of return and mortality rates:	

		g = $\dfrac{(1+r_y)}{(1+\tilde{r}_y)(1+q_y)}$ which is exactly the formula seen in the original drawdown rule. Depending on the force of mortality and the assumed and actual interest earnings, consumption growth in a drawdown program will turn negative at a projected age. But this is valuable information because it acts as a guide to choosing if and when to exercise the option to annuitize. Consumption for an annuitant is fixed at the time of annuity purchase at: $c_x^{(a)} = \dfrac{F_{x0}}{a_{x0}}$ By contrast, consumption in a drawdown strategy is dynamic and follows: $c_x^{(d)} = \dfrac{F_x}{a_x}$ The task is to compute the ratio of drawdown consumption to annuity consumption for different ages, where the change in the ratio (g) equals $\dfrac{(1+r_y)}{(1+\tilde{r}_y)(1+q_y)}$. But this is merely the first formula: $\dfrac{c_x^{(d)}}{c_x^{(a)}} = \varpi \prod_{y=x_0}^{x-1} \dfrac{(1+r_y)}{(1+\tilde{r}_y)(1+q_y)}$ Under many cases, the drawdown strategy produces higher income initially. The longer-lived the individual, the more attractive the	

		drawdown strategy, all else equal. However, at more advanced ages, the drawdown strategy produces a level of income substantially below that obtainable from an annuity. Thus, drawdowns may be more attractive to retirees concerned with maximizing immediate cash flow. Enhancing early retirement cash flow occurs at the potential cost of lower income at later ages. Individuals with a high preference for front loaded retirement incomes (high personal preference discount rates) may prefer the drawdown strategy. However, such a personal time preference rate must be high for the drawdown strategy to have a positive net present value over the entire planning horizon. [Note: Gordon Pye (2012) develops this insight in much greater detail]. One consequence of these observations is that the optimal strategy for an individual might be to initiate a temporary drawdown strategy followed by annuitization at a later age ("individuals…have the ability at any point to buy an annuity which locks in their current level of consumption"). The author develops a table for different mortality assumptions showing the extra return needed per year in order to make the drawdown strategy attractive relative to annuitization (e.g., "…a 65 year old needs to earn between 1.5% and 1.9% extra per year in order to make the strategy work"). Numerical simulations illustrate the interesting result that under an assumed interest rate of 6%, a mortality table at male age 60, actual mean returns of 9% with 20% standard deviation (assuming a normal distribution), the mean annuitization age is 73 but the modal age is only 61—one year only in the drawdown program! Some people do well in the stock market because of the higher expected return and volatility to the upside. These fortunate investors drag the average to 73. However, the high volatility means that many do not do well and this group annuitizes (drags the mode towards 61) at an early age in order to preserve consumption.	
2002	"Portfolio Choice and Retirement Income Solutions"	Absent a bequest motive, the authors assert that the yield from fixed annuities will strictly dominate the yield from bonds (the underlying instrument) because of the extra return from the redistribution of capital	The authors develop the concept of the 'annuity premium'—the spread between annuity yields and long-term government bond yields. Capturing this

	Sandeep Kapur and J. Michael Orszag [Watson Wyatt Technical Report 2002-RU05]	from decedents to survivors. They create a chart to illustrate the concept of annuity yield premium by comparing, at various points in time, insurance industry annuity yields against long-term government bond yields. This spread is termed the annuity premium. To capture the annuity premium, the investor must willingly embrace "capital sacrifice." The allocation decision is multi-dimensional; and arises due to the fact that the investor/retiree can elect to invest in assets at various levels of risk and can elect various levels (changing through time) of capital sacrifice. For income-oriented investors, the allocation choice, however, boils down to annuities vs. equity portfolios because annuity yields will strictly dominate bond yields assuming that the cost of annuities plus profit loading imposed by insurers is less that the "death" credits. The greater the costs and profit loading, the more likely a rational investor is to delay the option to annuitize. Investment in equity also has an expected equity premium (return from equity – risk free rate). The expected equity risk premium is not age dependent; while the annuity risk premium is increasing with age--i.e. as death rates become higher. The authors develop a mathematical expression for the amount of wealth (w) at age x that the investor will commit to equity in an "optimized portfolio." The fraction devoted to equity is a function of individual (relative) risk aversion (γ), and the risk premium between expected (but random) equity return (μ) and the annuity premium (bond yield = r plus death credit = δ(x)): $W(x) = \frac{1}{\gamma}\frac{Max(\mu - r - \delta(x),0}{\sigma^2}$ The formula predicts the age at which an individual will cease to hold any equity. This age does not depend on market volatility or on the degree of individual risk aversion. Rather it is a function solely of the difference between the equity premium and the annuity premium where the magnitude of the annuity premium depends on the force of cohort	spread requires "capital sacrifice." Thus, a trustee decision to acquire an immediate payout annuity to protect the current beneficiary's standard of living involves defending capital sacrifice. A position in an annuity might be seen in terms of the annuity "crowding out" bonds because of the annuity premium. If equity is used to purchase an annuity, the decision is seen as an exchange of the expected equity risk premium for the annuity premium. Using a variation on the Merton Optimum, the authors contend that whenever the annuity premium exceeds the expected equity risk premium, the option to annuitize should be exercised. This is a preference-free calculation—a question of stochastic dominance rather than investor utility.

		mortality. Market volatility and risk aversion influence the positive proportion of equity held. However, it does not influence the absolute time of complete crossover to a 100% annuity portfolio. The authors present a chart assuming an equity premium of 7%, market standard deviation of 25%, and relative risk aversion of 2. The chart depicts a gradual movement from equity to annuities over time which the crossover year sooner for males than for females. Note: assuming (absent any preference for bequests) a person with quadratic utility [1÷.5 = 2]; expected return from equity = 8%; yield from annuity = 6%, and 20% market SD, the optimal percent of equity in the portfolio at age x is equal to: $\frac{1}{2}(\frac{8-6}{.04}) = \frac{2}{.08} = 25\%$	
2002	"Developments in Decumulation: The Role of Annuity Products in Financing Retirement," Olivia S. Mitchell, Chapter One, Ageing, Financial Markets and Monetary Policy (eds.) A. J. Auerbach et al. Springer-Verlag (Berlin, 2002), pp. 97 – 125.	Annuities are valued based on an expected present discounted value method: $V_b(A) = \sum_{j=1}^{600} \frac{A * P_j}{\prod_{k=1}^{j}(1+i_k)}$ Where 'b' is the age of purchase, and i_k is the term structure of interest rates. For real annuities, the interest rate is replaced with a corresponding real interest rate: r_k. $V_b(A_r) = \sum_{j=1}^{600} \frac{A_r * P_j}{\prod_{k=1}^{j}(1+r_k)}$ The "money's worth" of an annuity is defined as the ratio of EPDV [Expected Present Discounted Value] to the purchase price.	The pre-tax EPDV of an annuity is, per regulation, generally negative (assuming that the insured and the insurer both have access to the same capital markets). Tax benefits may make the post-tax EPDV positive—but this assumes no future changes in tax law. Important to use the correct mortality table: "…mortality processes may be heterogeneous across subgroups of the population." The money's worth calculation is one way to estimate annuity "loads." When using an annuitant mortality table, the annuity load is in the 7 to 9 percent range. "…the types of loads currently charged by insurers would not be expected to offset the substantial utility values associated with insurance against longevity risk."

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Risk averse investors place a value on annuities that is higher than the “money’s worth” value because they derive utility from longevity insurance. The author measures the extra value according to a concept called Annuity Equivalent Wealth (AEW). This is the amount of wealth that an investor without access to an annuity market would need in order to achieve the same expected lifetime utility as he could achieve by purchasing either a nominal or a real—i.e., constant dollar—annuity. The utility function that the investor maximizes with a given survival vector (P_t) is:

$$\underset{C_t}{Max}\sum_{t=0}^{T}\frac{P_t U(C_t)}{(1+\rho)^t}$$

Where ρ = the investor’s preference discount rate. Without access to an annuity market, the investor’s budget constraint is limited to initial wealth—i.e. the present value of future consumption discounted by the riskless rate must equal initial wealth:

$$W_o = \sum_{t=0}^{T}\frac{C_t}{(1+r)^t}$$

If the investor can purchase an actuarially fair annuity, the budget constraint becomes:

$$W_o = \sum_{t=0}^{T}\frac{P_t C_t}{(1+r)^t}$$

Without an annuity, the retiree’s *actual* present value of future consumption cannot exceed initial wealth; with an annuity the *expected* present value of consumption cannot exceed initial wealth.
</td><td>
Note: The analysis of nominal vs. real (constant dollar) annuities and utility (increasing with longevity insurance but decreasing with loss of purchasing power) is a good example of how asset management involves an exchange of risks. Trustee challenge is to document the prudence of the decision making process.
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		To evaluate the utility function (value of utility for the annuity vs no-annuity consumption paths), the author assumes a one-period utility function exhibiting constant relative risk aversion: $$U(C_t) = \frac{C_t^{1-B}}{1-B}$$ Where B is the Arrow-Pratt coefficient of relative risk aversion, and 1/B is the elasticity of intertemporal substitution in consumption. Comparing the utility values with and without annuities leads to the calculation of the dollar amount of wealth required to bring utility values into equilibrium. The article provides tables illustrating the magnitude of AEW over a range of CRRA coefficients for investors with access to nominal and real annuity income streams: Consumer with No Pre-Existing Annuity Wealth:	

Coefficient of Relative Risk Aversion	Real Annuity	Nominal Annuity: iid inflation	Nominal Annuity: Persistent Inflation
1	1.502	1.451	1.424
2	1.650	1.553	1.501
5	1.855	1.616	1,487
10	2.004	1.592	1.346

Consumer with Half of Initial Wealth in Pre-Existing Real Annuity (e.g. Social Security)

Coefficient of Relative Risk	Real Annuity	Nominal Annuity:	Nominal Annuity:

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<tr><th>Aversion</th><th></th><th>iid inflation</th><th>Persistent Inflation</th></tr>
<tr><td>1</td><td>1.330</td><td>1.304</td><td>1.286</td></tr>
<tr><td>2</td><td>1.441</td><td>1.403</td><td>1.366</td></tr>
<tr><td>5</td><td>1.623</td><td>1.515</td><td>1,450</td></tr>
<tr><td>10</td><td>1.815</td><td>1.577</td><td>1.451</td></tr>
</table>

At higher levels of risk aversion, AEW value indicates that a real income guarantee is worth double initial wealth. The data suggests that real annuities are valued more highly than nominal annuities but the valuation differs with the inflation process. For nominal annuities, utility is pulled in two different directions. The longevity insurance increases utility while the erosion of purchasing power under conditions of inflation decreases the utility of a fixed income stream. As risk aversion increases, the second effect becomes stronger.
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<td>2002</td>
<td>“Optimal asset allocation in life annuities: a note,” Narat Charupat & Moshe A. Milevsky Insurance: Mathematics and Economics vol. 30 no. 2 (April, 2002), pp. 199 – 209.</td>
<td>The authors seek to derive the optimal asset allocation within an annuity structure for an investor with a stochastic life span. The optimal allocation is the one that maximizes the expected utility of consumption—i.e., the investor lacks a bequest motive. What is new in this study is that the allocation is made within an annuity framework. This means that the element of longevity risk is eliminated to the extent that the annuity provides for longevity insurance. The goal is to determine the optimal allocations both pre and post retirement. Investors may choose to allocate all wealth to a fixed annuity, a variable annuity or to a combination. The functional form for calculating utility is CRRA and the price process for the risky asset portfolio is geometric Brownian motion.

The pre-retirement asset allocation maximizes an objective function of the form:</td>
<td>Authors use simplifying assumptions to demonstrate that the utility maximizing asset allocation within an annuity framework is the Merton optimum (growth maximizing portfolio) both pre and post retirement. However, this conclusion is a function of the model’s assumptions and may change for investors with utility functions other than CRRA and for investment processes other than iid normal.</td>
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$$MAX_{\alpha_t}\ E\left[\frac{1}{1-\gamma}W_T^{1-\gamma}\right]$$

The solution to the objective function is the Merton optimum:

$$\alpha^* = min\left[\frac{\mu - r}{\gamma\sigma^2}, 1\right]$$

The authors use the following expressions to depict post-retirement asset management strategies:

1. Buy & Hold: $W_t = aX_t + (1\text{-}a)Y_t$ where X_t and Y_t are the stochastic asset class values at time t.
2. Constant Mix: $dW_t = adX_t + (1\text{-}a)dY_t$
3. Dynamic Allocation: $dW_t = a(w,t)dX_t + (1\text{-}a(w,t))dY_t$.

The authors review the well known actuarial pricing for a fixed immediate annuity. The annuity pricing factor is:

$$A_x(r) = \int_0^\infty e^{-rt}({}_tp_x)dt$$

Where the conditional probability of survival is a function of the instantaneous hazard rate (λ_s): ${}_tp_x = e^{-\int_x^{x+t}\lambda_s ds}$
Therefore, $A_x(r)$ is the present value cost of $1 of lifetime income per period. The higher the interest rate (r), the lower the value of $A_x(r)$ and the higher the value of the annuity payout.

If the investor's lifespan is assumed to be exponentially distributed (${}_tp_x = e^{-\lambda t}$), then the annuity payout for any level of wealth equals $W(\lambda + r)$. This illustrates how the payout is a function of both a mortality premium and an interest rate. The article calculates the fixed immediate annuity payout under both an exponential distribution and a Gompertz-Makeham mortality distribution.

The annuity pricing factor for a variable annuity (purchase of a fixed

		number of units per period instead of a purchase of a fixed number of dollars per period) is expressed as: $$Ax(h) = \int_0^\infty e^{-ht}({}_tp_x)dt$$ Where h is the assumed interest rate (AIR). If risky asset returns are above h, the unit payout increases; if they fall below h, the unit payout decreases. The payout is expressed as: $$\text{Annuity payout} = \frac{W}{a_x(h)} exp\left\{\left(\mu - h - \frac{\sigma^2}{2}\right) t + \sigma B_t\right\}$$ Where B is the risky asset Brownian motion process. In order to maximize post-retirement utility, the investor maximizes the discounted expected utility of consumption: $$U(a) = \int_0^T e^{-pt} \frac{1}{1-\gamma} c_t^{1-\gamma} dt$$ Where p is the subjective discount rate (impatience or consumer time preference) and c is the annuity payout. The optimal solution under either exponential or Gompertz-Makeham mortality is, once again, the Merton optimum. Given CRRA utility and the assumption of Geometric Brownian motion, the utility maximizing allocation both pre and post retirement is the Merton optimum. However, the authors note: "…there is some preliminary theoretical evidence to suggest that decreasing relative risk aversion preferences, insurance fees and expenses, strong bequest motives and pre-existing (such as government) annuities, might dramatically alter the optimal mix."	
2002	"Equity Allocation and the Investment Horizon: A Total Portfolio Approach" R. Douglas Van Eaton & James Conover Financial Services	The authors survey various schools of through regarding financial planning advice focused on retirement savings objectives. They claim that advice based solely on an individual's age (length of planning horizon) may lead to poor results. Advice should reflect a consideration of the investor's total wealth: a "total portfolio" approach. The total wealth concept underlies one of the classic life-cycle economic	Article illustrates how a simple change in cash flow assumptions (e.g., lump sum vs. periodic contributions) can significantly change the results of historical back tests.

	Review, vol. 11, no. 2 (Summer, 2002), pp. 117 – 133.	models developed by Bodie, Merton & Samuelson. This model recognizes that wealth consists of financial assets (stocks, bonds, etc.) and human capital (the ability to save a percentage of wages above those needed to finance current consumption). For younger workers, the value of their human capital is often many times the value of their financial capital. Over time, however, human capital is converted to financial capital. The classic Bodie/Merton/Samuelson model assumes that human capital is less risky; and, therefore, one implication is that workers with a higher PV of future human capital can better afford to tolerate equity risk. As labor (earnings) flexibility decreases with age, the proportion of equities should be cut back in the investor's portfolio. Other schools of thought advance the following general arguments: 1. Equity risk diminishes as the planning horizon lengthens; and, therefore, stocks are ideal assets for long-term investors (a "time diversification" argument); and, 2. Shortfall risk is the appropriate measure for retirees; and, as the planning horizon increases, the risk of shortfall decreases with increased exposure to equity. However, there are difficulties with each of these arguments. The variance of total dollar wealth in all-equity portfolios increases as time horizon lengthens; and, bond-oriented retirement portfolios evidence a greater probability of shortfalls in the face of consumption withdrawals than do all equity portfolios. (Alternatively, the cost of a put option guaranteeing the risk free rate of return increases over time). Recent articles suggest that the total portfolio approach should also consider Social Security entitlements (if S.S. is risk free, then investors should be able to tolerate increased equity risk, all else equal), and home ownership (with wealth offset by the value of a mortgage). The authors, while acknowledging the wisdom of including all assets within the portfolio optimization process, attempt to work with a more tractable model incorporating only two assets: (1) a retirement investment	

		account; and (2) a wage-linked present-value of future contributions account (i.e. financial capital in the first account and human capital in the second account). The authors present several models suggesting that investors, even if they have conservative risk preferences, will own 100% equity accounts at younger ages assuming a high present value of human capital. Financial accounts funded with 100% equity will usually be of modest size at younger ages and, relative to total wealth, will be only a small fraction of the whole. However, other important factors may work to modify such rule-of-thumb financial advice. For example, a young entertainer or athletes may already have a large financial account and may face considerable uncertainty regarding the PV of future contributions (human capital). This circumstance suggests that the percentage commitment to equities should be decreased despite the presence of a long planning horizon. Additionally, the authors back test (overlapping 25-year holding periods from 1926 through 1996) certain allocation strategies. For example, final wealth is equalized under four asset management strategies: ➢ periodic contributions to a constantly rebalanced portfolio; ➢ periodic contributions to a 75% equity portfolio with equity reduced 1% each year; ➢ periodic contributions to a 90% equity portfolio with equity reduced 2% each year; and, ➢ periodic contributions to a 100% equity portfolio with equity reduced 2.67% each year. By construction, the mean ending account dollar values are equal. However, the standard deviation of the ending dollar value drops as the initial commitment to equity increases. This result is different than the back test for a lump sum investor over the same period. For the lump sum investor, the SD of final account value increases as the commitment to equity increases. Again, one size fits all advice may not produce good results: "…holding periods have different effects, depending on	

		individual financial circumstances and savings." The authors conclude "strategies seeking to reduce risk by decreasing equity allocations over time have their greatest value when future contributions represent a large portion of total retirement assets. Strategies seeking to reduce risk by decreasing equity allocations over time are least valuable (or possibly ineffective) when no future contributions will be made."	
2002	"Withdrawal Patterns and Rebalancing Costs for Taxable Portfolios," J. Christopher Hughen, Francis E. Laatsch, Daniel P. Klein Financial Services Review vol. 11 (2002), pp. 341 – 366.	This study extends the research of Cooley, Hubbard & Walz [CHW] to portfolios operating in a taxable environment. In CHW, portfolios are back tested to determine which allocations promoted portfolio sustainability. CHW concluded that the most favorable allocation had at least a 75% equity weighting for retirees seeking 4 to 5 percent inflation-adjusted withdrawals. ["Sustainable Withdrawal Rates From Your Retirement Portfolio," 2001] The Hughen, Laatsch, Klein [HLT] article incorporates the effects of rebalancing and taxes within the CHW model. A portfolio with an initial value of $1 million is invested under five asset allocations: 100%, 75%, 50%, 25%, and 0% equity. Equity returns are calculated "…using historical total returns on common stocks; bond returns are total returns on long-term government bonds as provided by Ibbotson for the period 1926 through 1999. Portfolios are rebalanced annually. Withdrawal rates range from 3% to 12% of the initial value per year. A successful asset allocation/distribution set is defined as one which has a positive value after the end of the designated planning horizon. The study considers rolling 20, 25, and 30 year periods. Ordinary income and capital gains tax liabilities are adjusted for the current [28% OI / 20% Cap Gain] rates with losses up to $3,000 offsetting ordinary income and the unused loss balance carried forward to future years. Basis is calculated by the FIFO method which limits losses in bear markets and reduces recognized gains in bull markets. Calculations are expressed in both nominal dollar terms and in constant	Authors argue that, for taxable portfolios, the results from overlapping historical periods of 20, 25 and 30 year length indicate that a 100% equity allocation is generally preferred when withdrawals are adjusted for inflation. The article provides a good case for the argument that the asset allocation decision and the withdrawal decision must be made jointly. The authors put forth several preferencing criteria in terms of cash flows: highest mean cash flow, lowest standard deviation of cash flow and highest ratio of mean cash flow divided by standard deviation. The last criterion is a reward-to-risk metric. The authors view positive terminal wealth as a potential "opportunity cost." This argument is akin to that of Sharpe, et al.—the existence of a surplus suggests that the optimal income target was not funded in an efficient manner ["The 4% Rule—At What Price?" 2008]. The article, using fixed time horizons such as 20 years, translates the nominal dollar surplus into "the equivalent payment value." Note: the "opportunity cost" theme reappears in

		dollars. Withdrawals occur annually (at the time of portfolio rebalancing) and the withdrawal amount factor is 1 + the ratio of the current year's CPI number to the previous year's number. The total data set encompasses 55 overlapping 20-year periods, 50 overlapping 25-year periods, and 45 overlapping 30-year periods. For 20-year periods, there is a high success rate for all allocation/withdrawal combinations provided that the withdrawal rate is 5% or less. For withdrawals at a higher rate, the 0% equity allocation exhibits dramatically increasing failure rates. The 100% equity allocation exhibits the lowest failure rates for high withdrawals—6% to 12%: "For all three retirement periods, the 100% equity allocation has the lowest percentage of failures at withdrawals above 7%." The authors also record the minimum, mean, and maximum after-tax cash flows for each allocation / withdrawal rate set. Additionally, they calculate the standard deviation of the cash flows. "For withdrawals under 6%, the 100% bond allocation provides the highest mean cash flows and the 100% stock allocation provides the second highest mean cash flows. …In addition, asset allocation becomes more important at the higher withdrawal amounts….Throughout all time horizons investigated, the 100% stock allocation has the lowest standard deviation [of cash flow] for withdrawals less than 5%, and this allocation generally produces the lowest standard deviation for withdrawals above 8%. The results are mixed for the withdrawals between 5% and 8%." One reason for this is that high bond allocations often result in portfolio depletion. This, of course, means that with cash flow goes to zero. The 100% equity allocation dominates other allocations at all withdrawal rates with respect to mean remaining terminal wealth. After calculating the tabular data based on historical investment and inflation paths, the authors advance three preferencing criteria for the joint asset allocation / withdrawal rate decision: 1. Highest mean after-tax cash flow	the 2012 – 2013 of Blanchett and others. The terminology morphs into the preferencing metric WER—withdrawal efficiency ratio. Note: the calculation of the opportunity cost of terminal wealth manifests similarities with the calculation methodology advanced in Gordon Pye's The Retrenchment Rule published in 2012. Note: Contrast the implied recommendation to tilt the portfolio allocation to equity with the article "Is The Retirement Plan with the Lowest 'Risk of Failure' Really the Best Choice?" Michael Kitces [2012].

		2. Lowest standard deviation of after-tax cash flow 3. Highest ratio of mean after-tax cash flow divided by standard deviation of cash flow. Each criterion is applied for both nominal and constant dollar withdrawals. For low nominal withdrawal rates a pure bond allocation is preferred over all horizons. However, for constant dollar withdrawals the demand to hold equity increases dramatically. For example, a 100% equity allocation works best for inflation-adjusted withdrawals of 4%+ over a 30-year horizon, while a 0% equity allocation is preferred for nominal withdrawals at a rate of 5% or less. The lowest risk [standard deviation] to after-tax cash flow for nominal withdrawals at all horizons is the 100% equity portfolio for low withdrawal rates (less than 5%) and for high withdrawal rates (more than 9%). Interestingly, when the study measures risk for constant dollar withdrawals the high equity allocations produce high standard deviations. However, this is primarily due to the number of failures in portfolios weighted heavily towards bonds—once a portfolio runs out of funds all future years have a constant cash flow of zero. The authors maintain that the best preferencing criterion is the ratio of the mean after-tax cash flow to the standard deviation of the after-tax cash flow: "Without adjusting for inflation, the 100% equity allocation has the highest reward-risk ratio in 16 of the 30 withdrawal amount / retirement period combinations. Using inflation-adjusted cash flows, this ratio rises to 27 of 30. Thus a 100% equity allocation is generally preferred and is clearly the best allocation when the withdrawal is below 5% or above 8%." The authors develop a concept called "the equivalent payment value." This is a way to express terminal wealth in terms of an extra payment that could have been received throughout the planning horizon. It is "…calculated using an interest rate equal to the total return on equity over the particular time period." This value is then expressed as a	

		percentage of the initial portfolio value. [EXAMPLE: Assume $1 million portfolio with terminal wealth of $1.4 million. Total annual return on equity for a 20-year period is 10%. If this had been converted into a nominal annual payment the amount of extra yearly income would be $24,443. As a percentage of initial portfolio value, terminal wealth was sufficient to support a 2.4% increase in the nominal withdrawal rate. HP-12 keystrokes are FV = -1,400,000; I = 10; n = 20; and PMT = answer]. Finally, the authors present several examples illustrating the importance of advising investors based on after-tax results. For example, for a low risk investor: "…compare the after-tax cash flows from a 100% equity allocation and a 50%/50% allocation with a withdrawal of 3% and a 20-year retirement period. The 100% equity allocation has a significantly lower standard deviation. Assuming a normal distribution of cash flows, 68% of the cash flows from the 100% equity position would be between $21,435 and $21,939. But the expected range for the 50%/50% allocation would be $14,433 to $23,819, which is considerably larger due to taxes. "	
2002	"Ruin in Retirement: Running Out of Money in Drawdown Programs" Mike Orszag Watson Wyatt Technical Paper 2002-RU06	The author compares the level of income (as well as the level of growth) available to a retiree through an annuity (lifelong consumption guarantee) to an available consumption level in a drawdown strategy providing an equivalent income stream. That is to say, the author calculates the year in which the fund underlying the drawdown strategy will run out of money. The year of bankruptcy is compared to the distribution of deaths faced by the retiree at the time of the initial funding decision in order to quantify the risks of the drawdown election. The analysis begins with a consideration of an individual wishing to achieve a target level of consumption growth $\lambda(x)$ at each future age x. Given this consumption growth target, the task is to select a percentage withdrawal $\tau(x)$ from a fund of money (F). Thus, the consumption function can be expressed as:	For a modest sized trust, what is the dollar-value level at which a trustee should consider annuitizing wealth in order to provide minimum threshold support to a current beneficiary? Although annuities are expensive in a low interest rate environment, the *ratio* of annuity income to bond income is attractive because of the "mortality credit." The lower the interest rate, the greater the relative impact of the credit. Most commentators argue that annuity purchases in low interest rate environments are likely to be detrimental because they lock in low payout amounts over lengthy planning horizons. Contrast Orszag with, for example, "Merging Asset Allocation and Longevity

		$c(x) = \tau(x)F(x)$. Absent new contributions, the fund of money increases with the rate of return and decreases with the rate of withdrawal: $$F(x) = F(x_0)\, e^{\int_{x0}^{x} (r(s)-\tau(s))ds}$$ Or, $$F(x) = F(x_0)[r-\tau]$$ Which is equivalent to: $\lambda = \frac{\dot{\tau}}{\tau} + r - \tau$ where, 't dot' in the numerator is future income withdrawal rate. The author claims that this is a Bernoulli differential equation, and that he wishes to solve the equation to determine the reciprocal of C where C is the boundary condition which determines the level of consumption. Thus the income withdrawal rate target is the reciprocal of the Consumption target that will drive the fund to zero after x number of years [$\tau(x) = 1/C$]. [Note: The task is to determine a wealth/consumption ratio that will support the fund for a stated number of years]. Defining 'g' as $1/\tau$, the equation becomes an ordinary linear differential equation: $$\dot{g} = -(\lambda(x) - r(x))g - 1$$	Insurance: An Optimal Perspective on Payout Annuities" Peng Chen & M. Milevsky (2003).

		which has the solution for given boundary conditions: $$g(x) = Ce^{\int_{x0}^{x}(r(s)-\lambda(s))ds} - \int_{x0}^{x} e^{\int_{v}^{x}(r(s)-\lambda(s))ds} dv$$ where 'v' is the lower level of integration in the exponent of 'e' in the second term on the right. And: $$\tau(x) = \frac{1}{Ce^{\int_{x0}^{x}(r(s)-\lambda(s))ds} - \int_{x0}^{x} e^{\int_{v}^{x}(r(s)-\lambda(s))ds} dv}$$ When the consumption growth rate is zero (λ=0) and the return 'r' is a fixed or deterministic return, integrating the above equation for the income withdrawal rate, yields: $$\tau(x) = \frac{1}{\frac{1}{\tau(x_0)}e^{r(x-x_0)} + \frac{1}{r} - \frac{1}{r}e^{r(x-x_0)}}$$ If the withdrawal rate τ is larger than the earnings rate r, the fund will eventually go to zero: $$\tau(x_0) = \frac{r}{1-e^{-r(x^*-x_0)}}$$ For example, at r = 3% and τ = 10%, the fund is exhausted in 11.89 years. The author next defines the following ratio: ϕ = the ratio of the income available to an individual provided that the individual is willing to run out	

		of money at some future year x, to the income available to an individual that would sustain consumption forever. For example, if an individual could earn 5% and was willing to exhaust the fund after 10 years, he or she could achieve an income stream 2.541 times higher than an income stream that would preserve capital forever. If an individual could earn 7% and was willing to deplete the fund to zero after 30 years, the ratio equals 1.139. Rather than avoiding fund depletion by preserving capital forever, individuals can purchase an annuity. The annuitant with income guaranteed for life achieves a consumption rate equal to $1/a_{x0}$ where a_{xo} is the annuity factor which, in turn, is a function of interest rates 'r' and mortality. The force of mortality is the reciprocal of life expectancy, or: $1/\varepsilon$. Thus, annuities provide income based on both 'r' and $1/\varepsilon$. The ratio of annuity income to long-term sustainable income is thus $1+1/r\varepsilon$ which is strictly greater than 1. In low interest rate / long life expectancy environments, annuities are particularly attractive. An annuity produces more income than a drawdown strategy as long as: (Targeted depletion age – current age) $\geq \frac{\ln(r\varepsilon+1)}{r}$ But the threshold time (T) for running out of money is always lower than life expectancy. That is to say, an individual can achieve a retirement income higher than an annuity only by risking running out of money at a point prior to life expectancy.	
2002	"Optimal Asset Allocation and The Real Option to Delay Annuitization: It's Not Now-or-Never," Moshe A. Milevsky and Virginia R. Young	The authors develop normative guidelines to help investors decide how to act in the face of current investment and annuity markets. Once an annuity is purchased, it is irreversible because it cannot be traded or resold. The lack of tradability makes the option to annuitize similar to a real option (as opposed to an option on a tradable financial asset). Each investor, therefore, is faced with the	Note: without annuities, the PV of consumption must not exceed the PV of wealth. Annuities (fairly priced) are a mechanism through which the living inherit the income of the dead (payment is conditional upon survival) and, therefore, enable a consumption stream greater than the PV of wealth.

	Pensions Institute Discussion Paper PI-0211 (2002).	real option to defer annuitization (RODA) of wealth. This option is "akin to exercising an American-style mortality-contingent claim" and should have time value. After a short review of the annuity puzzle literature, the authors state that one "common sense" explanation is that "most people shun life annuities simply because they want to maintain control of their assets." The purpose of the paper is to determine when individuals can benefit from exercising the option to annuitize (assuming no loads or commissions on the annuity and no bequest preference on the part of the investor). The authors' methodology indirectly answers Brown's criticism of an early paper [Self-Annuitization and Ruin in Retirement," Moshe A. Milevsky and Chris Robinson, 2000] in which Brown stated that an annuity-payout-linked consumption path may, in fact, not be the optimal path in terms of maximizing utility. In this paper, the retiree is allowed to determine and follow a preferred consumption path until such time as the option to delay annuitization loses value. The authors term the optimal time for annuitization of wealth the "optimal stopping time." The value of the option is defined as the percentage increase in an investor's wealth that would substitute for the ability to defer. Discussions regarding annuitization in classic finance literature often assume that the decision to annuitize is made in the face of only a single risk-free alternative with a pre-determined rate of return. De facto, this means that the investor's budget constraint is also fixed. Investment in a risky asset portfolio, however, means that there is a chance of improvement in the future budget	Optimal consumption decreases with increased probability of survival. The article is a good example of the argument that the optimal stopping time for an investment program is a function of option value. This approach contrasts with an active monitoring approach that considers the Wealth/Consumption ratio value as it unfolds dynamically. However, it is the level of wealth that must support future consumption—not the value of an option. The article contains a series of tables for various ages, gender, and risk aversion that calculate the optimal age of Annuitization, the Value of RODA, the Probability of a poor outcome if deferral is elected, and the probability of a successful outcome (annuity income > 20% more).

		constraint. When the interest rate used to price annuities also is subject to change it may be worth waiting in order to have the chance of buying a cheaper annuity with a larger amount of wealth. A discrete (3-period) time model is used as a base illustration of the option's valuation. A consumer aged x with initial wealth of $1 faces the following probability of death (q_x) during each period: 0.10, 0.25 and 0.60. [Note: authors state that "if the individual is fortunate to survive to the end of the third period, she consumes and immediately dies"]. The consumer's subjective time preference rate equals the risk-free rate (10%). The consumer is faced with the task of maximizing the expected discounted utility of consumption given an initial endowment (i.e. budget constraint) of $1: $\$1 = \frac{(1-0.1)c_1}{1.1} + \frac{(1-0.1)(1-0.25)c_2}{(1.1)^2} + \frac{(1-0.1)(1-0.25)(1-0.6)c_3}{(1.1)^3}$ = 1.5789 where the utility of consumption reflects constant relative risk aversion. After forming the Lagrangian and setting the first derivatives to zero, the optimal consumption (c*) equals $1/a_x$ where a_x is the actuarial fair value of an immediate annuity priced under the above conditions (a one dollar per period income stream costs $1.5789 and optimal consumption equals 1/1.5789 = $0.6334 or the amount consumed in the absence of a market for annuities). If the time preference is not equal to the risk-free rate, or if the mortality beliefs are not symmetric, the optimal consumption path (non-annuitized) will differ. With a coefficient of relative risk aversion equal to 1.5, u(c) = $-2/\sqrt{c}$. [Note: log of ½ = -	

2] The utility of consuming \$0.6334 per period equals –3.9679.

Thus the investor can invest \$1 into a risky asset portfolio, consume \$0.6334 at the end of period one, and hope to have a sufficient amount of wealth to secure a larger annuity stream in a future period. This amounts to taking a chance on the next period's budget constraint. In the authors' model, the investor is faced with a 70% probability of attaining a 45% return in the next period and a 30% probability of a 0% return [expected return = +31.5% which is greater than risk free rate of 10%]. In the up market, after consumption, the investor's period 2 budget constraint is \$0.8166; in the down market the budget constrain is \$0.3666. However, the expected discounted utility of annuitizing at the end of the first period is greater:

$$\frac{1-0.10}{1.1}\left[0.7u\left(\frac{0.8166}{a_{x+1}}\right)a_{x+1}+0.3u\left(\frac{0.3666}{a_{x+1}}\right)a_{x+1}+u(0.6334)\right]$$

The equilibrium amount required for an investor to be indifferent between annuitizing immediately and deferring one period is an additional \$0.0249. Thus, in this simple case, the RODA is 2.49% of initial wealth. With the coefficient of CRRA equal to 1 (logarithmic utility) the option value is 4.27%. A higher risk aversion coefficient decreases the value of the deferral option and pushes the investor to annuitize sooner.

Interestingly, the authors point out that any difference between the objective mortality rate (used by the insurance industry to price annuities) and the individual's subject assessment of personal mortality, leads toward increasing the value of the option

		to defer. A healthy person will delay to try to "out-earn" the annuity and may be able to buy a cheaper annuity in the future. Individuals in poor health will find the "cost" of the annuity (expected value discounted by probability of death) to be too high and will also prefer to wait. All else equal, higher loads and expenses will also increase the value of the option to defer because they increase the cost of the annuity. <u>The Continuous Time Model</u> The authors extend their model to a continuous time framework with stochastic interest rates and equity premiums. They model values in nominal rather than real terms due to the rarity of inflation-adjusted annuities. In a continuous time model, the actuarial present value of a life annuity of $1 per year payable continuously is: $$\overline{a}_x = \int_0^{\infty} e^{-rt}\,{}_t p_x dt$$ where ${}_t p_x$ is the conditional probability of survival. If the subjective mortality rate is used, then the present value of the annuity becomes $\overline{a}_x^s$ instead of the objective price of $\overline{a}_x^o$ which is the market price of the annuity (without loads or expenses). The model assumes that the retiree consumes at the rate of c_s between time t and the time of annuitization (τ). The model indicates that the individual will annuitize at time T for which μ = r+ λ_{x+T}^0 [the expected return of the investment portfolio equals the fixed rate used to price the annuity plus the objective—i.e., based on mortality table— "mortality credit" of λ_{x+T}^0 offered by the annuity]. This is equivalent to stating that the option to defer has	

lost its time value. At, and following, the time of annuitization, the retiree consumes at the rate of $\frac{W_t}{\overline{a}^0_{x+\tau}}$ by virtue of having purchased a commercial annuity.

The model becomes a "2 stage" utility of wealth model where stage one is the optimal consumption path pre-annuitization and stage two is the income stream post annuitization:

U(w,t) =
$\sup_{c_s, \pi_s, \tau} E\left[\int_t^{\tau} e^{-r(s-t)}\,{}_{s-t}p^s_{x+t}\,u(c_s)\,ds + e^{-r(\tau-t)}\,{}_{\tau-t}p^s_{x+t}\,u\left(\frac{W_\tau}{\overline{a}^o_{x+t}}\right)\overline{a}^s_{x+\tau}\right]$ given that W_t = w

Where π equals the percent of wealth allocated to the risky asset portfolio and the preference discount rate equals the objective discount rate.

The utility function is the standard CRRA (negative second derivative / positive first derivative) which is an increasing concave function:

$$U(c) = \frac{c^{1-\gamma}}{1-\gamma}$$

Substituting U(c) into the above equation, determines the value function V(w,t,T). In the continuous time model, V solves the Hamilton-Jacobi-Bellman equation (where V(r+ λ^s_{x+t}) is the utility of the objective rate 'r' plus the subjective force of mortality which acts as an extra mortality credit):

		$V(r+\lambda^{s}_{x+t}) = V_t +$ $\max_{\pi}\left[\frac{1}{2}\sigma^2\pi^2V_{ww}+(\mu-r)\pi V_w\right]+rwV_s+\max_{c\geq 0}\left[-cV_s+\frac{1}{1-\gamma}c^{1-\gamma}\right]$ Where V(w,T;T) = $\frac{1}{1-\gamma}\left(\frac{w}{\overline{a}^{o}_{x+T}}\right)^{1-\gamma}\overline{a}^{s}_{s+T}$ The authors produce a further series of equations calculating the optimal consumption and asset allocation policies that satisfy first-order conditions. [Note: the optimal allocation to equity is the Merton optimum (reward/risk) ratio multiplied by $1/\gamma$, or: $\Pi^* = \frac{\mu-r}{\sigma^2\gamma}W^*_t$ where W^*_t is the optimally 'controlled' wealth prior to annuitization]. The authors demonstrate that the optimal time to annuitize wealth is as soon as the instantaneous force of mortality, λ_x is greater than $(\mu\text{-}r)/2\sigma^2\gamma$. As stated, whenever the subjective force of mortality differs from the objective tabular mortality, the option to delay annuitization will increase in value.	
2002	"Who Should Buy a Nonqualified Tax-deferred Annuity?" William Reichenstein Financial Services Review vol. 11 no. 1	Study deals primarily with deferred annuities rather than immediate single-premium annuities. Reichenstein argues that low-cost, passively managed mutual funds generally provide larger ending wealth than annuities. This is largely because of high annuity costs: "The big loser in the analysis is the insurance industry's staple—annuities with expenses around 2%. These products are not in the best interest of investors."	A detailed analysis of the annuity v. mutual fund debate.

	(Spring 2002), pp. 11 – 31.	Reichenstein explains the large volume of annuity sales (the average age of buyers is 63) in terms of incentives paid by insurance companies to commission-based brokers. The conclusion is tempered by the observation that "…an annuity often makes sense for individuals in the distribution stage of life who are interested in an annuity's ability to reduce longevity risk."	
2002	"Analyzing and Managing Retirement Risks ," Zvi Bodie, P. Brett Hammond & Olivia S. Mitchell Chapter One: Innovations in Retirement Financing eds. Olivia S. Mitchell, Zvi Bodie, P. Brett Hammond, and Stephen Zeldes Pension Research Council The Wharton School of the University of Pennsylvania (2002), pp. 3 – 19.	Retirement risks commonly manifest themselves in the form of: ➢ Unexpectedly low income; and, ➢ Unanticipated health shocks. Risk management literature distinguishes between individual risk (life style choices, personal mortality expectations, personal savings elections, etc.), and systematic risk (government reduction in benefit entitlements, inflation, general declines in asset values, etc.). One of the largest sources of idiosyncratic risk is "inadequate knowledge about financial market processes." The literature discusses two general approaches to risk. One approach is customarily referred to as risk management (i.e. oriented towards insuring and hedging), and the other as investment management (optimizing risk/reward tradeoffs through diversification). The authors believe that effective risk management may involve using hedging, insuring and diversifying techniques: <u>Hedging</u> is the elimination of the risk of a loss by sacrificing the potential for gain. For example, equity portfolio risk can be hedged by selling short a stock index future; transitioning from equities to bonds can reduce a retiree's stock market exposure. <u>Insuring</u> is the payment of a known sum of money to eliminate the risk of losing a larger sum (substituting an acceptable small loss to prevent a catastrophic large loss). Put options, for example, are a method of insuring against stock risk. <u>Diversification</u> is the investment in many different risky assets instead of	Offers a nice conceptual structure regarding the dimensions of risk management. Retirement income portfolio management lies at the intersection of several disciplines including finance, actuarial science, financial engineering, and so forth.

		placing all funds in a single risky asset. The power of diversification is limited to the extent that risky assets exhibit positive correlation. Likewise, diversification is limited by the correlation between asset values and labor income value—a stockbroker holding a diversified equity portfolio is not diversified with respect to wealth. Usual retirement planning advice is limited to diversification analysis rather than to an integrated program of risk management. In some cases, however, hedging and insuring may be more appropriate management tools. Purchase of a life annuity is insuring against longevity risk. [Note: existence of reverse annuity mortgage options may decrease demand for commercial annuities]. The authors point out that fixed annuities and bonds may or may not be a good way to manage or hedge retirement income risk. Three requirements must be met: 1. the financial instrument must be default free; 2. the cash flows must match the maturity and time pattern of the spending targets; and, 3. the cash flows must match the unit of account of the spending target (i.e. pay off in required units of consumption). However, insurers have sometimes defaulted on annuities, and nominal annuities may not keep pace with inflation. Likewise, annuities may not provide adequate cash flows to cover certain unexpected income shocks. Annuities are illiquid and irreversible and may siphon liquidity that is needed to meet unexpected costs.	
2003	"Merging Asset Allocation and Longevity Insurance: An Optimal Perspective on Payout Annuities" Peng Chen & M.	Portfolios based on modern portfolio theory consider risk/reward tradeoffs in financial markets. Retirees, however, face a second risk that the authors term "longevity risk." The standard MPT tools used by financial planners may not yield optimal results because they ignore the second risk. The article illustrates (using the RP-2000 table) the conditional survival of	The risk to consumption (longevity risk) as well as the risk to bequests parallels the trustee's risk for administration of assets for current and remainder beneficiaries. Although this paper does not raise the issue of whether mortality-contingent claims are an "asset

	Milevsky Journal of Financial Planning vol. 16 no. 6 (June 2003), pp. 52 – 62.	age 65 single males, single females, and at least one member of a couple to ages ranging from 70 through 95 (18.4% probability of one member living to at least age 95). Annuity purchases, however, are not without risk. Although annuities are designed to provide a cash flow stream that cannot be outlived, fixed annuities lock the annuitant into payments that have declining purchasing power, while payouts from variable annuities fluctuate according to market performance. Additionally, the lack of liquidity creates additional risk. Payouts on annuities are a function of the prevailing interest rate at the time the contract is executed: "locking-in a fixed annuity is implicitly a market timing play." The authors divide past literature on the topic of annuities into three catagories: (1) "The first category consists of the theoretical insurance economics literature that investigates the equilibrium supply and demand of life annuities in the context of a complete market and utility-maximizing investors;" (2) "The empirical annuity literature examines the actual pricing of these products, and whether consumers are getting their money's worth;" and, (3) "a third and final strand attempts to create normative models that help investors decide how much to annuitize, when to annuitize and the appropriate asset mix within annuities." The remainder of the article (following track #3) describes and reports the results of a model developed to determine the optimal asset allocation between annuity (fixed and variable) and non-annuity (other financial instruments) assets. The model incorporates: ➢ Investor risk tolerance; ➢ Age; ➢ Subjective Probability of Survival; ➢ Objective Probability of Survival; ➢ Relative weights placed on consumption and bequest ➢ Utility derived from consumption and bequest; and, ➢ Risk/Return characteristics of assets.	class," its language is suggestive of the debate to come: the authors wish to study "…the total asset allocation decision in retirement, which includes both conventional asset classes and immediate payout annuity products." This vocabulary implies that an annuity is a product not an asset class: "A lifetime payout annuity is an insurance product that exchanges an accumulated investment into payments that the insurance company pays out over a specified time…." Later, the paper clarifies that asset classes can be wrapped in products to form two distinct "categories:" "The two categories are annuitized assets and non-annuitized assets." Or, "…a 60 year-old male…would like to allocate his portfolio across the two investment asset classes and the two mortality- contingent claim classes."

		The authors claim that "the greater the desire for creating an estate, or bequest value, the lower the demand (or need for) payout annuities. This is because life annuities trade-off longevity insurance against the creation of an estate." The illustrated results are for a single male age 60 with mortality defined by the IAM (not the RP-2000) Annuitant population mortality table. Assets are allocated across a risk free asset paying a fixed 5%, a risky asset with an expected return of 10% and standard deviation of 20%, a fixed payout annuity, and a variable annuity. The model derives Utility from a constant relative risk aversion function. Assets are rebalanced after the end of 20 years. If the retiree has no weighting on consumption (i.e. 100% altruistic), all assets are allocated outside of annuities. The mix of risky and risk-free assets is merely a function of the retiree's risk aversion value. Conversely, if the retiree has 100% weighting on consumption (i.e. no bequest motive), all assets are annuitized. The mix of fixed and variable annuities is a function of risk aversion. When the retiree places positive weighting on both bequest and consumption, all four asset classes (over risk aversion values ranging from 1 through 6) are present in the allocation. The macro allocation between equities and risk-free does not change appreciably (bequest motives do not change risk preferences). However, allocation to annuities decreases as the investor's risk aversion increases: "It seems that higher aversion to risk increases the implicit weight on the utility of bequest." The essay concludes that the decision process should be top-down where the first step "…is to locate a suitable global mix of risky and risk-free assets independently of their mortality-contingent status…. The annuitization decision should be viewed as a second-step 'overlay' that is placed on top of the existing asset mix."	
2003	"Reducing Retirement Income	The authors list the risks faced by retired investors:	The article provides a rationale for trustee purchase of annuities within a private family trust. The

	Risks: The Role of Annuitization," John Ameriks & Paul Yakoboski Benefits Quarterly vol. 19 no. 4 (December 2003), pp. 13 - 24.	1. Longevity risk 2. Rate of return risk (uncertainty about future investment results) 3. Inflation risk 4. Medical risks (uncertainty about medical care needs or long-term care costs). They begin with a discussion of longevity risk. Given the actuarial characteristics of immediate annuities, purchase of these contracts "…lets annuitants trade off risk and returns in the financial markets, while simultaneously reducing longevity risk." They describe the pooling mechanism that enables the annuity provider to provide a rate of return higher than that available from similar risk-adjusted financial instruments: "This has nothing to do with the investment acumen of the annuity provider; it's simply basic principles of insurance. Dealing with longevity risk is in fact not a matter of choosing the right investments—it's an issue of pooling and thereby reducing longevity risk." Individuals with substantial wealth can mitigate rate of return risk through purchase of stripped U.S. Treasury bonds wherein the maturity schedule matches the investor's income requirements. Inflation risk can be mitigated through ownership of TIPS. However, "…in the absence of a life annuity arrangement, there is no provision to hedge longevity risk, even if the rate of return risk can be controlled via financial mechanisms. It can be costly in that if withdrawals are held low enough to be supported indefinitely, spending will be significantly below what would otherwise be possible." Note: Compare with Sharpe, et al: "The 4% Rule—At What Price?" [2008]. Annuities may also provide a benefit to remaindermen in the sense that they free up money for investments. A cost of a lifetime dollar of income is less with an annuity. The authors develop this topic further by noting that "…there is an inherent tradeoff between maintaining a stock of assets and supporting	authors' argument is a variation on the theme that the trustee must make conscious decisions regarding the risk/return tradeoff, and must generally follow a course of action that will reduce unsystematic risk unless there is a reasonable expectation that the trust will be adequately compensated for retaining it. Longevity is a key systematic risk factor. [Note—the article can be used to justify annuities as a "safety net" product which the trustee has the option to utilize if an investment program is not successful]. Annuity purchase decisions are compared to asset allocation decisions. The annuity provides the investor with a choice regarding the amount of longevity risk that they assume. Note: Compare to Chen & Milevsky [2003] "Merging Asset Allocation and Longevity Insurance: An Optimal Perspective on Payout Annuities"

		a flow of income…." The annuity contract is valuable in so far as it allows retirees to achieve the greatest efficiency in spending money throughout retirement. The authors' vocabulary suggests that they consider annuities to be an asset class that may improve the Markowitz efficient frontier: "…the annuity allocation is as important as asset allocation in maintaining a stable income level and target portfolio balance…. An annuity is not an all-or-nothing decision-but more like an asset allocation decision, in which the participant needs to decide how much longevity risk they will take…." Furthermore, the authors note that the annuity product is primarily a "middle-income product." Annuities maximize cash flow when financial resources are small relative to consumption demands.	
2003	"The hurdle-race problem," S. Vanduffel, J. Dhaene, M. Goovaerts, R. Kaas <u>Insurance: Mathematics and Economics</u> Vol. 33 No. 2 (October, 2003), pp. 405 – 413.	The essay considers the problem of how to determine the amount of "provision" or "reserve" that is required at the current time to pay amounts α_i at times I (I = 1,2,…n). The provision is designated as R_0. The authors point out that if one wishes to meet the payment obligations with certainty, the replicating portfolio consists of n zero-coupon bonds assuming that the liability is deterministic. The paper, however, calculates the optimal reserve when the investment portfolio generates stochastic rather than certain returns. In this case, the objective is to determine the provision "…such that the probability that we will be able to meet our future obligations will be sufficiently large. Conversely, if the level of the provision is given, our methodology will enable us to compute the probability that we will be able to meet our future obligations under the given investment strategy." When the level of the provision or reserve is given—such as the case for a retirement income portfolio or funding of an testamentary trust—"…the optimal investment strategy could be determined as the one leading to the maximal probability that we will be able to meet our future obligations."	This approach to portfolio management forms a solid basis for monitoring portfolio sufficiency and for implementing an initially conservative allocation to protect against early downside returns while maintaining the option to become more aggressive if early results are positive. The approach justifies a "safety first" preferencing criterion for asset management. Portfolios that are initially conservative might be better suited to meet interim goals (wealth targets and consumption) at a higher probability. If future surplus develops, then the investment strategy can become less conservative. This approach contrasts sharply with Gordon Pye [2012].

		The paper details how to calculate the provision when the future payments are known and when the provision is invested in a stochastic return process. Given a stream of liability payments, the provision must have a value equal to or greater than zero at the end of the applicable planning horizon: One could determine the initial provision as the minimal amount such that Rn will be non-negative with a probability of at least $1 – \varepsilon_n$, with ε_n sufficiently small: $$\bar{R}_0 = \inf\{R_0 \mid \Pr[R_n \geq 0 \mid R_0] \geq 1 - \varepsilon_n\}.$$ The reserve is adequate if its level is greater than the stochastic present value of the payments to be made. The liability is deterministic but the present value of the liability payment stream is subject to changes in the discount rate. Therefore the PV of the liability is also stochastic. $$\Pr[R_n \geq 0 \mid R_0] = \Pr[S \leq R_0]$$ Where the random variable S equals the sum of the stochastic payments. Given that we want to assure the ability to meet the payment obligations at a sufficiently high degree of probability, the optimal reserve is defined as: $$\bar{R}_0 = \inf\{R_0 \mid \Pr[S \leq R_n] \geq 1 - \varepsilon_{n\}}.$$ Assuming that the stochastic return generating process is iid normal, this means that the initial reserve is the $(1 – \varepsilon_n)$-quantile of S. Assuming that the distribution of the sum of the payments is also iid normal, it is possible to calculate the probability of reserve adequacy. However, the authors note: "In general, it is impossible to determine the distribution function and the quantiles of S analytically, because in any realistic model for the return process…the random variable S will be a sum of strongly dependent random variables. Approximations for the distribution function of sums of dependent random variables have been	

		considered extensively in the actuarial literature. These approximations are based on the concept of 'comonotonicity' which describes a strong positive dependency between random variables…." A limitation of defining the optimal reserve in terms of "reaching the finish" is that there may be situations where interim value of the reserve falls below a threshold level which may violate regulatory requirements or which may, in general, represent an undesirable situation. The problem becomes one of calculating the optimal reserve in terms of not only the ultimate goal but also in terms of the period-to-period reserve value: "…the conditions that year-to-year the provision R_j is larger than a given deterministic value V_j with a sufficiently large probability. These additional requirements are the 'hurdles' that have to be taken." Thus, the reserve at time zero is determined by: $\bar{R}_0 = \inf\{R_0 \mid R_0 \geq V_0;\ \Pr[R_j \geq V_j \mid R_0] \geq 1 - \varepsilon_j;\ j = 1,\ldots n\}$. The authors note: "In situations where year-to-year adjustments of the level of the reserve are possible, the probabilities of taking the hurdles in the first years could be chosen larger than these probabilities in the later years." Or, "In practice, one will often choose the ε_i in the first years lower than the later ones because the conditions in the immediate future have to be met with the highest probability." For a stochastic lifetime, the longevity tail risk has a lower probability than the risk of running out of funds at life expectancy (the mean). Assuming that investment returns for each asset are lognormal, the portfolio has a multivariate normal distribution in which the evolution of returns depends on the weighting of each asset within the portfolio as well as the Pearson's correlation coefficient between the returns. The optimal reserve portfolio depends, of course, on the distribution of the Present Value of the Stochastic Liabilities—which is the PV of a Sum of payments plus any required interim additions to the reserve (V).	

		Although the distribution of this sum cannot be determined exactly, the authors provide approximation formulas for calculating the liability sum at a lower bound and at an upper bound. The "safest" strategy is to use the upper bound calculation However, provided that the actual return generating process is close to the lognormal assumptions in the authors' model, or provided that portfolio variance is sufficiently small, the upper bound—i.e, highest liability value--and lower bound—i.e., lowest liability value—converge to a value with a reasonably small standard error.	
2003	"The Relation Between Portfolio Composition and Sustainable Withdrawal Rates," Rory L. Terry The Journal of Financial Planning vol. 16 no. 5 (May 2003), pp. 64 – 72.	Terry uses Monte Carlo simulation to test conclusions, made by authors of earlier articles, that adding equity to portfolios increases sustainable withdrawal rates. Terry argues that, given the consequences of retirement portfolio depletion, the acceptable probability of failure must be very small: "My opinion is that most investors would find failure rates in the five percent and higher ranges described in Ameriks, Bengen and Pye, for example, to be unacceptable. I believe that most investors would find even a one percent probability of failure to be excessively high when dealing with irreplaceable assets and considering the extreme costs of failure." Terry creates a highly stylized asset allocation model that generates outputs by means of a Monte Carlo simulation. He is not interested in the "best allocation" or the most favorable withdrawal strategy according to historical results. Rather, "…defining a best methodology for determining a sustainable withdrawal rate seems much more relevant than defining a single best withdrawal rate." Given some fixed assumption such as a constant 6% earnings rate on fixed income and a constant 3% inflation rate, and a 30-year withdrawal period, Terry tests allocations to gain an understanding of the minimum of the failure rate function. In cases where the retiree has insufficient initial capital to fund a constant withdrawal rate, an all debt portfolio has a 100% probability of failure. In this case, betting on equity provides the only chance of generating the target income stream. However, assuming that the	Author argues that only a vanishingly small risk of portfolio depletion is acceptable given the catastrophic consequences of outliving retirement income. When current Wealth is insufficient to support target consumption, the investor must bet on equity. However, the higher the equity exposure, the greater the likelihood for failing to achieve the income target despite the fact that equity has a higher expected return than debt. Terry's conclusions largely depend on presence of cash as an asset class, a constant fixed-income earnings rate higher than the withdrawal rate, and on the assumption of a constant rate of inflation. Terry is interested in a Min/Min strategy that focuses primarily on the extreme left tail—it minimizes the probability of the minimum possible outcome. Argues that myopic strategies are not prudent.

		investor has sufficient initial capital, a 4.9% withdrawal rate, adjusted for inflation, can be amortized over a thirty-year period at a 0% rate of failure—debt return is always a certain 6% in Terry's model. Keeping initial capital constant, portfolio failure rates increase with an increase in equity weighting. A simulation suggests that an all-equity allocation can sustain a 1.85% annual payout with a failure probability of 10.02%. If the all-equity portfolio attempts to match the 4.9% constant dollar withdrawal target, the failure rate increases to 27.64%. "…sustainable withdrawal rates and percentage of equities are inversely correlated…." Terry's argument focuses primarily on a 2001 study by Ameriks, Veres and Warshawsky ("Making Retirement Income Last a Lifetime") who built Stock/Bond/Cash portfolios and, after considering the outputs of a Monte Carlo simulation, concluded that increasing equity weighting decreased failure rates monotonically given a constant dollar withdrawal rate of 4.5% of the initial portfolio value. For Terry, the inclusion of cash in the asset allocation distorts results. Terry asserts that the cash position is the economic equivalent of a 0% yield bond. The conservative portfolio (20% stock / 50% bond / 30% cash) thus had only 70% of its assets earning returns different from 0%. The most aggressive portfolio (85% stock / 15% bond / 0% cash) had a full 100% of its assets earning a non-zero return. The decrease in portfolio failure rates, therefore, is not due to increased equity weighting, but rather to the crowding out of the cash drag on portfolio performance. What would be the sustainable withdrawal rate for an all-equity portfolio at a 1% failure rate? Terry suggests that the rate declines to 0.19 percent: "debt allows a larger withdrawal percentage than equity, for a given level of risk, even though equity has historically provided a higher expected return." However, debt has drawbacks: 1. Debt heavy portfolios are an all-or-nothing proposition, while equity offers a substantial upside potential.	

		2. Debt is far less likely to immunize the investor against unexpected inflation shocks than is equity.	
2003	"Mortality risk and real optimal asset allocation for pension funds," Francesco Menoncin and Olivier Scaillet archive ouverte UNIGE University of Geneva (2003) http://archive-ouverte.unige.ch/unige:5785	This article presents a closed form solution, assuming complete markets and frictionless trading, for optimal asset allocation under mortality and inflation risks. The work expands previous research by (1) assuming a vector of stochastic state variables—instead of just a single variable like the real interest rate or the equity risk premium, and (2) the ability to borrow. The objective for the institutional investor—i.e., pension plan—is the maximization of the intertemporal utility of real wealth assuming a power utility of wealth function. The authors point to previous research in the case of a budget constraint prohibiting borrowing. The optimal solution is to invest a part of wealth in the unconstrained strategy and the remainder of wealth in an American put option. The authors explore the impact of inflation, the effect of plan-population mortality risk, and the asset allocation decision in both the accumulation and decumulation stages. A closed form solution to utility optimization is feasible if markets are complete and if the price of risk—equity risk premium—is independent of the other state variables such as the stochastic interest rate, inflation rate, and investment opportunities. The authors define a "prospective mathematical reserve" for the pension arrangement as the expected value of all future contributions—a stochastic variable dependent on labor income—less the expected value of all future pensions—i.e., distributions. At any time t, the pension fund's total wealth is its current wealth plus its prospective reserve. This modified wealth is termed "disposable wealth." Thus, a pension arrangement seeks to maximize the utility of its real disposable wealth. It is clear that the model for an institutional investor such as a pension arrangement parallels, in some respects, a model for an individual investor endowed with human capital. In the institutional model, however, the personal discount factor applied to intertemporal utility is replaced by an actuarial discount factor which incorporates mortality risk. The model generates asset prices evolving according to a Brownian motion process and the riskless interest rate evolving as an Ornstein-	Develops the concept of "the feasibility condition" for corporate pensions. This parallels the condition that the market value of wealth must be equal to or greater than the PV of liabilities in order to make retirement feasible. The authors make the observation that the value function for the Hamilton-Jacobi-Bellman equation "inherits its functional form from the utility function." That is to say, the model determines outcomes and, therefore, any conclusions must be model dependent. This is a key insight because many normative articles fail to acknowledge "model risk." Sources of model risk include the choice of the utility of wealth function as well as the functional form used to model inflation and asset price evolutions.

		Uhlenbeck process according to the Vasicek model. They cite the previous [2002] paper by Charupat and Milevsky who solve for the optimal asset allocation in both the accumulation and decumulation phases of an individual investor's portfolio. The accumulation phase maximizes the expected utility of terminal wealth, while the decumulation phase maximizes the expected utility of consumption during retirement. Under certain assumptions, the Charupat-Milevsky model's asset allocation strategy is the same in both states—the portfolio is allocated according to the Merton Optimum. A pension plan, however, must continuously manage wealth even after the retirement of some of its participants. Therefore, the Menoncin-Scaillet [MS] model solves for asset allocation simultaneously in both phases from the perspective of the institutional investor. The MS model has two sources of investment risk: (1) the change in wealth due to the change in market value of investments, and (2) the change in wealth due to changes in asset allocation. Additionally, the MS model introduces a profit sharing feature which distributes nominal surplus to plan members. The authors develop the concept of "the feasibility condition." In order for a contract between a plan sponsor and participants to be economically feasible, "…the expected discounted value of all contributions is equal to the expected discounted value of all pensions." This assumes that at time zero, the initial pension plan balance is zero. This concept maps to the feasibility condition for a successful retirement for an individual investor. The paper solves the Hamilton-Jacobi-Bellman equation for the optimal utility. The authors observe: "…it is well known that the value function usually inherits its functional form from the utility function." Indeed, the optimal allocation to the risky asset, although independent of both real wealth and price level, is, nevertheless, heavily dependent "…on the choice of both the utility function and the functional form chosen for the	

		drift and diffusion terms of assets, price level, and inflation." The solution finds that, unlike the Charupat-Milevsky model, optimal asset allocation differs in the two phases. In contrast to the classic two-fund Tobin model or the three-fund Merton life-cycle allocation model, the MS model has a five-fund solution: • A minimum variance fund • A growth optimal fund which is scaled on total wealth (real current wealth + prospective mathematical reserve) • Two funds to hedge against changes in value of stochastic state variables—one as a hedge against changes in the prospective mathematical reserve and the other as a hedge against changes in the value of state variable vector components such as the riskless interest rate, etc. • A fund consisting solely of the riskless asset. The authors acknowledge that model assumptions are strong but they point out that the solution to the model "…can be thought of as a benchmark that can give some practical insights of the actual investments of pension fund managers." Assuming that the pension arrangement begins with zero wealth and with no immediately retiring participants, the authors provide some general rules for optimal institutional investing. For a CRRA investor, most models suggest that the optimal portfolio hedging component is the only fund that is dependent on time horizon—other investment fund components should "stay-the-course" according to the re-optimized Merton Optimum. However, in the MS model, the argument for the CRRA utility function contains the function Δ(z,t) which reflects changes in the prospective mathematical reserve that are dependent on changes in time as well as in the value of the state variable vector components: "This means that, in some sense, the risk aversion depends on the time horizon as well." The MS model suggests: "the speculative activity of the pension fund (i.e., investment in risky assets) must decrease when the pension date approaches. Thus, investments should be concentrated on the riskless	

		asset in order to provide a safer revenue for paying the pensions whose payments approach. Finally, after T, when the death probability becomes higher and higher and the pensions start being paid, the speculative investments increase since the relative weight of the total amount of future pensions that must be faced by the fund becomes lower and lower.”	
2003	“Optimal portfolios for Different Holding Periods and Target Returns,” Sandip Mukherji Financial Services Review vol. 12 no. 1 (Spring 2003), pp. 61 – 71.	This paper reviews the time-diversification issue from the point of view of minimizing the ratio of downside risk divided by mean real value at the end of the holding period. One school of research points out, for short planning horizons, that stocks have a higher standard deviation than bonds. For longer planning periods, however, the reverse is true. Other research suggests that the volatility of total returns on stocks increases with the holding period and that the standard deviation of the holding-period return rises faster than the mean return as the holding period increases. The author’s position is that most of the long-term variation in stock returns flows from upside movements which, from the investor’s perspective, represent reward rather than risk: “…most of the greater long-term variability of riskier asset consists of uncertainty about the extent to which riskier assets will outperform safer assets….investors are more concerned about downside risk than about overall volatility….” The author discusses preferencing criteria for the optimal asset allocation across various planning periods. Rather than selecting the portfolio that minimizes downside risk, the author makes the case for minimizing the “proportion of downside risk to mean real terminal value for each holding period.” His risk-to-reward model uses a bootstrap sampling methodology for six U.S. asset classes over the period 1926-2000. The real [inflation adjusted] returns are calculated for samples of 1, 5 and 15-year holding periods. Downside variance is, of course, calculated with respect to the target return. For each of the three holding periods, the author designates a low target (2% real) a moderate target (5% real) and an aggressive target (8% real). Only returns less than target appear in the downside variance [DV] calculation:	This study does not take into account interim cash flows. It provides an interesting view on the time diversification debate—however, if standard deviation reflects period-to-period return variations, then using only a returns-below-a-target method may “throw away” much useful information regarding risk. For long horizon investors, loading for equity is optimal in the absence of cash flows. Compare the preferencing criterion in this essay to “Withdrawal Patterns and Rebalancing Costs for Taxable Portfolios,” J. Christopher Hughen, Francis E. Laatsch, Daniel P. Klein [2002].

		$$DV = \sum(TV - V_h)^2 / (H - 1); \;\; V < TV$$ Where TV is the minimum target value, V_h is the real value of the investment at the end of holding period h, and H is the total number of holding periods used to calculate the DV. The square root of the Downside Variance is termed Downside Deviation [DD]. The coefficient of downside deviation [CDD] is the DD divided by the mean real value of the portfolio. Subject to the constraints (1) the weight of each asset must be equal to or greater than zero within the portfolio, (2) the sum of the weights equals unity, and (3) the terminal value of the portfolio provides a real return equal to or greater than the target rate of return, the author uses the Excel 2000 Solver program to determine the optimal weighting of each asset class relative to the planning horizon and targeted return. The author investigates the changes in CDD for each asset class as the planning horizon increases. His data suggests that stocks are most risky in the short term but are the safest assets—i.e., lowest CDD—over the long term: "For long-term investors with a low target return, the optimal allocation is 58% intermediate government bonds, 14% large stocks, and 28% small stocks. For medium and high target returns, the optimal portfolio of long-term investors is fully invested in small stocks. These results indicate that T-bills dominate the portfolio only for short-term investors with a low target….For medium and high target returns over a long holding period, the optimal portfolio consists solely of small stocks." For investors concerned primarily with minimizing DD rather than maximizing the CDD ratio, there are changes in the optimal portfolio: Over shorter planning horizons, intermediate bonds have a higher weighting. "The most striking difference between the optimal portfolios of investors minimizing DD and those minimizing the CDD is for the 8% target return over 15 years. Whereas the former have 100% in small stocks, the latter have 76% in large stocks and 18% in small stocks."	

2003	"Pensionmetrics 2: stochastic pension plan design during the distribution phase," David Blake, Andrew J.G. Cairns, Kevin Dowd, Insurance: Mathematics and Economics vol. 33 no.1 (August 2003), pp. 29 – 47.	The authors point out that annuities are not risk free investments. Annuity purchasers face (at least) two risks: 1. Purchase of an annuity in a low interest rate environment may lock the annuitant into a permanently low retirement income stream; and, 2. Unanticipated inflation can substantially decrement future purchasing power. Insurers offering annuities also face (at least) two risks: 1. Mismatch between asset (cash inflows) portfolio backing reserves and cash outflow payment requirements; and, 2. Risk of annuitants living longer than expected (mortality risk). Although the fall in bond yields have made fixed income vehicles such as annuities relatively unattractive in recent years; it is, nevertheless, true that "…any income drawdown programme that draws a fixed income from a fund heavily invested in equities has a positive probability of running down to zero before the plan member dies." The risk of portfolio depletion prior to death "…can be ameliorated by requiring that the amount drawn from the fund be linked to the fund size at each point in time…." Factors determining the amount of the annuity's periodic payment include: 1. The size of the fund 2. Long-term bond yields on the purchase date 3. Annuity type 4. Age, sex, and, in some cases, health of the annuitant 5. Margin for profit, administration, investment expenses and marketing costs Method The paper compares three alternative distribution programs for a male retiree aged 65 with a fund of 100,000 pounds:	Article suggests that prudent retirement spending is linked not to a predetermined rule—e.g., 4% withdrawal rule—but, rather, to the size of the assets supporting such spending "…at each point in time…." If the fund does well, retirees with greater risk tolerance may wish to permanently defer annuitization. The optimal age to annuitize depends on bequest utility and on the performance of the investment fund during retirement. This is an example of a 'top-down' approach to the annuity purchase decision. The authors demonstrate that, for retirees, the demand to annuitize increases as the value of the investment fund decreases. Trustee has a duty of impartiality between interests of the current and remainder beneficiaries—unless trust document instructs otherwise, in which case the trustee may favor one beneficiary class over the other. If the value of the trust corpus decreases because of poor investment results, how does the trustee "balance" a beneficiary's interest in the face of "shifting utility values?" See Venter [1983] The authors find that it is the asset allocation—demand to hold equities—that has the greatest effect on the timing of annuitization Unlike the Milevsky et al decision rule regarding annuitization—annuitize when interest + mortality credits exceed expected risk premium of portfolio—the authors argue that optimal annuitization is a function of investment performance *and* the size of

		➢ Purchased Life Annuity (PLA). This is the benchmark against which all options are compared. The authors term this annuity the "benchmark pension." It is a single-premium, immediate annuity. ➢ Equity Linked Annuity (ELA). Retirement assets are held in an annuity funded by a managed stock and bond account (equity allocations ranging from 0% to 100%). Yearly annuity income fluctuates with the account's value. At the end of each year the recipient receives, in cash, a payment for any survival credit that may arise when those who die early create a fund that can be shared by the survivors. In return for the cash payments, the ELA participant leaves no bequest upon death. The participant purchases a PLA at age 75. ➢ Equity-linked Income Drawdown (ELID). The pension fund is invested according to a specified, constant (rebalanced yearly) asset allocation. The participant receives no survival credit payment nor does the participant forfeit his right to make a bequest of the account value if death occurs prior to age 75. Per UK law, the participant purchases a PLA at age 75. The above distribution alternatives were found to be superior to three alternatives considered in an earlier paper: 1. A fixed amount withdrawal which trades stability of income for a positive risk of ruin prior to death. 2. Use of derivative overlays to limit downside risk. 3. Purchase, at age 65, of a deferred annuity with a payout start date of 75 and consumption of the remaining fund prior to 75. Note: alternative three is a variation on the concept of pure longevity insurance. See [1998] Scott: "The Longevity Annuity." The bond investment grows at a deterministic continuously compounded	the wealth fund. Smaller sized accounts produce less income and the marginal utility of the excess income from annuitization has great value. This is a good summary of the investor's difficulty with making a prudent decision—"…the marginal utility of consumption gets large as the fund size gets small." The size of the fund relative to target consumption is the critical ratio that the investor must monitor: "the size of the fund is directly related to the propensity to delay annuitization: the larger the fund, the longer the delay." Note: this article uses a single-premium immediate annuity as the "benchmark pension." This theme is further developed in a trust context by: Collins, Patrick J. "Managing Modest-Sized Family Trust Portfolios: Issues in Income Adequacy and Portfolio Sustainability," ALI-ABA Course of Study Materials: Representing Estate and Trust Beneficiaries and Fiduciaries, (Chicago, 2011). Collins, Patrick J., Fast, Steven M. & Schuyler, Laura, "Well-Performing Portfolios and Well-Disguised Insolvency," ALI-CLE Course of Study Materials: Representing Estate and Trust Beneficiaries and Fiduciaries, (Chicago, 2014) pp. 499-522. The theme is developed in an investment portfolio context by: Sexauer, Peskin & Cassidy "Making Retirement Income last a Lifetime." [2012]. Cassidy, Peskin, Siegel & Sexauer, "Be Kind to your Retirement Decumulation Plan—Give it a Benchmark" [2012].

		rate, while the equity investment satisfies the classic geometric Brownian motion such that the yearly returns are independent and identically distributed log-normal variables. Thus, for the PLA, the retiree uses his funds at time zero [F(0)] to purchase a life annuity at a price of a_x per pound for a lifetime income of F(0) / a_x. The annuitant receives no cash survival credits because such credits are implicitly incorporated into the pricing of the annuity. The ELA adjusts the payout [P(t)] each year (prior to 75) depending on the value of the fund [F(t)] at the beginning of the year. The equity weighting selected by the retiree remains constant and the payout in any year equals F(t) / a_{x+t}. Furthermore, the yearly survival credit depends on mortality realizations vs. expectations; while the value of the Fund depends on the asset return generating process less payouts and survivor credit payments. Note: unlike a pure unitrust portfolio distribution solution, the ELA adjusts both the numerator and the denominator [a_{x+t}] each year. The ELID provides a level draw down only if the return on the assets is equal to the risk-free rate adjusted for mortality [return of fund = e^r / p_{x+t}]. The programs are ranked according to their expected discounted utility. Utility takes the form of the following two-part value function with the first part representing the expected discounted utility of the lifetime payouts and the second part the expected (lifetime) value of a bequest to be made at some future year: $$V = E\left[\sum_{t=s}^{K} e^{-Bt} J_1(P(t) + k_2 e^{-B(K+1)} J_2(D(K+1))\right]$$ Where	The authors provide a helpful summary of previously suggested decision rules: • Yaari (1965): Annuitize (PLA) immediately. • Merton (1983): Purchase an annuity immediately; never opt for PLA—mandatory at age 75 in the UK. • Milevsky (1998): Annuitize (PLA) when mortality drag ≥ equity risk premium. • Kapur and Orszag (1999): Gradual annuitization (PLA) with full annuitization when mortality drag ≥ equity risk premium. • Milevsky and Young (2002): Switch to PLA at deterministic time T. ELID before T managed under an optimized dynamic asset mix. T depends on risk aversion and model parameters. • This paper (2003): Switch to PLA at a stochastic stopping time T. ELID before T includes optimized static asset mix. T depends on risk aversion and bequest utility.

$$J_1(P(t) = h_1(\gamma_1)\left(\frac{P(t)}{P_B}\right)^{\gamma_1}$$

$$J_2(D(t)) = h_2(\gamma_2)\left[\left(\frac{D(t)+d_2}{d_2}\right)^{\gamma_2} - 1\right]$$

P_B is the survivor credit and D = change in value of the Fund from t to t+1: D(t+1) = δF(t+1).

J_1 is a constant relative risk utility function (power or log utility) while J_2 is a hyperbolic absolute risk aversion function (HARA). β is the retiree's subjective rate of time preference, and k_2 (0 < k <100) specifies the appropriate balance between the desire for income and the desire to make a bequest. A male with minor children may have a much higher value of k than a single male without children. When k_2 equals 0, the utility functions become irrelevant. Otherwise:

$$h_1(\gamma_1) = \frac{1}{1-d_1^{\gamma_1}}$$

Where d_1 is between 0 and 1 (model assumes value of 0.75); and,

$$h2(\gamma 2) = \frac{1}{((F(0)+d_2)/d_2)^{\gamma_2} - 1}$$

Where the value of d_2 in the model (bequest) is 10,000.

The model examines and ranks the retirement election choices across risk aversion parameters ranging from 0.25 to 25.0 with the time preference value of β fixed at log 1.05 [4.88%] and k_2 at 5. For relative risk aversion [RRA] less than 1.25, the best program is the ELA with 100% equity allocation; with an RRA between 1.25 and 10, the best program is

		the ELA with decreasing equity exposure, and for RRA greater than 10, the best program is the PLA. The authors state, however, that the classic utility of consumption model is not intuitively easy to understand and that a better metric might be a use of a type of AEW (annuity equivalent wealth) measure that determines how much extra (proportional to the initial value of the fund) a retiree needs at time zero to be indifferent between any two options. Utility measure is critical, however, for at least two reasons: 1. Because those with low risk aversion have a strong demand to hold equities and would need substantial compensation for participating in a bond-oriented investment strategy (and visa-versa). Indeed, the authors find that their model suggests the choice of the equity proportion in the retirement account dominates the choice of retirement program; and, that this finding is not particularly sensitive to the size of the equity risk premium. 2. Decision rules based solely on timing (e.g. Milevsky's rule to switch to an annuity at the time when the expected excess return on equity equals the mortality credits offered by annuities) are suboptimal for any investor who is not risk neutral. Interestingly, the authors find that the annuity decision is likely to depend on the fund size because the bequest utility does not exhibit CRRA: "Our results indicate that a plan member is more likely to prefer to delay (bring forward) annuitization if his investments have been performing well (badly). To illustrate this, suppose the fund size is almost zero and a plan member is considering a switch from the ELID programme to the PLA programme. On the one hand, the negative impact on the bequest utility will be negligible because the fund size is very small. On the other hand, the payment of survival credits through the PLA will have a strong beneficial impact on the utility of consumption, because the marginal utility of consumption gets large as	

		the fund size gets small." Likewise, the authors note that increasing the value of k (relative balance between desire for income and desire for bequest), has only a very gradual effect on the utility of annuitization. Also, the existence of other annuity income does not materially alter the conclusions. Conclusions: ➢ The optimal choice of program is insensitive to the weight attached to making a bequest; ➢ The critical variable is the proportion of equity underlying the distribution program; ➢ The optimal age for annuitization is very sensitive to the degree of risk aversion and moderately sensitive to the bequest motive ["Our more general analysis demonstrates that decision making is much more complex than the Milevsky rule suggests…."]; ➢ The size of the fund is directly related to the propensity to delay annuitization: the larger the fund, the longer the delay.	
2003	Yonggan Zhao, Ulrich Haussmann, and William T. Ziemba, "A Dynamic Investment Model with Control on the Portfolio's Worst Case Outcome," <u>Mathematical Finance</u>, Vol. 13, No. 4(October 2003), pp. 481 – 501.	Generally, a rational investor seeks to maximize the expected utility of wealth by trading off "…profits in good future states against losses incurred in bad states." The authors assert, however, that "…explicit downside risk control approaches seem preferable for investors with a liability stream." They are interested in modeling a utility function that reflects both the weighted sum of expected wealth (which increases with allocation to the risky asset because of the expected equity risk premium) and the worst-case wealth outcome (which worsens with an increase in allocation to the risky asset). "How do we compromise between these two competing aspects to develop a dynamic investment portfolio?" The article asserts that the structure of the optimal portfolio is equivalent to a call option where the strike price is the "worst outcome that may have occurred," and a riskless asset. Given HARA utility and geometric Brownian motion, the optimal investment strategy is a two-fund strategy where the risky	The article begins by noting Robert Merton's 1971 proof that independent and identical asset [iid] returns, characterized by geometric Brownian motion in continuous time, allow for closed-form solutions to determining investor demand to hold a risky asset assuming a utility function from the HARA class—where utility (decreasing or absolute) is also a linear function of wealth. The stochastic dynamic programming theory of continuous-time optimal investing usually follows either *a stochastic control methodology* or *a martingale analysis* approach. In the stochastic control approach, given the continuous time context, the 'state-of-nature' or 'system state' is known at each instant; and, the

		portfolio is a leveraged growth optimal portfolio with the leverage changing with passage of time. Initially, the article reviews the (sometimes complex) mathematics required to transform a stochastic price process model into an equivalent martingale measure under the assumption of an iid. geometric Brownian motion process [often termed a Wiener Process]. The article's next section models an investment strategy that is "…a trade-off between the overall wealth and the worst outcome." The optimal solution is "…to maximize a convex combination of 'utilities' received from both the actual outcome and the worst outcome of wealth." The revised objective function becomes: $V(x,y) = \rho U(x) + (1-\rho)U(y)$ where 'x' is the actual wealth outcome, 'y' is the worst possible outcome and 'U' is the investor's utility function. According to the authors: "this is equivalent to maximizing expected utility with a different probability measure—that is, reducing the probabilities (density) of all states by the ratio ρ and adding the total reduced probability weights to that of the worst case outcome. The control intensity of downside losses increases as ρ decreases." The following example illustrates the concept. An investor exhibiting log utility and endowed with $100 is offered a gamble that pays $1 with .80 probability and -$2 with .20 probability. Terminal wealth is $101 with 80% probability and $98 with 20% probability. Without a downside control variable the gamble's expected utility is 4.6091 [0.8ln(101) + 0.2ln(98)] which, because it is greater than ln(100) = 4.6052, induces the investor to take the bet. With the downside control variable the utility value becomes: 0.8ρln(101) + 0.2ρln(98) + (1-ρ)ln(98). In order for this equation to generate a value greater than ln(100), ρ must be greater than 0.8375. In this case, an 80% probability of gain is insufficient to induce the investor to accept the bet. When confronted with the need to fund a liability, "…a downside-control	goal of the stochastic control approach is to find the value of the asset allocation variables that maximizes an integral value over the relevant time period. The integral could be a concave function—i.e., utility from time zero to time T. As new information becomes available, the investor adjusts the control variable(s). Adjustment is expressed as a differential equation (e.g., change in wealth per change in control variable) which, in turn, is a function of the random returns of the portfolio's asset allocation weights (as well as other relevant variables). The portfolio's asset weighting is continuously solved for its optimal value. However constraints on the controls—i.e., restrictions on short positions in assets, or eliminating the possibility for negative consumption—make it difficult to arrive at solutions. This problem is known as the "compactness of controls" problem. A martingale measure transforms a stochastic continuous time process into discrete analogues which have useful properties under various probability [P] measures. A martingale process [M] exhibits the property that the best estimate of the future value for M is its current value. This type of martingale models a fair game or a zero-mean process (white noise). A super-martingale models a game with a process leading to an expected unfavorable outcome (time spent in a casino); a sub martingale models a game with a process leading to an expected favorable outcome (expected value of a stock). Thus, when applied to risky assets, a martingale approach can accommodate both drift and diffusion elements within the price process. A stochastic process with probability P may, with

		investor will not maximize the expected utility with the objective probability measure but with an adjusted one." The investor accomplishes the adjustment by shifting a portion of probability mass at higher outcomes to that of lower outcomes. This results in a shifted probability measure Q which has the effect of changing the marginal rate of substitution for the risk/reward tradeoff. This shift, in turn, changes the optimal investment strategy. Where A is the set of all admissible investment strategies, α is the allocation to the risk-free investment, and θ is the allocation to the risky investment, the dynamic optimization problem is to maximize expected utility with respect to (α, θ) ∈ A: $\max E\left[\rho U\big(W(T)\big) + (1-\rho)U\big(W^L(T)\big)\right]$ The problem can be divided into two problems: an optimization of terminal wealth problem and a "replication problem deriving the optimal portfolio strategy." At the limit, if ρ equals one, the problem becomes the traditional utility maximization problem; if ρ equals zero, the investor will prefer to keep all funds in a risk-free account. The authors restate the problem using the Kuhn-Tucker Criterion for calculating a constrained optimum. The authors point out that the martingale transform enables them to use standard methods for valuation of contingent claims. For a given level of wealth [W= K with K a constant > 0; and K = call option strike price] they solve the model for the optimal asset allocation weighting (the riskless asset weighting in the portfolio required to guarantee K; the risky asset weighting; and the leverage factor on the risky asset fund). At K, the marginal utility of increasing expected terminal plan wealth is exactly equal to the utility loss induced by changing the option's strike price to a lower level of W: $U_x(K) = \rho E[U_x(L)]$ Where L is a random variable.	suitable transforms, be modeled as a Martingale process under probability Q. This is known as the 'equivalent martingale measure' of the price evolution process. The Girsanov Theorem permits a transform of a stochastic process under an historical probability measure P to a martingale process under a risk neutral measure Q. It proves that every P-measure sub-martingale is also a Q-measure sub-martingale thus permitting calculation of a definite integral under a more tractable martingale approach. Given the stability and the finite bounds of a martingale process—indeed a pure martingale is a constant—this means that it is possible to (1) "identify the subspace of attainable wealth," and (2) choose the best allocations to satisfy portfolio constraints. This is the underlying rationale for Zhao, Haussmann, and Ziemba's use of a martingale analysis approach for calculating the value of the integral over a finite planning period. Note: The shifting of the density in a probability function (or, intensification of the downside risk control factor ρ) is comparable to the utility penalty cost factor (c) in "Liability Investment with Downside Risk," Andrew Ang, Bingxu Chen & Suresh Sundaresan, National Bureau of Economic Research Working Paper 19030 http://www.nber.org/papers/w19030 (May 2013). Both studies use an option valuation approach.

		In order to solve for the optimal portfolio management strategy, the authors employ numerical methods to solve a partial differential equation with boundary conditions. This method involves calculating the optimal weighting of the risky asset without a wealth-level constraint, and then deriving the optimal strategy at the optimal wealth level (a solution to a partial differential equation). An options strategy closely replicates the optimal portfolio management (i.e., trading) strategy. The authors follow a four-step process: 1. Find the optimal initial leverage (λ_0) and threshold wealth level (K); 2. Create the risky asset portfolio using available assets; 3. Calculate the hedging portfolio for a call option on the risky asset portfolio at strike price K; 4. Transform the hedging portfolio into an optimal portfolio of primary assets. The four-step process is difficult to calculate and implement for a number of reasons: investors exhibit differing utility functions, it is unlikely that exchange-traded options will exactly match the strike price and investment horizon required by the investor, the risky-asset price process ("drift rate") may not be a deterministic function of time unless the model incorporates simplifying assumptions; and so forth. However, when the utility function is HARA and the asset price process follows geometric Brownian motion, there is an identifiable optimal trading strategy; and, this strategy is identical "…if investors have [the] same risk aversion and same risk skewness…." This solution takes the form of owning (1) a call option on a mutual fund "synthesized" from primary assets available in the marketplace with leverage factor λ, and (2) a riskless asset. A corollary is that the risky asset fund, assuming a multivariate Brownian motion process, should be managed under a constant mix strategy. The study ends by discussing a new risk measure [τ] that, unlike the Sharpe Ratio, evaluates downside risk only, and expresses a Variance-at-	

		Risk measure in the denominator: $$\tau = \frac{E[W(T)] - W_0 e^{eT}}{VaR[W(T)]}$$ An investor can use the performance measure to calibrate the optimal downside control factor ρ.	
2003	"A Comparative Analysis of Retirement Portfolio Success Rates: Simulation Versus Overlapping Periods," Philip L. Cooley, Carl M. Hubbard & Daniel T. Walz, Financial Services Review, vo. 12 no. 2 (Summer, 2003), pp. 115 – 129.	Withdrawal planning requires a balance between withdrawing too much (retiree dies broke) and withdrawing too little (unnecessarily sacrifices a higher standard of living). Withdrawal planning benefits from simulating future security returns or from making use of historical returns ["Referred to as the overlapping (or rolling) periods method…"]. The authors contend: "…if the simulation faithfully incorporates all meaningful market properties, the simulation methodology should produce highly robust results…." However, "if actual security returns are generated from unstable distributions or different distributions over time, it might not be possible to build a realistic simulation." If this is the case, the authors assert that a rolling-periods methodology is a more accurate predictor of sustainable withdrawal rates. Rather than simulate an aggregate portfolio with a hypothesized mean and standard deviation, the authors simulate stocks, bonds and inflation separately. They include correlations and a factor for mean reversion, and they model inflation as a serially correlated variable. They test for the success rate (money remaining in the portfolio at the end of the planned payout period). Their risk model considers various withdrawal rates (nominal and constant dollar) as well as asset allocations under the assumption that monthly returns are normally distributed. Equity returns are modeled as an autoregressive moving average process of order ARMA (1,1) with a monthly intercept of the ARMA model of 0.0103 and a standard error of 0.0414 based on historical returns for the S&P 500 from 1946 through 2001. However, "we found that the autoregressive coefficient is not significantly different from zero after 1969." The model assumes a constant correlation value between stocks and bonds equal to 0.39.	Good discussion of risk modeling methodologies. However, the simulation model assumes a stable log-normal distribution with constant parameters. Continues the assertion that a heavy loading to equity results in enhanced sustainability—especially at higher withdrawal amounts. See also: [2002] Hughen, Laatsch & Klein ; [1994] Larry Bierwirth; [2004] Gerrard, Haberman & Vigna.

		The rolling period methodology uses 56 years of monthly returns. For example, there are 42 overlapping 15-year payout periods [56-(15-1)]. The authors acknowledge that the boundary years are used only once in the analysis, while the middle years appear multiple times. The observations, therefore, are not independent. However, because the middle years in the historical period were the 1970s and early 1980s, the results of the overlapping period methodology should be considered to be conservative because the period with the heaviest weighting had two major recessions. Interestingly, the overweighting of the historically performing poor periods results in the Monte Carlo simulation methodology producing higher success rates than the historical rolling period methodology. On a nominal dollar basis, portfolios allocated heavily to equity can sustain a 7 to 8 percent withdrawal rate over a 30-year period. The success coefficients [approx. 75%+] however drop off substantially as bonds are included. A 90%+ success nominal-dollar coefficient requires a reduction to a 6% withdrawal rate. Inflation-adjustments for a 30-year period assuming heavy allocation to equities exhibit a 75% coefficient of success at 4 to 5 percent withdrawal rates; and a 90% coefficient at 3 percent rates.	
2003	"Asset Allocation Advice: Reconciling Expected Utility and Shortfall Risk," Michael Stutzer Working Paper, University of Colorado.	Asset allocation advice is often based on a maximization-of-expected-utility criterion. Common assumption are an investor with (1) a CRRA function (power utility), or (2) quadratic utility (mean-variance utility function). Asset allocation advice may, however, be based on the alternative target shortfall probability approach. This approach "…obviates the need to assess a risk aversion parameter." However, it requires that the investor specify a target return. Asset allocation advice that follows the shortfall probability approach requires a three step process: 1. Minimization of the probability that the long-term growth of real wealth falls short of a feasible long-run target return.	Stutzer defines optimal asset allocation in terms of the portfolio with a minimum probability of shortfall relative to a benchmark target. He does not consider the magnitude of potential shortfalls. Compare to "Shortfall-Risks of Stocks in the Long Run," Peter Albrecht, Raimond Maurer & Ulla Ruckpaul [2001] Article advances portfolio preferencing criteria under a shortfall risk metric. The author's take on equity is that the lower the

		2. Determining the efficient frontier of portfolios designed to minimize the probability of shortfall associated with the range of feasible long-term returns on wealth—in general, the higher the target return, the greater the probability of a shortfall). 3. Matching the set of portfolios on the efficient frontier to the specific investor. The conventional use of CRRA utility "…holds the risk aversion parameter fixed, rather than varying it to maximize expected utility." When the conventional use of CRRA is combined with the further assumption that asset returns are identically and independently distributed across time, utility is maximized "…by initially choosing a vector of value weighted, optimal asset allocation proportions or weights 'p' that maximizes [power utility], and then rebalancing the portfolio at the beginning of each subsequent period back to those initial weights." Under these assumptions, a constant mix portfolio is superior to a buy and hold portfolio when the investor's objective is to maximize the utility of terminal wealth. However, the exact asset return distribution is uncertain. Therefore, it is common to use historical returns to develop an "historical time average estimator." Under a CRRA utility maximization approach—assuming mean / variance optimization—the vector of stock weighting changes significantly as risk aversion increases. The author cites several general population "questionnaire" studies indicating high degrees of relative risk aversion. Investors with a coefficient of risk aversion of 4 [survey indicated that two-out-of-three investors had risk aversion higher than 3.76] would prefer a maximum exposure to equity of only 39%. This creates a problem because a shortfall minimization approach--where the target real rate of return is 4.5%--indicates that an 80% stock allocation achieves a shortfall probability of 21% at a 25 year horizon as opposed to a shortfall probability of approximately 34% for a 39% stock allocation weighting. The author suggests that there may be a scale problem with CRRA utility.	allocation to equity, the greater investor utility. However, the greater the allocation to equity, the less the risk of portfolio depletion. The article attempts to reconcile these observations. The portfolio management issue is whether to increase risk as wealth is decreasing. The high expected growth rate of equity makes it attractive when the investor falls below the terminal wealth goal. However, the increased variance of equity increases the probability that the goal will not be reached. The author acknowledges that shortfall probability is characterized by long-term persistence. This reflects the positions of Sid Browne, Mark Rubinstein and others. However, he argues: "…the persistence of target shortfall probabilities is unavoidable when the target is ambitious, and would only be worse if some other criterion function were used to choose a stock allocation."

		Theoretically, the scale of choice should have no effect on the value of the risk aversion parameter [e.g., CRRA investors exhibit risk tolerance that is invariant to changes is wealth]. However, in practice, investors distinguish between risks in the small and risks in the large, to paraphrase Kenneth Arrow. The author further suggests that any concave utility function may be problematic. For example, the exponential utility function "…implies that an investor whose total wealth grows rapidly through time will inexplicably keep a fixed *dollar* amount invested in stocks, rather than a fixed proportion of his/her growing wealth." The shortfall probability efficient frontier, by contrast, better illustrates a fundamental risk/reward tradeoff: "Higher growth portfolios are generally more volatile, and while their higher growth helps them exceed a target, their higher volatilities increase the chance they won't." The author notes that a financial advisor may try to "reverse engineer" a risk aversion parameter based on the investor's reaction to the risk/reward tradeoffs illustrated in a shortfall probability analysis. The reverse-engineered parameter becomes an input to a mean-variance utility analysis. However, such an attempt will probably generate results that are internally inconsistent. A mean-variance investor cares only about the first two moments of a distribution and does not take into account either outperformance or shortfall probabilities relative to a benchmark target. However, when a target is explicitly defined, it is possible to reconcile CRRA utility maximization with a long-term target shortfall probability minimization metric: "when the analyst *completely* maximizes an expected CRRA utility of *the ratio of the portfolio's return to the investor's target*, by jointly choosing *both* the portfolio's asset allocation weights and the CRRA utility's risk aversion parameter, the result is the same recommended long-run asset allocation derived from the target shortfall….probability criterion." Stutzer uses a binomial lattice with up/down probabilities—calibrated	

		from historical data—in order to calculate the probability-adjusted rate of portfolio return over the two states of nature—up and down. His model calculates the ratio of up-to-target return and down-to-target returns separately. He then adds the CRRA utility-adjusted results from each branch to arrive at total utility for any specific asset weighting. The asset allocation recommendations from this utility-based calculation mirror those from a shortfall minimization analysis because the target is built right into the calculation formula. "What determines the risk borne by a shortfall probability minimizing….investor is the specific target the agent wants to beat *and* the investment opportunity set the investor uses to try to beat it." According to the author, the primary difficulty with Paul Samuelson's utility-based argument—i.e., the long-term allocation should be the same as the short-term allocation—is that it lacks a target benchmark. He suggests that Samuelson's argument was directed towards those who use probabilistic arguments to advocate for expected growth rate maximization—an expected log utility criterion.	
2003	Asset Allocation And the Liquidity Premium For Illiquid Annuities S. Browne, M.A. Milevsky & T.S. Salisbury The Journal of Risk and Insurance, vol. 70 no. 3 (2003), pp. 509 – 526.	The authors point out that the flexibility to maintain a constant proportion allocation over time (e.g. periodic rebalance to a% equities or risky assets and 1-a% to bonds or risk-free assets) provides greater expected utility to investors exhibiting constant relative risk aversion (CRRA) than a simple buy and hold approach that eschews any rebalancing. However, in the retirement wealth *accumulation stage*, the surrender and/or market adjustment penalties of a fixed accumulation annuity render such an investment product illiquid. During the *payout stage*, the immediate payout annuity is, by definition, illiquid because it cannot be sold or returned to the insurance carrier for a lump sum. Two schools of through characterize current thinking on the topic of required compensation for holding illiquid assets. One school contends that, in equilibrium, investors should be compensated for liquidity	If all wealth is annuitized, the investor may demand a substantial illiquidity premium. The "mortality premium" may or may not offset the required extra compensation for illiquidity risk.

		restrictions. The other school contends that investors select instruments that match their investment horizons and, for long-term objectives, they can, without any great inconvenience, select illiquid investments without any required increase in yield to compensate for the illiquidity. Most of the academic literature suggests that, all else equal, the markets demand that illiquid investments provide a higher yield to maturity than that provided by liquid investments. From the product manufacturer's point of view, however, liquidity restrictions are necessary to preserve appropriate asset/liability matching. From the investor's point of view, if a sufficiently large pool of money exists outside the annuity framework, these other assets can be used for portfolio rebalancing. Thus, the extra yield required for illiquidity of annuities is a function of the proportion of total wealth committed to annuities. The article presents an investor who selects an allocation of 50% to risky assets and 50% to a risk-free, illiquid annuity. The investor's objective is to maximize the expected utility of wealth. Thus, if a_t^* = ½ = the optimal allocation to the risky asset, 1 - a_t^* = ½ = the optimal allocation to the risk-free annuity, and U_T^* = maximum expected utility at the end of the planning horizon (T). R. Merton demonstrates that an investor with CRRA [$u(w) = w^{(1-\gamma)} / (1-\gamma)$] and faced with an uncertain wealth process characterized by geometric Brownian motion, will select a time-invariant investment policy (asset allocation). A strict buy and hold investment policy is suboptimal in the Merton framework. However, the liquidity restrictions of the fixed annuity impede the investor's ability to maintain a constant proportion portfolio allocation in the face of dynamic market movements. The liquidity restriction is thus equivalent to a forgone opportunity cost. A rational investor will not incur such a cost unless he or she is adequately compensated by an increased annuity yield. The authors define this yield as the amount required to provide the same utility to investors not faced with such a liquidity constraint. They argue	

		that, all else equal, a longer time horizon, lower level of risk aversion, and higher forecasted growth rate from equity markets, imply a larger required liquidity premium. In the continuous time Merton framework, risky assets follow a drift and diffusion process in which their change in value can be modeled as : $$DV_t = \mu V_t dt + \sigma V_t dB_t$$ Where B_t is Brownian motion, μ is forecasted equity return and σ is forecasted volatility. Thus, wealth over time grows according to the following formulation: $$V_T = e^{(\mu-\sigma^2/2)T+\sigma B_T}$$ The fixed annuity, by contrast, assumes a flat, deterministic interest rate of 'r', and the growth of this risk-free asset over time is characterized as follows: $$A_T = e^{rT}$$ The end of the period utility function is the standard power utility function when the risk aversion factor (γ) does not equal 1 and the log of wealth function when γ equals 1. Maximal level of expected utility in a dynamic (i.e., risky asset) environment is modeled as: $$EU^*(r \mid dynamic) = \frac{1}{1-\gamma} e^{\xi(1-\gamma)T}$$ Where, $$\xi = r + \frac{(\mu - r)^2}{2\gamma\sigma^2}$$	

		And, as Merton derived, the optimal commitment to the risky asset is: $$a_t^* = \frac{\mu - r}{\gamma \sigma^2}$$ The expected utility from the static (i.e., non-rebalanced) account is the utility produced by ending wealth which is the linear sum of the returns on money committed to each account (1-a) = risk free account and a = risky asset account: EU*(r $\mid$ static) = EU[(1-a)A_T + aV_T] = EU $\left[(1-a)e^{rT} + ae^{(\mu-\sigma^2/2)T+\sigma B_T}\right]$ Where expected optimal Utility$_{static}$ is equal to or less than expected optimal Utility$_{dynamic}$.	
2003	"Optimal Timing of the Annuity Purchases: A Combined Stochastic Control and Optimal Stopping Problem," Gabriele Stabile International Journal of Theoretical and Applied Finance vol. 9 no. 2 (2003), pp. 151 – 170.	Article seeks to answer the question of the optimal time for annuity purchase by a risk-averse investor. The author explores a "free boundary" problem where investors of differing degrees of risk aversion determine a threshold amount of wealth such that an annuity is purchased when the threshold is reached, or purchased is deferred if wealth is above the threshold. The dynamic programming approach used in the article ignores bequests and assumes a complete market spanned by a risky asset and a riskless bond. The evolution of the investment portfolio is a combination of the bond's fixed interest rate plus a Brownian motion diffusion process. Mortality is modeled via an exponential function. Investment returns are compared to periodic annuity payments which are a function of the current interest rate, survival probability, costs, loads and fees, and the wealth available at time 't' to purchase the annuity contract. The model presents a binary choice—commit all wealth to an annuity or invest all wealth in a stock/bond portfolio. The investor has no wealth or income sources beyond the portfolio. Between the time	This is a "top down" approach to the issue of optimal time to annuitize. It is a part of a school of literature that includes Huang, Milevsky, and Wang's "Ruined Moments in Your Life" and `Vanduffel, Dhaene, Goovaerts and Kass's "The Hurdle-race Problem." This research thread provides justification for rigorously monitoring a portfolio as it approaches a lower-bound threshold amount beyond which it is unable to support minimum cash flow and terminal wealth objectives. If approaching the free boundary from below, the decision rule is to purchase an annuity at the boundary; if approaching from above, defer until the boundary is reached. If the investor does not immediately annuitize at the boundary, the situation might deteriorate further. This solution prevents the risk-of-increasing-risk described by

		of retirement and the annuity purchase, the investor must determine a "consumption rule (c)" and a "portfolio rule (π)" (asset allocation) such that wealth evolves according to: $dx(t) = x(t)[r + \pi(t)(\mu - r)]dt + \sigma\pi(t)x(t)dW(t) - c(t)dt.$ The investor compares the discounted expected utility of constant consumption generated by financial markets to that generated by the actuarial solution. The value equation of current wealth (x) as a function of asset allocation, consumption and time is the integral from time zero (now) to time T (stochastic date of death) of two periods: (1) the retirement period in which the investor participates in the financial markets, and (2) the retirement period in which the investor owns the annuity income stream. The author expresses utility of financial market participation in terms of a subjective interest discounting rate + a mortality rate so that investor preference is a function of age ["impatience" + hazard rate = investor preference discounting = β]. The utility of the actuarial solution is expressed as the annuity payout discounted (divided) by β. Total utility is calculated over c, π, and t and is expressed as the solution to the Hamilton-Jacobi-Bellman dynamic programming equation. At what threshold ("free boundary") is the investor best served by switching from financial market participation to annuitization? The time control factor is the earliest of death or bankruptcy, the consumption control factor is the risk-adjusted first derivative of wealth at time t. The portfolio factor (asset allocation) is the Merton optimum $[\frac{\mu - r}{\sigma^2}]$ times the Arrow Pratt Absolute Utility of Wealth function: first derivative of wealth ÷ negative second derivative of wealth. The dynamic programming solution exhibits a region D for which it is not optimal to annuitize provided that utility U is a subset of D [$U \subset D$]. Given the HARA utility functions the solution to the HJB equation is the time at which the value function exits D. In the case where the investor's risk aversion function shows no	Stutzer [2003]. Although the author's model assumes that annuities will crowd out bonds in the asset allocation, when wealth is above the threshold amount needed to support consumption, the investor continues to hold bonds in the financial markets. The author presents an asset management model for trustees and investors.

		preference for an income derived from the financial markets as opposed to from the annuity, if the financial market returns are greater than the annuity payouts then the investor never annuitizes and the dynamic programming problem becomes the Merton infinite horizon consumption/investment problem where the optimal asset allocation is the risk-adjusted Merton Optimum—best risk-adjusted rate of growth. If, for case one, the value function U lies outside of D [D = ø], then the investor annuitizes all wealth immediately—this is the Yaari solution. In the case where the investor's risk aversion for income derived from financial market investments is higher than for the annuity payout, then—according to the free boundary calculations—if initial wealth is greater than a threshold amount, then the investor continues to participate in the financial markets according to the optimal portfolio and consumption rules. If wealth increases faster than consumption, the investor rebalances towards bonds to preserve asset allocation. However, if financial market returns decreases wealth, the individual rebalances to risky assets and accepts increasing financial risk [CRRA: constant mix allocation]. "When the wealth reaches the threshold $\hat{x}$, the individual has a low rate of consumption but, because of her his aversion, she does not want to increase the exposure to the risky asset. Then, the phase of investment in the financial market ends, and the individual [applies] her wealth in the annuity purchase. If the initial wealth is lower than $\hat{x}$, the individual immediately buys the annuity."	
2003	Jarrod Wilcox, "Harry Markowitz & the Discretionary Wealth Hypothesis," <u>The Journal of Portfolio Management</u> Vol. 29, No. 3 (Spring 2003), pp. 58 – 65. Note: available in	Wilcox's paper synthesizes ideas from Markowitz (mean-variance optimization), John Kelly ('optimal growth theory'), and Wilcox's discretionary wealth management approach to multi-period investing. The article begins by providing background on the history of Kelly's studies in information theory at Bell Labs. Eventually, Kelly's work became popularized by the well-known mathematician Ed Thorpe for use in betting systems, and eventually found its way into finance [A comprehensive and entertaining narrative of the history of the growth-optimal system is *Fortune's Formula: The Untold Story of the Scientific Betting System That Beat The Casinos and Wall Street* by William	Wilcox is a proponent of asset/liability investment management [ALM] for individual investors. His approach explicitly incorporates the personal/household balance sheet through the "discretionary wealth hypothesis." The asset management approach was later expanded to encompass dynamic (multi-period) changes in the investor's financial position through examination of the "time series of implied balance sheets." The present value of current and future liabilities (car

	draft form under the heading "Managing Discretionary Wealth" June 4, 2011 posting on http://jarrodwilcox.com	Poundstone, Hill and Wang (New York) 2005]. Wilcox observes that many academics give the growth-optimal model a cold reception because (1) it fails to incorporate investor utility (preferences and constraints) explicitly; and, (2) the Kelly system's success often depends on making extremely small fractional "bets" over time periods that could easily extend beyond an investor's lifespan. Trading costs and planning horizon concerns may vitiate the benefits of the Kelly system. Under an optimal growth approach: "…the investor maximizes the expected rate of long-run return in an investment process with independent returns by maximizing the expected logarithm of each single-period return multiple." However, although long-term success probability tends towards one, during any time interval wealth may approach an extremely low level. Note: "return multiple" is the return relative. A 5% positive return has a return relative equal to 1.05. A 5% loss has a return relative equal to .95. Since the log of 1.00 is zero, positive results produce positive log values; negative results produce negative log values. The article continues with an explanation of the differences between multiplying (compounding) periodic returns and adding the natural logs of return relatives and applying the inverse 'antilog' function ('e') to the additive result. The first process is used to find the *mean* of a terminal wealth distribution; the second process is used to find the *median* (50th percentile) of a terminal wealth distribution. Given the properties of long-term multiplicative compounding, the mean is often substantially greater than the median simply because multiplicative results are often highly skewed. "Because long-run wealth outcomes are nearly log-normally distributed, average wealth is strongly influenced by low-probability sequences in which unusually high single-period returns are compounded. Consequently, mean wealth will be greater, often much greater, than median, or 50th percentile, wealth from a long-run compounded process. Median wealth is closer to what most outcomes will be." Wilcox continues with a critique of Variance—the second moment of a distribution—as an adequate measure of investment risk. The single-	payments, mortgage expenses, retirement, college expenses, etc.) is subtracted from the present value of current assets to arrive at the investor's discretionary wealth. This is the amount of wealth that the investor could lose without fatally compromising critical current and future objectives (quasi-liabilities). "Discretionary wealth is the amount one could afford to lose without suffering whatever one defines as a shortfall disaster." It is a form of consumer surplus optimization with the constraint that wealth must not drop below the critical level needed to cover expenses and fund critical objectives. Over time, the ratio of [(assets – critical liabilities) ÷ wealth]—i.e., the fraction of wealth that the investor considers discretionary—will likely change. Allocation weight given to risky assets at time 't' becomes a function of the composition of the personal balance sheet at time 't'. Example: Where the ratio of real expected return (.06) to variance (.04 = standard deviation of .20) is 1.5, and the ratio of discretionary wealth to assets is 30%, the optimal allocation to risky assets is D(E/V),or, 45%. Wilcox contends that dynamic allocation based on changes in balance sheet values is more practical than implementing an insured portfolio investment management approach. It is interesting to note that Laurence Booth ["Asset Allocation: The Long View," Chapter 12 in <u>Retirement Income Redesigned: Master Plans for Distribution</u> eds. Harold Evensky and Deena B. Katz, Bloomberg Press (New York, 2006), pp. 203 – 216]

		period Markowitz optimization algorithm ignores the higher moments of terminal wealth distribution: skew and kurtosis. However, Wilcox notes: "There is a deep relationship between the statistical moments that describe a distribution of returns and the successive terms of a Taylor series whose sum is mean log return." Most investors, in Wilcox's opinion, are concerned with downside risk; and, "the incremental information sought by many investors in avoiding "downside risk" or semivariance is captured by the third and fourth terms [in the Taylor Series]." $$\text{Expected } \ln(1+r) \approx \ln(1+E) - \frac{V}{1(1+E)^2} + \frac{SV^{\frac{3}{2}}}{3(1+E)^3} - \frac{KV^2}{4(1+E)^4}$$ Where 'E' is the mean return 'V' is variance of return 'S' is Skew 'K' is Kurtosis By contrast, the Markowitz mean-variance approach considers the two left terms of the series expansion only. If the Markowitz approach pays too little attention to lower probability financial reversals, the infinite-life Kelly system may give downside catastrophes too much weight. Wilcox provides an example: "Suppose the annual probability of an automobile driver fatality per year in the US is about one-one-hundredth of a percent. If a driver were to drive for thousands of years, the median driving outcome would be grim. But since the cumulative probability of a driving fatality over a realistic lifetime is only about 0.5%, and since driving helps us with many other goals, most of us rationally decide to drive." Wilcox adapts both Markowitz's mean-variance optimization algorithm, and the Kelly system--as expressed by a Taylor series expansion for expected log return--to the needs and circumstances of an individual investor faced with a finite planning horizon and the need to maintain a	presents asset allocation argument that focuses on observation similar to those made by Wilcox. Booth provides a clear explanation of the difference between multiperiod arithmetic and geometric returns: "The arithmetic return is approximately the geometric return plus half the variance in the arithmetic rate of return. In the case of equities, the volatility is 20.31 percent; for simplicity, call it 0.20, in which case the variance (square of the standard deviation) is 0.04 and half the variance is 0.02, or 2 percent. The approximation indicates a difference of 2 percent between the arithmetic and geometric returns…." The impact of compounding over time can be significant. The median value is simply the middle (50^{th} percentile) wealth value in the distribution of terminal wealth. The mean, however, is the average value where the average is influenced by the outliers from a skewed distribution. After a number of years "…most of the 'probability mass' is below or to the left of the mean." This means that the probability of earning the expected—i.e., average or mean return—is less than 50-50. Furthermore, the probability of earning a return below the mean increases with time. After a financial planning exercise which calculates the expected return required for an investor to reach a target wealth goal, Booth demonstrates that it is important to understand the distinction between mean and median. Given the lognormal distribution of terminal wealth, however, Booth estimates that the hypothetical investor in his financial planning example has only a 41% likelihood of equaling or exceeding his wealth goal because of variance term's detraction from

		wealth level sufficient to fund critical objectives. "We will assume that risk aversion is caused by the need to avoid shortfalls, not only in some far-off ending period, but all along the way." The discretionary wealth hypothesis is the key. Examination of balance sheet values enables the investor to set the shortfall boundary at "…the zero-point of discretionary wealth…." Risk measurement is shifted away from the statistical characteristics of the risky asset portfolio in favor of assessing the risk to the investor's discretionary wealth: "Classical financial utility theory represents conservative investors as having utility functions that are more strongly curved. What we do here is the alternative, varying the apparent distance between two outcomes by changing the scaling of returns from that on risky assets to that on discretionary wealth. For the utility theorist, we have said that all investors will be better off if they act as thought their utility was given by the log of their discretionary wealth. Every investor is advised to have the same-shaped utility function, of the form log (w-c), where w-c is discretionary wealth." If, for example, an investor has a portfolio consisting of 100% risky assets and a discretionary wealth ratio of 20%, there is an implicit leverage factor of 5. The higher the leverage factor, the greater the investor's sensitivity to both variance and to higher moments of the return distribution. Incorporating the leverage factor [L] into the terms of the Taylor series expansion, directly accounts for the investor's concern with shortfall risk. $\text{Expected } \ln(1+Lr) \approx \ln(1+LE) - \frac{L^2V}{2(1+LE)^2} + \frac{SL^3V^{\frac{3}{2}}}{3(1+LE)^3} - \frac{KL^4V^2}{4(1+LE)^4}$ The interaction of each moment of the distribution with the leverage factor determined by the investor's shortfall constraint (the boundary condition) determines the optimal point on the efficient frontier. It is an easier calculation than trying to determine the curvature of an investor's	expected [mean] wealth. Booth indicates that there are three solution paths: Create a growth optimal portfolio [The Merton optimum] in which the optimal equity asset allocation is the expected risk premium divided by the variance of the equity-risk premium. A Markowitz mean-variance solution in which the variance is multiplied by the investor's risk aversion coefficient in order to reflect the specific investor's risk tolerance—the greater the risk aversion, the smaller the equity allocation. Replace a difficult-to-calculate risk aversion coefficient with a shortfall probability constraint in the objective function. For example, for an investor who wishes to have no more than a 30% risk of failing to meet the return sufficient to fund his goal, the probability target becomes: $N\left(\frac{Tk - TE(r)}{\sigma\sqrt{T}}\right) < 30\%$ Where N is the standard normal distribution, T is the time horizon, k is the target rate of return, E® is the compounded expected return of the portfolio and σ is return volatility. "In the solution with the probability target, the target plays the same role as risk aversion in the mean-variance model. As the investor wants more assurance of meeting his retirement target—for example, with an 80 percent rather than a 70 percent probability of meeting the target—more is subtracted from the risk premium. As a result, the equity allocation is smaller." For the

		utility function. Where is the optimal point? "…the ideal implicit leverage is approximately the ratio of expected real return to variance, of E/V, and thus the proportion of wealth allocated to risky assets should be about D(E/V), where D is the fraction of assets considered discretionary….the best point on the efficient frontier will be obtained if we set the Markowitz risk aversion coefficient equal, not to ½, [i.e., quadratic utility], but to 1/(2D)." Wilcox concludes: "To account for the needs of conservative investors who must avoid shortfalls along the way to the long run, we re-mapped returns on total wealth to amplified returns on discretionary wealth, the wealth available before shortfall. We represent conservatism not by greater curvature of an abstract utility function, but by greater distance between investment outcomes based on measuring returns against fractions of wealth considered discretionary. For example, a 10% loss for an investor who can lose no more than 20% of total assets without shortfall is represented as a 50% loss."	70% likelihood of success probability target [a z-score of -0.5244], the optimal equity allocation is: $$\alpha = \frac{(Equity\ risk\ premium) - 0.5244\sigma T^{-0.5}}{\sigma^2}$$ Booth's shortfall probability constraint—a variation on the shortfall avoidance risk control literature—plays a role similar to Wilcox's leverage on discretionary wealth in determining an appropriate asset allocation. The two authors seem not to be aware of each other.
2003	"Portfolio choice with endogenous utility: a large deviations approach," Michael Stutzer Journal of Econometrics Vol. 116, (2003), pp. 365 – 386.	The author reviews several portfolio preferencing criteria including the Kelly "growth-optimal" investment strategy and the Sharpe ratio maximization strategy. He proposes that investors seeking to achieve the maximum probability of avoiding a shortfall in target return / target wealth should construct a portfolio based on minimizing the likelihood that the selected portfolio fails to exceed the target growth rate. The article begins by reviewing the rationale for the log-optimal growth rate portfolio. Although a log-optimal portfolio's rate of return over an infinite planning horizon will almost surely lead to the highest growth of capital, the investor is likely to suffer significant periods of time in which the rate of return falls far short of the investor's goals. Growth-optimal strategies load for risky assets; and, investors with finite time horizons risk the loss of a substantial portion of their wealth. Stutzer also notes: "The problem is exacerbated when investors have specific, short to medium term values for their respective investment horizons." He cites Samuelson's well-known argument for time-invariant	If an investor's target rate of growth in wealth is the risk-free rate of return, the investor seeking the highest probability of avoiding shortfall risk invests all financial wealth in the risk-free asset. This is similar to Roy's safety first criterion. This is an investor exhibiting high risk aversion. If the argument of the utility function changes from wealth growth to consumption, this may be an argument for tilting the portfolio towards annuities. See, for example, "Age-dependent investing: Optimal funding and investment strategies in defined contribution pension plans when members are rational life cycle financial planners," David Blake, Douglas Wright and Yumeng Zhang, [2014]. Stutzer acknowledges that the shortfall from target probability calculation is subject to criticism in that

		asset allocations; but reminds the reader that Samuelson's argument is valid only under the assumption that asset returns are IID, and that Samuelson himself cites six modifications of his stay-the-course allocation advice which can result in horizon-dependent allocations. Although Stutzer does not refer to the "hurdle race" problem, the introduction of short-horizon goals adds to the complexity of the analysis. Likewise, the problem of horizon-dependent portfolio asset allocation is also complicated whenever the horizon is unknown: "The advisor may be unable to determine an investor's exact horizon length when it exists, while other investors may not have a specific investment horizon length at all." This is, of course, the situation for most retirement income portfolios tasked with providing an adequate lifetime cash flow. Irrespective of the horizon, the investor's risk aversion parameter greatly influences the preferred asset allocation, and it is important to develop models with utility functions exhibiting an appropriate form and argument. Stutzer develops a model in which the investor seeks to minimize the probability of falling short of the target rate of return at any given planning horizon: By choosing a portfolio that results in a higher expected growth rate of wealth than the target rate, the investor insures that the probability of not exceeding the target growth rate decays to zero asymptotically, as the time horizon $T \rightarrow \infty$ provided that certain feasibility conditions are met. But the probability that the realized growth rate of wealth at a finite time 't' will not exceed the target might vary from period to period and portfolio to portfolio. Which portfolio should be chosen? Without adopting a specific value of t, a sensible strategy is to choose a portfolio that makes this probability decay to zero as fast as possible as $T \rightarrow \infty$. Stutzer uses the Gartner-Ellis Theorem from the field of large deviations theory as the mathematical basis for calculating the decay rate maximizing portfolio. In addition to the investor's risk aversion parameter and the investor's opinion regarding the investment opportunity set which with which he is	it ignores the magnitude of the shortfall conditional on its occurrence. He suggests that adding an additional criterion based on expected shortfall may resolve some elements of this criticism. However, a portfolio designed to minimize the probability of a loss may differ considerably from a portfolio designed to limit the magnitude of a loss should a loss occur. There may be a considerable difference between a portfolio designed to minimize the "decay rate" vs. a portfolio designed to minimize the dollar value of the loss. See, also, Sid Browne, "The Risk and Rewards of Minimizing Shortfall Probability," [1999].

faced, the form and argument of the utility function determines the preferred asset allocation. Stutzer's utility-based model differs from the standard expected power utility model in several important respects:

1. The argument of the function is a ratio—the ratio of the realized growth of invested wealth to a benchmark level of wealth that assumes growth at the target rate of return.
2. The value of the risk aversion parameter reflects both the investor's target rate, and the rate at which the selected portfolio's probability of failing to achieve the target decays the quickest. This means that risk aversion is endogenous and that it can reflect, unlike Sharpe ratio or log-normal growth optimal portfolio, preferences for portfolios exhibiting positive skew.

Thus, Stutzer's utility function leads to the investor to prefer a decay rate maximization portfolio. Such a portfolio nests the expected growth optimal criterion. However, the investor's choice of target implicitly determines the value of the risk aversion parameter—a high growth target implies low risk aversion. This differs from standard models that assume a fixed risk aversion parameter and then test portfolios to determine which allocations produce the highest expected utility at each value of the risk aversion coefficient. "If the investor's target growth rate is lower, the investor uses a higher degree of risk aversion, and the associated decay rate minimizing portfolio is more conservative, with a lower expected growth rate, but a higher decay rate for the probability of underperforming that target growth rate (and hence a higher probability of realizing a growth rate of wealth in excess of that target)." The author notes that this observation characterizes the tradeoff between seeking growth optimal portfolios generating the highest expected feasible returns and shortfall risk exposure.

In the simplified case of IID normal returns, Stutzer uses a Z-score analysis to determine the probability of shortfall for a given asset allocation relative to its target—the expected mean of the distribution.

		After pointing out that the growth of compound wealth function decrements the mean by subtracting one-half of variance, he notes that the growth-optimal portfolio maximizes $E[\log R_p]$. The Sharpe ratio maximizing portfolio maximizes: $$\frac{E[R_p] - i}{\sqrt{VarR_p}}$$ However, the outperformance probability maximizing portfolio, by incorporating a benchmark constant return target in the numerator and a variance of the log return in the denominator, has the following expression: $$\frac{E[logR_p - logr]}{\sqrt{Var[\log R_p}}$$ Thus, for any given moment of a distribution, the optimal allocation may differ significantly depending on the form of the utility function under consideration. Stutzer illustrates this by plugging in various fixed distributional parameters to determine the likelihood of exceeding a target rate of return after T years. The article's accompanying graphs demonstrate that although the growth optimal portfolios have the highest long-term expected returns, unless the target is at the highest feasible return level they also generally present the greatest risk of falling short of the target. Their high variance makes it likely that they will experience a sequence of poor returns that will substantially decrement wealth within any finite horizon. Stutzer also generalizes his arguments to the case in which asset returns are independent but not necessarily identically distributed—i.e., not a symmetric bell-curve distribution. In this case, the auto-covariance of returns enters the denominator as an important term in calculating the decay rate optimal portfolio.	

2003

“Annuity Risk: Volatility and Inflation Exposure in Payments from Immediate Life Annuities,” Chris Soares and Mark Warshawsky Research Paper No. 2003-01 (January 2003).

The authors examine the range of payouts offered by immediate annuities during the period 1983 through 2002. They calculate the average monthly payout per $100,000 annuity purchase, and they measure variance by the standard deviation of annualized monthly payments. The authors assert” immediate annuities “…are costly for many potential annuitants, and they are risky.” The risk comes whenever current interest rates are low—price of the annuity contract may be too high; or, whenever future inflation undercuts the purchasing power of the nominal payment.

The study considers increasing annuities—payments increase by a fixed percentage set at the time of contract purchase, and inflation-adjusted annuities that link the payments to the rate of realized inflation. Although these contracts mitigate purchasing power risks, they offer substantially lower initial payments when compared to nominal annuity payouts. The authors simulate contract terms over the period under evaluation (1983-2002) using Social Security cohort mortality tables. The study evaluates life-only immediate annuity contracts issued to a male age 65, a female age 65 and for a joint and survivor annuity issued to a 65-year old couple. Additionally, they compare monthly income streams and the variability thereof from all-at-once purchases to a phased purchase over a three-year period. They summarize their findings as follows:

	Nominal Annuity	Increasing Annuity	Phased Purchase	Inflatio Adjusted A
Average	$742	$526	$736	$492
Standard Deviation	$131	$72	$104	$21
Range	$535 - $1,148	$406 - $779	$578 - $1,014	$437 - $
Max. One-Year Difference	-28% / +21%	-33% / +26%	-13% / +5%	-10% / +

The inflation-adjusted data spans the period April 1998 to December 2002. They authors note that annuity risk can be mitigated through a phased purchase strategy.

Annuity risk is the risk that the payments generated by an annuity contract will cease to become attractive in future economies especially if the inflation rate increases. One component of annuity risk is “regret” which comes about whenever annuity prices fall after acquisition of the contract.

This risk has been quantified under the heading “term structure of annuities” [see, for example, “The Role of Government in Life-Cycle Saving and Investing” Alicia H. Munnell (2008)]. Further development of this theme is found in Ralph S.J. Koijen, Theo E. Nijman and Bas J.M. Werker [2009].

Discussions of annuity timing risk are also found in “Albrecht & Maurer, [2002]; as well as in studies by Milevsky [1998], Mitchell, Poterba, and Warshawsky [1999] and Milevsky [2000].

		Finally, they note: "there is no volatility in the initial payments from immediate variable annuities, as the initial payment is based on a fixed assumed interest rate, usually four percent. The volatility from such annuities occurs after the initial payment, when the underlying asset portfolio changes value, and indeed that is the advantage of fixed annuities, whether minimal or inflation-adjusted—they offer more predictability in the stream of income received during retirement."	
2004	"The Strengths and Weaknesses of Various Financial Simulation Models," Joseph H. Davis, Nelson W. Wicas and Francis M. Kinniry Journal of Wealth Management vol. 6 no. 4 (Spring 2004), pp. 33 – 42.	The authors divide simulation approaches into two general categories: (1) historical simulation which assumes the future events will repeat the chronological patterns evidenced in past events, and (2) Monte Carlo simulations which assumes that future events will occur according to an hypothesized likelihood function. The approaches can lead to significantly different assessments of investment risk. Historical simulation has three assumptions: 1. The future mean return will approach the past mean return of the sampling period; 2. Future investment conditions will approximate past investment environments; and, 3. The pattern of future investments (i.e., sequence of returns) will approximate past return patterns. Thus, historical simulation is also termed "Time-Path Analysis." The two most common types of time-path analysis are rolling period and looping period. Although both methods begin at a fixed starting date, the rolling period method produces only a limited set of returns because it stops at the last data point. The looping method returns to the initial data point and reapplies the initial sequence of returns. The rolling time-path method does not re-use historical returns; the looping time-path method creates a greater number of scenarios because it can re-use historical data. The other historical approach is known as "bootstrapping." Bootstrapping samples historical data randomly with replacement. This produces a greater number of possible scenarios from the historical	If prudent investment decisions are probabilistic in nature, then an investor must have credible risk assessments. This is a helpful review of the strengths and weaknesses of some commonly used retirement risk models. Note: it is instructive to read this article with Elton McGoun's 1995 article: "The History of Risk Measurement."

		data. Unfortunately, each method has serious statistical flaws. For example, the rolling period approach over-weights the returns from the middle of the data sequence and under-weights the returns from the beginning and ending periods. The bootstrapping method destroys asset class dependency relationships. This includes both serial correlation within an asset class and cross correlations between multiple asset classes. The most serious statistical issue with historical simulation approaches lies in the fact that all future results are based on historical results—the bootstrap, rolling period and looping period approaches use the same set of realized returns. The distribution of simulated results is not based on independent samples and, as a result, understates the potential risks inherent in investment positions: "…"historically based simulations reflect merely the range of observed outcomes, rather than the range of possible outcomes." Although Monte Carlo simulation is also based on historical results, the range of outcomes is not limited by empirical return realizations. Rather, an a-priori, hypothesized probability distribution (the parameters of which are calibrated to reflect the time series of realized returns) determines the likelihood that any given return appears in the simulation process. Most hypothesized distributions are normal (bell curved). Unfortunately, the normal distribution assumes (1) time-invariant mean and standard deviation parameters, (2) independence of period-to-period returns, and (3) symmetrical distribution of returns around the mean. Forming a co-variance matrix under these assumptions results in a single, static value for correlation among asset classes. The greatest advantage to Monte Carlo simulation using a normal return distribution is that it generates a far greater range of probable range of returns. That is to say, the tails of a Monte Carlo simulation have greater length than the tails of a historical simulation. However, Monte Carlo simulations usually do not allow for time variations in risk premiums, variances, and correlation values. Its bell-shaped distribution distorts the actual risk of	

		an investment position. The authors suggest that a vector autoregression [VAR] approach mitigates the above-listed problems. The approach demands that the model user identify the variables that are key drivers of asset price changes. When the variables include macroeconomic factors, the model assumes a linear relationship between the macroeconomic variables. Additionally, a VAR model also posits a linear relationship between an asset's expected return and volatility and the past values of these parameters. Basically, a VAR approach is a-theoretical in that it the model's outputs emerge as a result of linear statistical relationships rather than as a result of the structure of an imposed distribution. Simulation paths can exhibit higher moments (skew & kurtosis). The lagged relationship between past and expected return and volatility create path dependencies for price evolutions and, therefore, produces a less distorted assessment of investment risk.	
2004	"Can the Private Annuity Market Provide Secure Retirement Income?" G.A. Mackenzie and Allison Schrager IMF Working Paper WP/04/230 (December, 2004).	Although annuities can provide longevity insurance, their premiums vary as a function of the yield curve's level and slope. This variance creates risk for retirees: "The DC plan participant cannot be certain before he annuitizes what premium he would have to pay for a given stream of annuity income." The authors investigate this issue by simulating the distribution of bond yields to estimate the distribution of annuity premiums. Further, they assess the strategy of staggered annuity purchases in order to see if an investor seeking to implement an annuity portfolio can significantly decrease cost variance. The study focuses on a life-only, single-premium fixed nominal annuity contract. No loads, taxes or fees are considered. The analysis begins with the standard actuarial formula for calculating the cost [PPD = Premium Per Dollar] of purchasing $1 of lifetime income, assuming an actuarially fair contract:	Investors face uncertain future annuity prices. Life insurers issuing annuity contracts are subject to systematic mortality risk (uncertainty regarding population life expectancy) and investment risk (the risk of funding annuities with fixed-income securities). The cost of an annuity contract is, in part, a function of the level and slope of the yield curve. Investors face risk because, over time, the premium per dollar of annuity benefit can vary significantly. The authors estimate that adverse selection adds approximately 8% to the cost of an annuity in the U.S. market. The article also explores topics of annuity portfolio diversification / dollar-cost averaging.

$$PPD = \sum_{i=1}^{T} \frac{P_i}{(1 + r_{0,1})} i$$

Where estimates of the probability of survival are based on the Social Security general population life table (65-year old men and women as of 2003). The maximum planning horizon [T] is 30 years.

The next step is to estimate the implicit forward one-year interest rates from term structure data. The mean and covariance matrix of these estimates are entered as parameters into a simulation model assuming a multivariate normal distribution of the mean of each one-year forward rate and the corresponding element in the covariance matrix. Each vector of simulated one-year forward rates is used to construct one observation for the term structure of interest rates.

A resulting table exhibits estimated costs considerably less that what major annuity carriers offer in the marketplace. This is due, in part, to the fact that the authors' calculations are for actuarially fair contracts. For a 65 year-old man, the estimated mean cost for one dollar of annual lifetime annuity income is \$8.99 with a standard deviation of \$0.68. However, the distribution of costs ranges from \$12.37 on the high end (90th percentile equals \$9.89) to \$6.93 on the low end (10th percentile equals \$8.15). Substituting the annuitant population mortality table for the Social Security general population table illustrates the costs of adverse selection risk faced by insurers. The mean age 65 annuity cost rises to \$9.74 with a standard deviation of \$0.78.

The analysis next turns to an assessment of the strategy of making staggered annuity purchases over time. The model makes several simplifying assumptions: (1) annuity contracts begin payments at the same time; and (2) bond yields are uncorrelated over time. By making staggered purchases, the investor is likely to significantly decrease cost variance: "Even if the annuitant buys just two annuities, the standard deviation of the premium drops by nearly 30 percent, while the expected

		value does not change. The more annuities the retiree buys, the smaller the variance, which converges to zero as the portfolio grows." However, the authors caution that a strategy of "annuity diversification" may not be optimal if the "fixed cost element" in an annuity purchase makes the acquisition of many contracts prohibitively expensive.	
2004	"The New Retirement Challenge," Jeffrey R. Brown White Paper for Americans for a Secure Retirement (September, 2004). Available at www.paycheckforlife.org.	Written at a basic level, Brown defines the retirement challenge. The challenge is two-fold: (1) accumulate a nest egg, and (2) convert it to a sustainable income for life. "…being a careful saver is not sufficient to ensure retirement security. One must also be a careful "dis-saver," meaning one must be a careful manager and spender of one's savings." Providing adequate lifetime income is the new retirement challenge: "On the one hand, if a retiree spends her nest egg too quickly, and then goes on to live longer than anticipated, she may find that she has too few resources to maintain the standard of living to which she had become accustomed. In the extreme, she may 'outlive her resources'…. On the other hand, if she goes the conservative route and tries to stretch out her resources to be certain that she will have some money left even if she lives to age 100, then she is forced to scale back her standard of living throughout retirement." A solution to the retirement challenge is an annuity: "The extra rate of return that a life annuity can pay is sometimes called a 'mortality premium' because it is essentially an extra rate of return that annuitants can earn in return for giving up their claim on their assets at death."	Concept of "mortality premium" offered to annuitants parallels that of "equity premium" offered to investors.
2004	"Retirement Income Solutions: Payout Annuities," June 14-15, 2004 Spring Meeting of the Society of Actuaries [Record, Volume 30, No. 2] Susan J. Sell, Steve P.	Susan J. Sell is a consulting actuary with Milliman. She makes the following observations based on a survey of insurance companies selling Single Premium Immediate Annuities: • "On average, for the longer payout options, the compensation is about 3 percent to 5 percent. For the shorter payout options, it's 1.5 percent to 3 percent…. The lowest average was in the wirehouse channel at 3.3 percent, and the highest was in the independent producer channel at 4.3 percent." • "Expenses are all over the place….On average this is about 80	Provides a helpful decomposition of annuity pricing factors.

	Cooperstein & Joel Jessen (panelists), pp. 1 – 20.	basis points on the premium side and $235 per contract. Similarly, on the maintenance-expense side, we typically see a per-policy and basis points of assets. Averages were $55 per contract there and 12 basis points." • For common target surplus assumptions, by far the majority of them hold a percent of statutory reserves that's about 4.3 percent. Some incorporate a premium component, and that averages 3.8 percent of statutory reserves, plus 4.84 percent of premium. On average, these levels represent about 250 percent of NAIC risk-based capital (RBC)." • "For pricing targets, we did not ask them what actual profitability they were realizing, but by far again, the majority use statutory internal rate of return (IRR) as the pricing measure. It averages about 12 percent. The second most common measure is GAAP return on equity (ROE), and it has similar ranges and averages as the statutory IRR." • "We asked our survey participants to report what their average asset mix was for their immediate annuities, and about 70 percent of the assets were in investment-grade corporate and commercial mortgages." • "Similar to expenses, the required interest spreads are all over the place. Because you have so many different product designs, some contracts don't have any loads; some have policy fees, annual loads and upfront percent of premium loads, so it's difficult to generalize spreads. They ranged in our survey from 50 basis points to 320 basis points….On average we saw a spread of about 118 basis points for a five-year period certain and 89 basis points for a single life option."	
2004	Jarrod Wilcox, "Risk management: Survival of the fittest," <u>Journal of Asset Management</u> Vol. 5, No. 1 (June	The title of the paper is explained by the fact that Wilcox, after presenting his views on risk management and the "discretionary wealth hypothesis," analyzes several well-known investment failures including Enron and Long-Term Capital Management. Wilcox begins by reviewing the mathematics of coin flipping to illustrate concepts like variance drain (risk decreases multiperiod return), and the	For a discussion of the concept of "discretionary wealth hypothesis, see: Jarrod Wilcox, "Harry Markowitz & the Discretionary Wealth Hypothesis," <u>The Journal of Portfolio Management</u> Vol. 29, No. 3 (Spring 2003), pp. 58 – 65. Note: available in draft form under the heading "Managing Discretionary

	2004), pp. 13 – 24.	impact of compounding a return series additively by summing the logarithms of the periodic return relatives. He states: "…consider what happens if for each period a fair coin is flipped to determine the outcome, with 'heads' giving a return of 100 per cent and 'tails' a return of -50 per cent. The median result is still an equal number of heads and tails, producing a 0 per cent return at the end of 10 periods. The expected wealth is far higher—it is 1.25^{10}, or over \$9, the same as if a constant return of 25 per cent per period were compounded." [Note: this is true only in the event of flipping exactly 5 heads and 5 tails. As the number of trials increases, the likelihood—i.e., expectation—of flipping exactly 50% heads and 50% tails decreases markedly. For example, the probability of flipping exactly 500 heads in 1,000 coin tosses is miniscule. This observation, however, does not invalidate Wilcox's point]. "Holding expected single-period return constant, one sees that the addition of riskiness in these returns has no effect on mean wealth achieved, but an enormous effect on median wealth achieved. In this case, the ending wealth distribution is so skewed that only a small proportion (less than 20 per cent) of the possible sequences of ten coin flips will produce wealth as great as the mean. The tiny percentage of outcomes with nine or ten heads, achieving wealth of \$256 and \$1024, disproportionately affect the averages, but has little relevance to the typical result." In any symmetric distribution—not just the normal distribution—maximizing the sum of log returns maximizes median wealth. Moving from coin flips to stock portfolios, Wilcox notes that higher moments of skew and kurtosis come into play. For such distributions, the expected log returns can be approximated by an expansion of a Taylor polynomial series up to the order capturing the fourth moment where the expansion of ln(1+r) is centered about its mean value: Expected ln(1+r) = $\ln(1+E) - \frac{V}{2(1+E)^2} + \frac{SV^{3/2}}{3(1+E)^3} - \frac{KV^2}{4(1+E)^4} + \cdots$	Wealth" June 4, 2011 posting on http://jarrodwilcox.com

Given the fact that Variance increases in a normal distribution by the square root of time, the greater the frequency of investment decision making, the less the impact of variance and the higher moments on median terminal results.

How does an investor determine how much wealth to allocate to the stock portfolio and how much to allocate to the riskless investment (cash)? This depends on another term 'L' which Wilcox designates as "leverage." Owning a portfolio with a leverage value greater than one suggests that the investor finds it optimal to purchase stocks on margin. Leverage less than one means it is optimal to have a positive weighting in cash. The decision is made by transforming the expected log(1+r) equation by inserting a term for leverage:

$$\text{Expected ln(1+r)} = \ln(1+LE) - \frac{L^2V}{2(1+LE)^2} + \frac{SL^3V^{3/2}}{3(1+LE)^3} - \frac{KL^4V^2}{4(1+LE)^4} + \cdots$$

Evaluating the above equation, Wilcox asserts:

1. There can be too much leverage: "…there is an optimum leverage beyond which expected log return, and therefore median ending wealth, will fall off sharply. Expected log return may be negative even if single period average percentage return is positive. Negative expected log return implies eventual failure with certainty."
2. If the ratio of Expected return to Variance [E/V] is less than one, optimal leverage is also less than one: "…absent portfolio insurance, one must hold cash reserves."
3. Leverage impacts each term in the Taylor series differently: "Lower leverage whenever there is a potential for negative skew or fat-tailed returns is very important in avoiding long-run catastrophe. High leverage in their presence is a red flag…."

Although employing a leverage factor greater than one "…allows for the possibility of catastrophic events," maximizing the value of the first four

		terms of the Taylor series makes such a possibility extremely small over the lifespan of a typical investor. However, over multiple lifespans (intergenerational wealth management) the probability of a catastrophic event increases towards one. How does an individual determine the optimal asset allocation? The first step is to calculate 'D' which, according to the discretionary wealth hypothesis, represents the ratio of discretionary wealth to total wealth where discretionary wealth is "wealth in excess of an amount that would trigger an intermediate shortfall and stop the sequence." It is akin to the floor value in a floor + multiplier insured portfolio asset management approach. In other words, D is the fraction of wealth that the investor can afford to lose without jeopardizing critical financial objectives and obligations. Instead of expressing risk aversion in terms of a derivative of a utility function, Wilcox assumes that the function's curvature is comparable to ln(w-c) where 'w' is wealth and 'w-c' is discretionary wealth [D]. For example, if discretionary wealth is one-half of total wealth, the leverage factor [L] is 2. A portfolio return of -10% is equivalent to a return of -20% on discretionary wealth [the surplus wealth in a pension ALM context]. Given mean, variance, skew, and kurtosis values for a portfolio allocated x% to stocks and y% to cash, a program such as Excel Solver can be used to calculate the optimal L. Optimal L is then multiplied by D to determine the weight of stock in the investor's portfolio. For example, if L equals 1.22 (the optimal leverage for a portfolio with an expected real return of 6% and a standard deviation of 20%), the optimal allocation to stock is 1.22 x D. A discretionary wealth ratio of 50% suggests that the best allocation to stock is 1.22 x 0.5 = 61%. Given this example, a shorthand calculation is simply a log-return adjustment of the Markowitz mean-variance optimization of E –V/2 to E – V/2D. Distributions that are markedly skewed or fat-tailed require additional adjustments for effective risk control. [Although the value of liabilities relative to assets constantly changes, Wilcox does not consider trading costs in his analysis]. The prudence of any particular portfolio management approach follows	

		from the values for L and D. Given a bear market, "…if one has low implicit leverage, it may make sense to let … losses ride or even to buy more when the market has dropped substantially…. Investors with high implicit leverage should be natural buyers of positive skew through portfolio insurance, while those with deeper pockets [higher D] should be natural sellers of portfolio insurance." Prudent asset management elections are dynamic and, in the event of a severe bear market, they assume particular importance: "…an initial loss leading to high implicit leverage can accentuate the later dangers of negative skew and fat-tailed returns, leading to a death spiral if risk positions are not promptly pruned." The lesson of Long Term Capital Management, for example, for a highly leveraged investor is "…that a loss should be followed by a reduction in the aggressiveness of the portfolio. That is, the risky assets should be reduced faster than their decline in market value. This would keep leverage on the reduced discretionary wealth from increasing."	
2004	"How Regimes Affect Asset Allocation," Andrew Ang & Geert Bekaert Financial Analysts Journal vol. 60 no. 2 (march/April 2004) pp. 86 – 99.	Analysis of international equity returns indicate that there are two regimes: Normal and Bear Market. The correlation between returns is significantly higher in the Bear Market regime. A regime-switching model captures the asymmetric correlations. The authors speculate that the economic mechanism behind regime shifts is the world business cycle. Global recessions create larger uncertainty and lower returns whenever a recession is anticipated. A regime switching model follows a Markov transition process with a probability [P] of remaining in a given state conditional on being in that state currently, and a probability [1-P] of transitioning to the other regime. This model allows the equity risk premium and return volatility to vary over time. The model also produces simulated results that exhibit higher order moments. The authors continue the article by developing a market-timing analysis based on the model.	Helpful example of development and use of a regime-switching risk model. Provides data and justification for a two-state model.
2004	"Optimal Investment Choices Post	The issue under consideration is the optimal allocation of a post-retirement investment fund until such time that the fund owner wishes	The model points out that the optimal time to annuitize is a function of both a maximum wealth

	Retirement in a Defined Contribution Pension Scheme," Gerrard, Russell, Haberman, Steven and Vigna, Elena Insurance: Mathematics and Economics vol. 35 no. 2 (October 2004), pp. 321 – 342.	to purchase an annuity, given that the fund's future value is sufficient to provide a certain target annuity income. The authors create a two asset class model (risky asset + risk free bond), assume that risky asset returns are lognormally distributed, and use the standard Brownian motion stochastic differential equation—as derived by Merton—to describe the evolution of the investment fund in a continuous time framework. The paper introduces a quadratic Loss (disutility) function: $L(t, X(t),y(T)) = L(t, X(t)) = (F(t) – X(t))^2 + \alpha(F(t) – X(t))$ Where F is the value of the fund required to buy a future annuity, y is the allocation to the risky asset, X is the value of the investment fund, and α is a measure of the investor's risk aversion. The higher the value of α, the lower the investor's risk aversion—the lower the risk aversion, the higher the target income the investor will pursue. This loss function assigns a penalty for deviations from the target when the target fund is underfunded. Finally, the authors note: "the targets are time dependent because as time passes the individual becomes older and the future life expectancy deceases: hence, the value of the annuity that would be purchased at the interruption of the income drawdown option decreases, ceteris paribus." A similar loss function is introduced for bequests (K) where the penalty cost is weighted by a constant. The retiree will defer purchasing an annuity in the hopes that an investment strategy produces a future sum of money sufficient to acquire a larger target income. This is a combination stochastic control and boundary problem: "…if the size of the fund allows the purchase of the high pension … the individual should stop investing the fund and lock it into an annuity. Therefore the existence of a finite maximum bound for the fund process would be realistic." Additionally, "a minimum limit would be intended to protect the retiree from outliving his/her assets and not being able to buy a minimum level pension at time T." However, adding finite bounds to the problem makes it difficult to solve	bound as well as a minimum bound—high wealth enables the investor to purchase a substantial lifetime income stream and, therefore, increases the demand for annuities; low wealth protects the floor income and also increases the demand to hold annuities. The model suggests a "load for equities" financial management approach.

analytically. Therefore, the authors' model assumes an unbounded process in which the individual will use the income drawdown option until time T—the UK mandated pension annuity purchase age—irrespective of the size of the fund.

The objective is to minimize the cost of the disutility of income plus bequest functions at the time of mandated pension annuity purchase. The control variable is the allocation to the risky asset at time t ($t < T$) conditional on the state x at time t.

Solving this problem involves application of the HJB equation. The authors arrive at a solution for both a constant income target and an exponentially increasing income target. Given the model's assumptions regarding quadratic risk aversion, it is not surprising that the optimal solution is the Merton optimum adjusted for the investor's risk aversion with respect to the income and bequest goals. As T goes to infinity, the optimal allocation to the risky asset is linear in the size of the fund. At each time period t, a simulation model calculates the optimal allocation to the risky asset which is then adopted in the growth of the fund for the next period. The probability of reaching the target is the probability that the target amount is reached over 1,000 simulated trials. The results suggest that the optimal allocation to the risky asset decreases as t approaches T. Additionally, the optimal risky-asset allocation increases as risk aversion decreases.

Simulation results are summarized in the following table:

Risk Aversion Coefficient [0 = low, 100 = high]	0	50	
Probability of ruin ($0 prior to mandated annuitization)	5.20%	9.10%	
Probability final annuity < target annuity	63.10%	28.40%	
Probability final annuity < initial potential annuity	14.00%	11.20%	
Probability of hitting target before age 75	40.70%	75.10%	
Average ruin time ıf ruin occurs	8.8 yrs.	7.9 yrs.	
Final annuity: 5th percentile (per $10 initial amount)	3.809	2.966	
Final annuity: 25th percentile	8.993	10.757	
Final annuity: 50th percentile	10.894	13.606	
Final annuity: 75th percentile	11.676	14.822	
Final annuity: 95th percentile	12.457	15.873	
Final annuity: mean	9.824	11.756	

		Final annuity: standard deviation | 5.357 | 6.684 | | 8.118 Assumptions are retirement at age 60, annuitize at 75, expected age of death is 85, risk-free rate = 5%, expected return on risky asset = 10%, standard deviation = 20%, Italian mortality data, target is exponential with annuity income equal to 1.5 times initial annuity potential. The authors conclude: “it is a remarkable fact that the probability of failing the target at retirement dramatically decreases when the risk aversion decreases (i.e., when riskier strategies are adopted….).”	
2004	“Ruined Moments in Your Life: How Good Are the Approximations?” H. Huang, M. Milevsky and J. Wang Insurance: Mathematics and Economics vol. 34 no. 3 (2004), pp. 421 – 447.	This paper provides a way to estimate the probability of ruin—the probability that “…a fixed retirement consumption strategy will lead to financial insolvency under stochastic investment returns and lifetime distribution.” The paper develops a partial differential equation [PDE] to estimate the probability of ruin. The authors conclude that a 65 year old male retiree requires 30 times the desired annual real consumption to generate a 95% probability of sustainability. The conclusion assumes that the portfolio is well diversified and is invested to earn a real (arithmetic mean) annual return of 7% with a standard deviation of 20%. The authors state: “…the 30-to-1 margin of safety can be contrasted with the relevant annuity factor for an inflation-linked income which would generate a *zero* probability of life-time ruin.” A second goal of the paper is to compare the PDE values with other approximations in order to test the robustness of the PDE ruin probabilities. Among the approximations evaluated are the Reciprocal Gamma and the Log Normal which provide accurate fit at annual volatility levels less than 30%. The wealth process which withdraws 1 unit of consumption in each period follows the well-known stochastic differential equation: $dW_t = (\mu Wt – 1)d_t + \sigma W_t B_t$, with $W_0 = w$	The article calculates the likelihood of ruin for various initial wealth to planned consumption ratios. For example, at a 5% probability bound, a 65 year old male requires a 30 to 1 wealth-to-consumption ratio. This means that (real) consumption should not be higher than 3.33% of initial capital. Market Value of Wealth ≥ PV Liabilities is the feasibility condition. Note: the 30:1 ratio is developed as an approximate solution for a 65-year-old male under the assumption of fixed consumption and a stochastic investment process. However, fixed consumption may not maximize utility—especially if the retiree exhibits a high degree of “impatience” with respect to consumption. Shortfall probabilities are calculated via a partial differential equation rather than via historical back testing, bootstrapping, or simulation.

		If µWt becomes small relative to the withdrawal of 1 unit of consumption, the wealth process may eventually hit zero despite the fact that "the classical geometric Brownian motion" is bounded away from zero in finite time. The paper is interested in the probability of the wealth process hitting zero before the investor's stochastic date of death. The equation for the net wealth process [investment growth – (real) consumption] is expressed as: $W_t = e^{(\mu-1/2\vartheta^2)t+\vartheta B_t}\left[w - \int_0^t e^{-(\mu-1/2\vartheta^2)s - \vartheta B_s}\,ds\right]$ This expression applies a compound growth factor against initial wealth less an integral of indeterminate sign. The exponential function of indeterminate sign is a proxy for the stochastic present value of lifetime consumption. When a level of wealth 'y' lower than initial wealth 'w' is specified, and 'y' is defined as equal to zero, the net worth process will not recover. Once wealth is lost, no amount of investment return can avoid ruin. Thus, when $\int_0^t e^{-(\mu-1/2\vartheta^2)s - \vartheta B_s}\,ds$ is equal to 'w,' the term in brackets goes to zero and stays there permanently. The authors define a random variable R_y^w which is the future ruin time—the "amount of time it takes for the net-wealth process W_t to 'die'—which is to hit the value of 'y'—assuming it starts at an initial value of W_t = w. Where λ is defined as the hazard rate function with Gompertz-Makeham parameters "m" (mode) and 'b' (scale), the probability of lifetime ruin can be expressed as follows: Pr [$R_y^w \leq T_x$] = Probability that the wealth process will ever hit the minimum (floor) value of 'y' or P(w,y,x,given λ,m,b,µ,σ). The probability that the wealth process hits 'y' is summed over the integral and weighted by the probability that the investor survives to	

		time t. Thus the expression $\int_0^T e^{-(\mu-1/2\vartheta^2)s - \vartheta B_s}\, ds$ must remain greater than 'w' throughout the investor's life span. The expression is a present valued stochastic random variable $\mathbf{Z_T}$. The distribution of this random variable can be approximated by various density functions including LogNormal and Reciprocal Gamma. Distributions are calibrated through Moment Matching techniques and the resulting probability of ruin outcomes are compared to the numerical solution of the PDE. Various distributional assumptions provide "quick-and-dirty' approximations to the PDE without the necessity for numerical—i.e., simulation—approaches.	
2004	Insurance Logic Moshe A. Milevsky & Aron A. Gottesman (Captus Press, 2004)	Retirees may decide either to transfer "longevity risk" or to retain it. Insurance Companies are in a better position to manage this risk. The book reminds a reader that an instrument designed to hedge risk is not necessary supposed to be a "good investment." Annuities provide both a rate of return and mortality credits—i.e., decedents forfeit their bequests to survivors at the hazard (mortality) rate. Annuities may have substantial "crediting rate" risk—the absolute cost of living increases while benefits per $1,000 premium cost decrease. Regression of AM Best credit ratings on Annuity payouts suggests that they are negatively correlated.	Book for general audience.
2004	"Optimal Asset Allocation and Ruin-Minimization Annuitization Strategies: The Fixed Consumption Case," Moshe A. Milevsky, Kristen S. Moore & Virginia R. Young 3rd Annual IFID Conference (April 28,	The authors derive the optimal investment and annuitization strategy for minimizing the probability of lifetime ruin which they define as "…the probability that a fixed consumption strategy will lead to zero wealth while the individual is still alive." The paper assumes an investor who does not have sufficient current wealth to buy an annuity that will fund the target—"exogenously desired" —fixed consumption level. The study notes two strands of research: 1. Models that assume a rational utility-maximizing investor with rigid inter-temporal preferences and pre-specified relative risk aversion; and,	This article switches from Milevsky's option valuation approach [ROR annuity > ROR risky asset portfolio] to a barrier control problem approach. The investor locks in the target income if wealth is equal to or greater than the cost of an annuity. The article is primarily concerned with the case where wealth is insufficient to purchase an annuity. Note: the barrier might be a free boundary problem where the boundary changes as a function of time, health status, inflation, etc. Low wealth, in

	2004).	2. Models that revitalize the "safety-first" rule to maximize the probability of achieving certain investment goals. [E.G. Sid Browne's optimal strategy for minimizing shortfall probability]. The paper models an investment strategy, as well as an annuitization strategy, that minimizes the probability of lifetime ruin. The model builds on 2003 research by Young ["Optimal investment strategy to minimize the probability of lifetime ruin"]. The Young paper assumes an investor with a fixed consumption target who seeks to invest in a complete financial market that does not offer the possibility of purchasing annuities. The essay uses an optimal stochastic control model which allows for pre-existing annuity income such as Social Security. The probability of lifetime ruin is a function of the Wealth / Consumption Ratio: z = w ÷ (c – A) where c = consumption, and A = pre-existing annuity income. In this model, utility is a zero/one function (all or nothing). If the wealth-consumption ratio is greater than or equal to the actuarial present value of a continuous annuity that pays $1 per year, then the individual will buy a lump sum annuity to lock in future consumption; if the ratio is less that the actuarial present value of the annuity, the investor's portfolio maintains a positive weighting to risky assets and the investor refrains from any annuitization. In a stochastic control model, this amounts to a boundary condition which determines whether to exercise an all-or-nothing annuitization option: "…the optimal annuity-purchasing scheme is a type of barrier control." The cost of guaranteeing a perpetual income stream in capital markets is 1/r where r = the real risk free rate. Annuity pricing is a function of interest and mortality (hazard rate). Furthermore, the median survival rate (m = 50-50 chance) is $0.5 = e^{-\lambda m}$ which implies that m = ln(2)/λ = .693/ λ. Where the death rate is 0.05, life expectancy (λ) equals .693 ÷ 0.05 = 13.86 years. Where λ = 0.1, life expectancy equals .693 ÷ 0.1 = 6.93 years. Therefore, given the current real rate of interest and the current hazard rate λ, the actuarially fair price of an annuity is the present value of the payments weighted for the probability of survival	this context, means a decreased demand for annuitization because the target income level is not yet achievable.

		over time: Cost of fixed annuity = $\int_0^\infty e^{-rt}\, e^{-\lambda t} dt = \frac{1}{r+\lambda}$ Wealth evolves according to the ordinary differential equation: dW(t) = (rW(t) – c)dt, W(0) = $1. The solution is: $W(t) = (1 - c/r)e^{rt} + c/r$, with t less than or equal to t* which is the time of death or portfolio depletion. <u>Deterministic Case</u> In the deterministic case with fixed initial wealth and fixed consumption, a self-annuitization strategy assumes a consumption rate of r + λ. The higher the value of 'r,' the greater the fixed consumption target. As a result, the rate at which wealth evolves (assuming that you use the annuity consumption rate as the benchmark) is negative: $-\lambda e^{rt}$. The derivative of the wealth function becomes more negative as 'r' increases—λ, of course, does not change. Thus ruin time—t*--is the point at which the wealth function reaches zero: $t^* = \frac{1}{r}\ln(1 + \frac{r}{\lambda})$ Thus, if r = .07 and λ = .05 (life expectancy of 20 years), the ruin time is: $t^* = \frac{1}{.07}\ln\left(1 + \frac{.07}{.05}\right) = 12.5 \text{ years.}$ Note: This formula assumes that consumption and interest rates are constant and that the consumption matches the payout provided by an annuity. <u>Stochastic Case</u>	

		In the stochastic case, the authors assume a decision maker attempting to minimize the probability of ruin given fixed consumption and an allocation to risky assets in a wealth process that follows a standard Brownian Motion model. The authors maintain that if current wealth is equal to or greater than (c – A)$\bar{a}^{o}_{x+t}$ where A is pre-existing annuity income, and $\bar{a}^{o}_{x+t}$ is the market price of an annuity, then the investor will buy an annuity sufficient to lock in the target (perhaps constant dollar) lifetime consumption. [Note the equivalency to the Ralf Korn & Martin Krekel paper ("Optimal Portfolios with Fixed Consumption or Income Streams," published by the Fraunhofer-Insitut fur Techno-und-Wirtschaftsmathematik in 2002) that recommended segregated "safety first" and performance portfolios]. However, the authors are primarily interested in the case where wealth is not yet sufficient to purchase such an annuity. The issue then becomes the optimal time to purchase an annuity. This is the solution to the Hamilton Jacobi Bellman equation where "…the optimal annuity-purchasing scheme is a type of barrier control." To the left of the barrier (wealth is below the stochastic PV of consumption) the investor makes *no annuity purchase*, to the right of the barrier, "…the individual will spend her wealth to guarantee an income rate of (c – A) + A = c to match her consumption rate of c." Following the earlier paper of Milevsky and Robinson ["Self-annuitization and ruin in retirement," 2000], "… the probability of lifetime ruin ψ is a function of the ratio z = w/(c – A) and time t." Numerical examples suggest that a market offering annuities, has a lower probability of lifetime ruin than that of a market without annuities: "…for very low values of z, the current wealth to the desired additional consumption, the probability of lifetime ruin is (obviously) close to 100%, but it is quite insensitive to whether or not annuities are available. Intuitively, the reason is that the costs of the annuity and the perpetuity are both relatively far from current wealth and are therefore probabilistically inaccessible. However, as the value of z increases, the probability of	

		lifetime ruin starts to decline, and the rate of probability improvement is much higher when the life annuity is available.”	
2004	“Asset Return Distributions and the Investment Horizon,” Haim Levy and Ran Duchin The Journal of Portfolio Management vol. 30 no. 3 (Spring 2004), pp. 47 – 62.	The use of the Markowitz mean-variance method for portfolio selection is based on the “…von Neumann and Morgenstern expected utility paradigm under two alternative cases: 1) quadratic preferences, and 2) normal distribution of returns in the face of risk aversion.” Later research established that the mean-variance rule with expected utility maximization is also valid for any elliptical distribution: “There, the question of whether rates of return are elliptically distributed, i.e., whether the M-V rule can be safely employed in the expected utility framework, is of crucial importance.” Return distributions depend on the length of the planning horizon. The distribution of daily returns differs from weekly, monthly and yearly returns. Thus, there is a further question of whether the same distribution fits returns at all horizons: “We demonstrate a methodology and the results of the goodness of fit of actual rates of return on various portfolios for 11 theoretical parametric distributions and various investment horizons.” The study briefly reviews the findings of a number of studies that examine the empirical distribution of stock returns. Fama [1965] argued that price changes are best described by a stable Paretian distribution; Officer [1972] found that empirical returns exhibit finite variance but had fatter tails and higher peaks than found in a Paretian distribution; Gray and French [1990] reject the normal distribution in favor of the exponential distribution; Harvey et al. [2002] argued that the skewed normal distribution best fits empirical returns. Levy and Duchin use monthly return data on five portfolios over the period 1926-2001. These include the S&P 500, small stocks, long-term corporate bonds, long-term government bonds and Treasury Bills as reported in Ibbotson’s SBBI. They also examine daily, weekly and monthly returns on the 30 stocks in the Dow Jones industrial average as	The study analyzes (1 + rate of return) in order to assume positive values throughout the domain. Thus, there is a positive skew in the empirical distributions. The study notes: “…for long horizons there is no particular theoretical distribution that best fits the empirical distributions of all assets.”

well as the composite index. All returns are nominal except T-Bills which are reported in constant dollars: "…they do not converge to any of the 11 theoretical distributions when represented in nominal terms, but converge very strongly to the logistic distribution when represented in real value terms."

The study follows the following methodology: "For goodness of fit tests, given the historical returns of the Ibbotson portfolios and the Dow Jones stocks, we first estimate distribution parameters using maximum-likelihood estimators (MLE). The estimates correspond to the 11 different theoretical distribution parameters. After obtaining the parameters of the theoretical distributions that best suit the actual data, we apply the Kolmogorov-Smirnov (K-S) goodness of fit test, defined as:

$$D = \max_{1 \le i \le N} \left| F(Y_i) - \frac{n_i}{N} \right|$$

Where F is the theoretical cumulative distribution of the distribution (CDF) tested; Y_i is the i-th observation (observations ordered by increasing values); n_i is the number of observations in the sample with values less than or equal to Y_i, where $1 \le i \le N$; and N is the number of observations in the empirical study. Thus, $F(Y_i)$ is the theoretical CDF and n_i /N is the empirical CDF.

The theoretical distribution functions under consideration are:

- Normal
- Beta
- Exponential
- Extreme Value
- Gamma
- Logistic
- Lognormal
- Student-t
- Skewed Normal

		- Stable Paretian - Weibull For the Dow Jones stock and index series the logistic distribution best fits the empirical data for most every measuring period—daily, weekly and monthly returns. Additionally, "…with a few exceptions, the logistic distribution best fits the empirical distribution whether for individual stocks or portfolios, and whether for equity or fixed-income securities. The dominance of the logistic distribution generally holds for most horizons ranging from one day to one year." The logistic distribution has fatter tails and a higher peak than the normal distribution. The authors note, however, that the single path of historically-realized returns is limited. They use a bootstrap methodology to expand the data set. Given the larger data set, "the logistic distribution best fits the returns on common stocks for all horizons of up to 11 months; it best fits the returns on small stocks for all horizons of up to 8 months; and it best fits the returns on long-term corporate bonds for all horizons of up to 9 months. For the other two fixed-income securities, the logistic distribution provides the best fit for horizons of up to 5 months. Distributions corresponding to longer horizons reveal a different story. The best fit for equity is provided by the lognormal and the extreme value distributions, while the gamma distribution is clearly dominant for fixed-income securities….For relatively short horizons, both equity and fixed-income assets are best described by the logistic distribution, while for longer horizons they diverge." "For the longer horizons, several distributions compete, and the lognormal and the extreme value are the winners in the equity market. It is interesting to note that for relatively long horizons, with 10,000 observations, the gamma distribution provides the best fit for returns on fixed-income assets." Because the logistic distribution is a member of the elliptical distribution	

		family, the Markowitz M-V rule remains the “optimal investment decision rule.” Additionally, “…any linear combination of elliptically distributed variables is still elliptical. Hence, one can use the M-V rule for individual assets as well as for portfolios.”	
2005	“Betting on death and capital markets in retirement: a shortfall risk analysis of life annuities versus phased withdrawal plans,” Ivica Dus, Raimond Maurer & Olivia S. Mitchell Financial Services Review vol. 14 no. 3 (Fall, 2005), pp. 169 – 196.	The authors distinguish between five types of annuity payments: (1) fixed in nominal terms, (2) graded according to a pre-specified fixed nominal escalation rate, (3) indexed to inflation, (4) reflecting the return of a benchmark index or portfolio, and (5) participating according to the insurer’s experience with mortality, investments and expenses. If an annuity is purchased, there is a loss of control over assets, liquidity, and the opportunity for making bequests. Additionally, there is the imposition of potentially high administrative costs. Many investors choose to self-annuitize under a phased withdrawal approach. However, a fixed amount withdrawal election carries the risk of outliving financial resources. A fraction of remaining wealth withdrawal strategy avoids this risk but the periodic amount withdrawn may be substantially higher or lower than the fixed benefit amount. [Note: the retirement income problem involves a tradeoff between budgetary certainty with an attendant risk of ruin and budgetary uncertainty with an attendant risk of insufficient periodic income.] The authors seek to compare alternative retirement income strategies. Traditionally, financial economists approach the problem by determining the plan that maximizes discounted expected utility of uncertain future consumption and bequests. Most models based on utility functions assume time separable utility and constant relative risk aversion. These limiting assumptions however may not represent explicit measures of a retiree’s risk preferences. Therefore, the authors elect a risk-value model where reward is defined as expected return from any retirement income strategy and risk is defined as the possibility of not reaching the desired level of consumption [“probability of consumption shortfall”]. The article considers a number of phased withdrawal strategies: 1. The fixed amount withdrawal strategy replicates the payout	Article puts forth the concept of the Expected Present Value of Shortfall [EPV Shortfall] as an alternative risk metric to utility-based AEW. The risk metric of interest is the probability of consumption shortfall—the inability to produce the period-by-period income relative to an annuity payout benchmark. A “risky” distribution strategy is one that does not match an annuity. The authors stress that valid risk metrics must account for both the timing and the magnitude of losses. They anticipate the argument made by Kitces in his 2012 article: “Is The Retirement Plan with the Lowest ‘Risk of Failure’ Really the Best Choice?” The article demonstrates the benefit of “payout modeling” as opposed to “investment risk modelling”—income patterns are graphed over time instead of dollar wealth patterns. Some income patterns provide “back-loaded” retirement benefits, some provide stable benefits, some exhibit benefits that decline over time. Depending on the nature of the investor’s utility of consumption function, one pattern may be preferred over another despite the fact that it produces a lower overall present value.

		from a life annuity for as long as the retirement portfolio permits. 2. The fixed fraction withdrawal strategy exposes the retiree to payout fluctuations as the value of the retirement portfolio increases or decreases. The fixed fraction strategy subdivides as follows: • A fixed percentage withdrawal strategy withdraws a constant fraction each period from remaining wealth [Unitrust formula]; • A 1/T withdrawal strategy sets T to the oldest age in an applicable mortality table. The retiree receives 1/T of the fund as the first payment, 1/T-1 of remaining assets as the second payment, and so on. This distribution formula rises with age. • A 1/E[T(x)] withdrawal strategy takes into account the retiree's remaining life expectancy given the conditional probability of survival to age x. The denominator is the conditional probability that a retiree surviving to age x will survive to age x + t. The shorter the remaining life expectancy, the higher the fraction of remaining wealth withdrawn. This formula is used for US 401(k) and IRA plans. The article analyzes the risks and rewards of the above-listed withdrawal strategies conditional on the retiree's survival. Historical data from German actuarial tables and the German equity and bond indexes is fitted to a probability distribution that assumes that assets follow a geometric random walk with drift. The model requires the assumptions that periodic returns are serially uncorrelated and are identically distributed with constant mean and covariance—the Normal distribution. The output deducts a 50bp expense charge from return and, where appropriate, discounts payout benefits by the German CPI. However, the model cannot accommodate the fixed amount withdrawal strategy because the benefits are path dependent and do not fluctuate with the value of the account. To overcome this difficulty, the authors use a Monte Carlo simulation for portfolio evolutions under this strategy. The withdrawal amounts under the 1/T and 1/E[T(x)] strategies are	

		functions of the annuity table values. All other strategies are set to meet the annuity payout benefits at the time of initial withdrawal. Assuming a portfolio of 50% stocks/50% bonds, the analysis demonstrates very different payout patterns under various strategies. The payout pattern is based on the mean or expected value of the payout. The unitrust formula's mean payout equals and then slightly exceeds the annuity benchmark payout, the fixed benefit rule mean payout matches the annuity payout initially but, under the pressure of constant amount withdrawals, the mean gradually declines. The 1/T strategy pays a much lower mean amount until a cross-over age of 80. Thereafter, the mean payout amounts increase until they reach 400%+ of the annuity benefit by age 100. Finally, the 1/E[T(x)] strategy starts with a slightly lower payout, provides increased benefits through approximately age 90 and thereafter declines to a very low average payout level. The authors calculate several shortfall probabilities of the self-annuitization strategies. At this stage of the analysis, the conditional probability is not included in the shortfall probability analysis. Rather, the article looks at the expected operation of each strategy over a time period starting at age 65 and ending at age 110. The shortfall probability measures are: 1. Payment [P] < Annuity payout (all strategies face a high probability of years in which the payouts are less than the annuity benefit); 2. Shortfall expectation Probability which considers both the probability of falling below the annuity payment benchmark and the average size of the shortfall when it occurs: $SE(B_t) = MEL(B_t) * SP(B_t)$ where SE equals the sum of losses weighted by their probabilities (unconditional average loss), MEL equals the Mean Excess Loss (how badly, on average, does the strategy perform) and SP equals Shortfall Probability (probability of payout < annuity).	

The 1/T strategy is initially the riskiest [relative to the annuity payout benchmark] with a 60% SE while the fixed benefit strategy has a very low SE through age 83. The fixed fraction (Unitrust) strategy has a modest 20% SE through age 80 and only increases slightly thereafter. The 1/E(T) has a modest SE through age 84 but has the highest SE for long-lived retirees. The 1/E(t) strategy also provides the lowest bequest expectation at every age. Initially the 1/T strategy provides the highest bequest expectations. However, for very long-lived individuals (100+)the ultimate bequest amount is low. The fixed fraction rule exhibits the most stability; the fixed benefit rule provides substantial mean bequests after age 90.

Obviously, shortfall probability analysis will be affected by survival probabilities and by asset allocation decisions. The authors extend the analysis by seeking the risk-minimizing static asset allocation given an uncertain lifetime. They introduce a summary measure of risk—The Expected Present Value of the Shortfall [EPVShortfall] which is the sum of expected shortfalls with respect to the annuity payout target, weighted by the probability of surviving to an age where such a shortfall manifests itself. The product of these terms is then discounted according to a static risk free interest rate:

$$\text{EPVShortfall} = \sum_{t=0}^{l-x} \frac{tPxSE(Bt)}{(1+R_f)^t} \quad \text{(l = maximum mortality table age)}$$

The authors explain that this may be interpreted "as the lump sum premium that would be required for the retiree to transfer this shortfall risk to an insurer, assuming actuarially fair pricing and no additional loading." Unlike other measures (e.g., relative frequency) of shortfall probability, the EPVShortfall measure accounts for both the timing and the magnitude of losses. Two other measures are also introduced: (1) EPVBenefits and (2) EPVBequest.

The authors develop two additional optimized withdrawal/asset

		allocation rules using German data: The Fixed Percent Optimized Rule (unitrust) and the 1/T Optimized Rule. The asset allocation and the withdrawal fraction are simultaneously optimized at the time of retirement. Once the rules are established, they remain static throughout retirement. By construction, the values of EPVShortfall and EPVBequest are both zero for the annuity. For a 65 year old male, the optimized unitrust strategy (withdrawal rate of 7% from a portfolio of 35% stocks/65%bonds) exhibits a €10.37 EPVShortfall per €100 of initial portfolio value. The EPVBenefits however exceeds that of the annuity (€102.43 vs. €97.291) with an EPVBequest of €56.25. The authors note that a US retiree would hold a much larger fraction of equity in the risk-minimizing portfolio. However, the optimal withdrawal fraction (7%) is not much affected.	
2005	"Normative Target-Based Decision Making," Ali E. Abbas and James E. Matheson Managerial and Decision Economics vol. 26 no. 6 (September, 2005), pp. 373 – 385.	In utility-based decision theory, the investor chooses the strategy that maximizes expected utility. In target-based decision theory, the investor chooses the decision that maximizes the probability of meeting the target. In certain cases, the two approaches are consistent. The authors review traditional utility theory represented in the work of Von Neumann and Morgenstern. When their utility axioms are applied to lotteries with monetary outcomes, the concept of the certain (or, certainty) equivalent is the "utility inverse of the expected utility." [Because payoffs are adjusted for risk aversion] The certain equivalent is the amount of money for which the investor is indifferent between (1) accepting the lottery and receiving an uncertain payoff, or (2) rejecting the lottery and receiving the certain equivalent payoff. "For our discussion, it is useful to think of the certain equivalent of a lottery as replacing the cumulative distribution of the lottery with a step cumulative distribution that jumps from zero to one at some specific value of x. When this value is set so that the expected utility of the step distribution function is equal to the expected utility of the original lottery, we call this value the certain equivalent and designate it as $\tilde{x}$."	Discussion of decision making in the presence of a minimum floor wealth level. The authors call this the "aspiration equivalent" level. Note that a second consumption level can also be designated as an acceptance or minimum threshold level. With suitable mathematical transforms the utility of a lottery is equal to the probability that the result of the lottery is above the aspiration level. See, also [2003] Stutzer for reconciliation of shortfall and utility based risk evaluations. However, the aspiration level itself changes with changes in the wealth level and the liability value. This is an excellent justification for the importance of portfolio monitoring (on an ALM basis) in the face of minimum cash-flow needs. The aspiration level provides a probability-oriented portfolio preferencing criterion. The article quotes Kahneman and Tversky: "a failure to adapt to losses or to attain an expected gain

		However, when the original utility function is normalized to a range within the zero to one interval, the normalized utility function can also be replaced by a step utility function. For example, the x-axis can be a dollar wealth target and the y-axis is the normalized range with zero at the origin and one at the top of the axis. If the step function is set "…so that the expected utility with the new step utility function is equal to the original expected utility, the x-y plane divides into two regions: Satisfactory and Unsatisfactory. The value at which the plan is bifurcated—i.e., the target wealth value—is designated as the aspiration equivalent, or $\hat{x}$. Given this calibration, "…the expected utility of a lottery is equal to the probability that the outcome of the lottery exceeds the aspiration equivalent." Note: the aspiration level is the boundary between satisfactory and unsatisfactory; the aspiration equivalent changes based on "changes in the lottery that the individual is facing." For investment issues, the aspiration level changes with factors such as wealth level and liability value. The exponential utility function can also be normalized on a zero-one interval [a,b]: $U(x) = \frac{e^{-\gamma a}\, e^{-\gamma x}}{e^{-\gamma a} + e^{-\gamma b}}$ where $a \leq x \leq b$. Gamma is the risk-aversion coefficient. The reciprocal of the risk-aversion coefficient is risk tolerance. "…for an exponential utility function, it suffices to assess the decision maker's certain equivalent of any single uncertain lottery in order to determine his risk attitude. An approximate value for the risk tolerance is equal to the value of ρ … that makes the decision maker indifferent between doing nothing or playing a binary lottery where he receives \$ρ with a probability of 0.5 and loses \$(ρ/2) with probability 0.5." The expected utility of a continuous lottery is:	induces risk seeking." This dovetails with the risk-of-increasing-risk observations of Milevsky & Robinson [2005] and Gabrielle Stabile [2003]. It has important implications for gauging the prudence of investor/trustee reactions to decreases in portfolio values the magnitude of which jeopardize the viability or sustainability of income [Probability of success increases with an increase in expected return but decreases with variance]. Does the investor/trustee increase risk / increase E(r)? Does the investor stay the course and hope that a market recovery will save the day? The article is also a variation on the boundary control or free boundary class of problems. In this case, the authors distinguish between satisfactory and unsatisfactory regions. Only strategies placing the investor in the satisfactory regions generate positive utility. This lays the groundwork for a target wealth / target income level preferencing metric for choosing among various investment strategies.

		U($\tilde{x}$) $\triangleq expected\ utility = \int_{-\infty}^{\infty} U(x)dF(x) = \int_{-\infty}^{\infty} U(x)f(x)dx$ Where F(x) is the cumulative probability function of the lottery, f(x) is the probability density function (pdf), U(x) is the decision-maker's utility function normalized to a zero-one range, and $\tilde{x}$ is the certain equivalent of the lottery. Thus, the certain equivalent is expressed as: $\tilde{x} = U^{-1}\left(\int_{-\infty}^{\infty} U(x)\, dF(x)\right)$ Or, $\tilde{x} = U^{-1}\left(\int_{-\infty}^{\infty} U(x)\, f(x)\, dx\right)$ Integrating by parts, expected utility is: $\tilde{x} = U^{-1}\left(\int_{-\infty}^{\infty} U(x)\, dF(x)\right) = U(x)F(x)\vert_{-\infty}^{\infty} - \int_{-\infty}^{\infty} F(x)dU(x)$ $= 1 - \int_{-\infty}^{\infty} F(x)dU(x) \triangleq 1 -$ expected disutility. The aspiration equivalent—the wealth boundary $\tilde{x}$ can be expressed in terms of the expected disutility: $F(\hat{x}) = \int_{-\infty}^{\infty} F(x)u(x)dx$ The utility density function is the derivative of the normalized utility function U(x). The above integral is similar to the integral defining expected utility except that the roles of F and U are reversed. Given the fact that the total area defined by the wealth axis and the normalized utility axis breaks into two regions, this means: Expected utility + expected disutility = 1, or, U($\tilde{x}$) + F($\tilde{x}$) = 1 which establishes a fundamental identity relating the certain equivalent and the aspiration equivalent: "The problem of choosing the lottery that has the highest expected utility is thus equivalent to the problem of choosing the lottery that has the lowest expected disutility." Rearranging terms gives: Expected Utility = 1 - F($\tilde{x}$) = G($\tilde{x}$) where G($\tilde{x}$) is the excess distribution function. Interpreted from an expected utility perspective, "The expected utility of a lottery is the probability that the outcome of the	

		lottery exceeds its aspiration equivalent. This result provides us with a new target-based method for choosing between lotteries; we choose the lottery that has the highest probability of meeting its aspiration equivalent. The aspiration equivalent is the point that divides the x-axis of the cumulative probability distribution into two portions, the probability of the portion to the right is numerically equal to the expected utility of the lottery, and the probability of the portion to the left is the expected disutility." After providing examples of how to calculate the aspiration equivalent assuming an exponential utility function—normalized so that the utility density function is a parallel to the probability density function—the authors observe that setting the aspiration equivalent close to the lower interval bound is the equivalent of acting with risk-averse behavior; setting it close to the upper bound is risk-seeking behavior. They point out that the observation is similar to Kahneman and Tversky's prospect theory: "There are situations in which gains and losses are coded relative to an expectation or *aspiration level* that differs from the status quo….a person who has not made peace with his losses is likely to accept gambles that would be unacceptable otherwise….*a failure to adapt to losses or to attain an expected gain induces risk seeking*." An investor setting high aspirations for a retirement standard of living must invest aggressively per unit of wealth. Mathematically, when the risk-aversion coefficient (γ) approaches zero, both the certain equivalent and the aspiration equivalent converge to the expected value of the lottery assuming a symmetric lottery. Probability distributions may, however, change through time and the targets may have to be revised to reflect updated information: "…pursuing a fixed goal may be operationally motivational when things are going smoothly, but when major impacts, such as setbacks or new opportunities, create a need to re-evaluate alternatives, the normative approach demands determining new targets…. Simply maximizing the probability of reaching the old target is no longer optimal."	

2005	"Life is Cheap: Using Mortality Bonds to Hedge Aggregate Mortality Risk," Leora Friedberg and Anthony Webb, Center for Retirement Research at Boston College, pp. 1 – 32.	Are insurance carriers systematically underpricing annuity contracts? If the answer is yes, this contributes to the "annuity puzzle" literature because consumers seem not to be anxious to buy low-cost contracts. It also suggests that insurance carriers may face insolvency risk if they issue large numbers of annuity contracts promising significant lifetime payments to annuitants. Although the central focus of the paper is on the potential benefits to insurance companies of using mortality-contingent bonds to hedge mortality/longevity risks in their life insurance and annuity business, the study also quantifies the mortality risk faced by insurance companies. They define this risk as "aggregate mortality risk" which occurs if annuitants, on average, live longer than estimated. Of course, there is no way to know the actual pricing assumptions (e.g., estimated inflation, investment, expense, and mortality estimates) underlying the cost of an insurance company's annuity contract. However, using a Lee-Carter model of mortality (a stochastic model predicting the future force of mortality including mortality improvement rates) and holding all other variables constant (or using static assumptions for parameter values), the authors conclude that the insurance industry may be issuing mispriced annuity contracts because of a significant underestimation of annuitant longevity: "We find that, by the Lee-0Carter benchmark, insurance companies systematically underprice annuities if they use the Projection Scale AA without making any compensating adjustment elsewhere in their pricing formulas. This finding is a corollary of our earlier results showing that actuarial life tables appear to understate aggregate mortality risk." This is longevity risk not from the perspective of the buyer; but rather from the perspective of the contract seller. The study calculates that "The extent of the underpricing ranges from 8.7 to 11.2 percent."	Authors point out that their approach is a complement to that of Mitchell, Poterba, Warshawsky and Brown (1999). MPWB calculate the Expected Present Value of an annuity by means of a ratio which divides the numerator (mortality-adjusted sum of discounted lifetime periodic payments) by the denominator (cost of the annuity contract). An annuity with an EPV value greater than one suggests that the buyer of an annuity contract will realize an economic benefit in excess of the premium paid. An actuarially fair annuity should have an EPV equal to zero; an annuity with a load should have an expected EPV of less than one. Thus, EPV is a measure of the economic value / cost to the consumer. However, to the extent that an annuity provides the consumer with an economic value ≥ 1, this suggests that the insurance company has mispriced the contract. Such mispricing produces increased insolvency risk for the carrier.
2005	"Annuitization and asset allocation with HARA utility,"	This paper extends the 2002 and 2003 research of Milevsky and Young [MY]—see "Optimal asset allocation and the real option to delay annuitization: it's not now or never" and "Annuitization and Asset	The authors' description of investor motivation to delay annuitization highlights the difference in approaches to the goal of securing stable and

	Geoffrey Kingston & Susan Thorp Journal of Pension Economics and Finance vol. 4 no. 3 (November, 2005), pp. 225 – 248.	Allocation." It reiterates and develops an underlying cause motivating individuals to delay annuitization of wealth: because risky assets carry the expectation of high return, a longer period of holding such assets "…offers people a chance to improve their budget constraint that evaporates after annuitization. So even risk averse individuals may decide to delay in the expectation of creating more wealth and enjoying a higher long-term income." Even individuals who (subjectively) anticipate a long life span may delay annuitization given the potential benefits of (1) lower future annuity costs—if interest rates increase, and (2) higher returns from exposure to risky assets. Given the fact that an annuity purchase decision is irreversible and that the real option to annuitize has time value, the MY research asserts that an investor will delay annuitization until such time that the expected return from the annuity contract exceeds that of other financial instruments exhibiting comparable risk. In contrast to the MY model which assumes CRRA, the Kingston and Thorp essay assumes Hyperbolic Absolute Risk Aversion—HARA. Further, they assume that the investor has a fixed consumption floor. Such a floor is a proxy for standard of living habit formation. The authors point out that a CRRA utility function is consistent only with a constant mix portfolio management approach. HARA utility functions however can accommodate a buy and hold approach as well as convex payoff approaches such as "portfolio insurance." The CRRA model assumes that investors derive utility from consumption irrespective of its absolute level. However, it is plausible to assume that only consumption above a threshold level generates positive utility. Consumption below a "subsistence" level does not generate positive value. When such a level ("non-zero conslumption floor") is introduced into the model, a HARA utility function is required; A commonly used utility function for a buy-and-hold portfolio management preference sets single-period utility as the log of consumption plus a constant. This is a linear risk-tolerance function known as the Stone-Geary utility function	sustainable income. One approach to the problem takes the form of a rate of return "trigger": as soon as the annuity ROR exceeds the expected return on the risky asset portfolio, the investor should elect the option to annuitize. This election should be made irrespective of the level of wealth. Hence, it is only appropriate for a CRRA investor. The other approach incorporates the concept of a consumption floor or standard of living target--see Abbas & Matheson [2005]. This model requires HARA utility. The presence of a "consumption floor" changes the decision making process. The goal can now be expressed in terms of surplus optimization. The minimum standard of living target is fully funded ("escrowed") by the annuity. The investor secures this floor income as soon as possible and, therefore, tends to favor early exercise of the option to annuitize. It is a variation of the 2-fund solution approach with surplus wealth invested in the risky asset portfolio. Monitoring is critical in that a key ratio is the level of available surplus relative to the changing costs of securing an "acceptance level" income stream. Compare with: "The hurdle-race problem," S. Vanduffel, J. Dhaene, M. Goovaerts, R. Kaas. [2003] Authors note that wealth is not an objective in and of itself: "The most common metric for the adequacy of an accumulation is the long-term income stream it can generate." Compare to Viceria & Campbell's book on Strategic Asset Allocation: "Wealth is an asset that pays

		or, in the language of financial economists, the generalized logarithmic utility model (GLUM). The authors note that Ingersoll's classic investment text solves a problem involving a HARA utility function by transforming it to a comparable problem involving a CRRA function. The key is to "…transform the state variable for wealth so as to reduce the problem to one of CRRA utility with a state variable net of an 'escrowed' wealth component that protects the consumption floor." The presence of such a floor brings forward the time at which it becomes optimal for an investor to exercise the annuitization option. Briefly, for a CRRA investor, the optimal annuity date is found at the point where the change (derivative) in the total value function (expressed in terms of the Hamilton-Jacobi-Bellman equation) with respect to the time of annuitization is proportional to the difference between expected rate of financial asset growth [the expected return from the risk-adjusted Merton Optimum [δ]—r + $\frac{(a-r)^2}{2\sigma^2\gamma}$] and the payout from the actuarial instrument: r + λ_{x+T} where λ is the hazard or mortality rate. However, for a HARA investor, the presence of a minimum consumption floor increases the utility value of securing this threshold. Thus the derivative of the HJB equation with respect to the choice of annuity timing changes to: $$dV/dT \propto \delta - \left(r + \lambda_{x+T}[1 + \frac{Surplus\ Wealth}{Minimum\ Wealth\ Escow\ Amount}]\right)$$ In the presence of a consumption floor, the last bracket is always greater than 1. This means that the optimal time for annuitization of wealth sufficient to secure the minimum target floor comes sooner rather than later. This also means that surplus wealth is a smaller percentage of the investor's total holdings and therefore the sacrifice of the expected risk premium is proportionately smaller. "It follows that introducing a positive consumption floor has a similar effect to raising relative risk aversion. In addition, the agent recognizes that it is 'cheaper' to store escrow wealth in an annuity rather than a bond portfolio (at least where	consumption as its dividend."

		there are small enough loadings), creating another incentive to switch into complete annuitization at an earlier date." [See Korn & Krekel "Optimal Portfolios with Fixed Consumption or Income Streams" 2002 for concept of "escrowed wealth"].	
2005	"Payout and Income Annuities," Stephen J. Abels Society of Actuaries 2005 Spring Meeting Record, vol. 31 no. 1 Session 5PD, pp. 1 - 24.	Stephen Abels is an actuary with Mutual of Omaha and was a speaker at the SOA 2005 New Orleans Life Spring Meeting. Among his remarks are: "It is difficult at above age 86 or so to find a company willing to underwrite a life-contingent payout…." "People with household incomes between $15,000 and $75,000 are the majority of the annuitization market. It is not simply for the affluent." "…people are not as inclined to lock in those low interest rates. They don't want to lock in the payouts that result from calculating them in low interest rates for the rest of their life." "…a single premium product. You set up a high reserve at the beginning as the interest rate that you must assume in your reserve calculation, and it plays heavily into the profit measurement. There are also the cost and risk of providing financial guarantees, longevity risk and concern over mortality improvement."	Author provides some helpful observations regarding single premium annuities.
2005	"Hot Topics in Fixed Annuities," David J. Weinsier Society of Actuaries 2005 Spring Meeting Record, vol. 31 no. 1 Session 64PD pp. 1 – 25.	David Weinsier is a senior consultant with Tillinghast in Atlanta. His remarks began with observations on the distribution channels for the fixed annuity market: "In terms of the banks, they continue to favor products that are standard commissions, but not too high; I would say 5 or 6 percent is on the low side. With your MGAs, historically there is more focus on the high-commission products, those in the double digits." [MGAs = Managing General Agencies] With regard to the companies underwriting fixed annuities, Weinsier makes the following comments: '…there are a few areas that we have noticed when you have one carrier who is able to credit a higher rate than another carrier. Some carriers are willing to take risks on the quality and/or duration on the asset side of the balance sheet." "There is an increased effort to reduce capital requirements….We see some companies trying to reduce such requirements using the	Insights into annuity costs and loading factors

		covariance factor, and some carriers are doing their C3 phase I analysis, even though they are not required to, to see if they can get away with some diversification and lower those all-important capital requirements." "In terms of profit measures, you have the traditional IRR measure….Profit margin is popular obviously, return on assets, GAAP ROE….Most folks are still shooting for that 12 percent IRR." "What about the impact of interest rates, obviously a key moving part to your fixed annuities? I think we all know that low rates cause spread compression. With a low sustained rate, your higher earning assets end by going over, you have to invest them low, that brings your portfolio yield down, and you are going to realize spread compression. On the other hand, a rapid rise in rates is no picnic either. If we see all of a sudden a very rapid rise, then you would likely get some surrender mediation occurring." "…nobody does complete hedging, because it is too expensive, and so I think almost everybody out there has some vulnerability to interest rates and movements thereof."	
2005	"A Sustainable Spending Rate without Simulation," Moshe A. Milevsky & Chris Robinson Financial Analysts Journal vol. 61 no. 6 (November / December 2005), pp. 89 – 100.	The authors contend that the literature on sustainable withdrawal and spending rates "…lacks a coherent modeling framework on which to base the discussion." They derive "…an analytic relationship between spending, aging, and sustainability in a random portfolio environment." The key concept is the Stochastic Present Value [SPV] of a given spending plan, at a given age, under a given portfolio allocation at an initial level of wealth. Retirement is feasible when the SPV of consumption is lower than current wealth. SPV is affected by age, asset allocation, spending target. The authors test which of these three levers of 'retirement sustainability' is of greatest importance in the prevention of "the probability of retirement ruin." The greater the positive differential between current wealth and the SPV, the lesser the probability of ruin: "In the language of stochastic calculus, the probability that a diffusion process that starts at a value of *w* will hit zero prior to an independent "killing time" can be	The thrust of the research question in this paper parallels that of "A Note on Parameter Elasticities in Monte Carlo Retirement Planning Simulations," by Walt Woerheide and Don Taylor [2006]. The conclusions are somewhat different. The critical measurement metric is the differential between the stochastic PV of wealth and the stochastic PV of liabilities. This leads to a "surplus wealth" optimization type of management approach. This represents a significant change from Milevsky's earlier option valuation approach. Monitoring is now focused on wealth in excess of a floor value. The value of the liability—i.e., the planned spending rate—is the key factor in long term sustainability.

		represented as the probability that a suitably defined SPV is greater than the same *w*." In a purely deterministic framework, the present value of consumption is the textbook formula for an ordinary simple annuity of \$1. If current wealth is greater than your consumption needs, then retirement is feasible. In a stochastic framework, retirement needs are a sum product where consumption must be summed over an uncertain number of years, and consumption requirements (discounted at the rate of investment return) must be discounted at a compound rate over the uncertain horizon. Finally, in a continuous time framework, the summation becomes an integral and the product of investment return discounting becomes a continuous time diffusion process: $$\text{SPV} = \int_0^{\infty} prob(\mathrm{T} > t) R_t^{-1} \mathrm{dt}$$ Specifically, the SPV depends on the investment generating process 'R' and the retiree's life span. If one assumes that the investment process is a log-normal process (exponential Brownian motion in continuous time finance) and that the remaining lifetime process is distributed exponentially, it is possible to develop a closed-form expression for the SPV. Finally, the SPV process itself can be modeled as a reciprocal gamma distribution that incorporates the mean and variance of the investment process, initial wealth, and the mortality rate. The mortality rate of an exponential process differs from the mortality table rates but, in the authors' opinion, the error is not great. This means that the closed form solution is a good approximation to results obtained under a comparable simulation. The article constructs tables of ruin probabilities for various mean/variance portfolios, retirement ages, and spending rates. The tables demonstrate, in general, that spending rates higher than 5% of initial wealth produce unacceptably high probabilities for ruin. The spending decision dominates the asset allocation decisions at a 5%+ rate: "No matter what reasonable portfolio is chosen, asset allocation will not turn a bad situation into a good one." This is because return and	Changing the asset allocation is not always feasible because increased expected return comes with increased return variance: The risk of increasing risk. A shortcoming of the analytical solution developed in this essay is the need to assume a log-normal distribution of asset returns.

		variance move together and any attempt to increase return will also increase the failure rate. The two most effective levers for controlling retirement success are postponement of portfolio distributions to a later age or reductions in consumption targets.	
2005	"Annuities and Individual Welfare," Thomas Davidoff, Jeffrey Brown and Peter Diamond The American Economic Review vol. 95 no. 5 (December 2005) pp. 1573 – 1590.	The classic Yaari life-cycle consumer with no bequest objective and with an uncertain date of death annuitizes all wealth under the assumption that the consumer is a von Neumann-Morgenstern expected utility maximizer with intertemporally separable utility, and with access to actuarially fair annuity products. The authors, however, contend that it is not necessary to assume that the consumer is an exponential discounter, or that he obeys the standard utility axioms, or for the annuity to be actuarially fair. Consumers under the Yaari model will annuitize all wealth provided that they have no bequest motives and that the net rate of return on the annuities is greater than the return on conventional assets of matching financial risk. Under other models, however, annuitization may or may not be optimal. The analysis begins with an examination of annuity demand in a two-period model with no uncertainty other than mortality. It assumes that all resource allocations are made at the beginning of the period. The rationale for this assumption lies in the complete market Arrow-Debreu model -- consumers are willing to commit to a fixed plan of expenditures at a starting time if they are able to trade goods across all periods and all states of nature (i.e. trading is a means of reversing or revising initial decisions); or, if the consumer lives for only two periods. The model in this article assumes that the consumer is definitely alive in period one and alive at probability 1-q in period 2 (without a bequest motive). The Utility of lifetime consumption is, therefore: $U = U(c_1, c_2)$. The authors note that the utility of second-period consumption may depend on the level of first period consumption (a "standard of living" utility function). Their methodology departs from other studies in that they analyze consumer choice in terms of minimizing expenditures (E) subject to attaining a given level of utility. Indeed, lowering the cost of attaining	The authors find that a consumer may opt to annuitize wealth even if some key assumptions of the Yaari model are relaxed. The article explores annuitization under the Yaari model and under several additional models. They introduce a new preferencing metric: a utility-at-the-lowest-cost metric. However, whenever the market is incomplete—i.e., risks cannot be spanned by financial assets—even a small amount of annuitization may be suboptimal. Conclusions regarding the utility of annuitization are highly dependent upon model assumptions. The authors execute a simulation model for a single male, age 65, assuming power utility function for both additively separable and standard of living utility. The model assumes exponential discounting at a deterministic rate. Mortality is maxed at age 100 and they use a Social Security general population table. It assumes a real interest rate of 3% and varies γ [γ = coefficient of risk aversion] and δ [δ = discount rate]. Age 65 wealth is normalized to a value of 100. Simulations divide into three groups. The base case for each group assumes the following: $\gamma = 1$ (log utility) and $\delta = 1.03^{-1}$ (discounting rate approximates the real interest rate). The value of α is set equal to one. Holding all else fixed, the simulation is then run with an increase in the

		utility means an increase in that utility either from a "lowered cost" perspective or from a "higher attainable consumption" perspective. Within the budget constraint, the consumer selects the desired level of consumption in period 1 and measures the utility thereof. They assume the availability of a bond return (R_B) that provides return in period 2 in exchange for foregoing each unit of consumption in period 1. The Bond pays in period 2 irrespective of whether the consumer is dead or alive. Additionally, they assume the availability of an annuity return (R_A) in period 2 that pays the annuity cash flow only if the consumer is alive. For an actuarially fair annuity, the return is $R_A = R_B / 1-q$. In the commercial world, adverse selection or transaction costs may drive the annuity return below this level. However, given a positive probability of death in any future period, there is a weak assumption that $R_A > R_B$. This means that the expenditure for total consumption equals: $E = c_1 + A + B$ where A is the cost of the underlying Annuity portfolio and B is the cost of the Bond portfolio. If the consumer holds a positive amount of bonds, it is possible to reduce the cost of consumption by swapping the bonds for annuities. Therefore the solution to the minimization of expense problem is to set the level of bonds equal to zero. Absent a bequest motive, all that is required is for the annuity payout (i.e. price) to dominate that of the conventional bond. All utility maximizing consumers will annuitize as long as second period consumption is positive. Finally, the welfare gain from releasing a constraint on the availability of annuities is measurable by the formula $[1 – (R_A / R_B)] < 0$. The cost of period 2 consumption falls from $1/ R_B$ to $1/ R_A$ with the result that period 2 consumption can adjust upwards—hence, increased welfare. Consumers, however, face many periods of potential consumption under many states of nature (poor health, uncertainty of survival, etc.). Additionally, rates of return on many assets are stochastic. The authors extend the simple 2 period model by adding a third period so that probability of survival is now $1-q_2$ to period 2 and $(1-q_2)(1-q_3)$ to period 3. The cost of financing consumption over the three periods is, therefore:	discount rate to $\delta = 1.10^{-1}$, and again with γ = to 2. • Group one assumes a consumer with separable utility. • Group two assumes a consumer with standard of living utility, retirement wealth of 100 and s_1 equal to 5 (i.e. consume 5% of initial wealth in first period). • Group three assumes a consumer with standard of living utility, retirement wealth of 100 and s_1 equal to 50 (consume half of initial wealth). Four welfare measures are calculated: ➢ Increase in wealth required to hold utility constant while moving from a constant real annuity to conventional bonds (analogous to AEW); ➢ Fraction of wealth optimally committed to real annuities instead of bonds; ➢ Increase in wealth required to hold utility constant while moving from optimal annuity position to conventional bonds; and, ➢ Gain in utility from selecting the optimal payout trajectory (bonds or annuities) with no requirement for either. An additional series of graphs plotting optimal consumption profiles with different levels and types of annuities are also produced. For group one (base case), all wealth is annuitized (i.e. a constant real annuity provides the optimal consumption path with AEV = 44% increase in initial

		$$E = c_1 + A_2 + A_3 + B_2 + B_3$$ With $$C_2 = R_{B2}B_2 + R_{A2}A_2, \text{ and,}$$ $$C_3 = R_{B3}B_3 + R_{A3}A_3.$$ The results of the above analysis extend trivially. The authors designate the Bonds as "Arrow Bonds" that pay off across different states of nature; and designate the annuities as "Arrow Annuities." They extend the model further by assuming that c_1 is a scalar; but that c_2, B_2 and A_2 are vectors with entries corresponding to arbitrarily many future periods within arbitrarily many states of nature ($\omega \leq \Omega$). The annuity and bond returns constitute a matrix (with columns corresponding to payouts in various states of nature and rows corresponding to periods). Combinations of bonds allow for completeness of markets; likewise, separate (theoretical) annuities with payouts in each year can also be combined to replicate a standard commercial annuity that provides income throughout each period during life. As long as annuities pay a positive premium, the model will lead to complete annuitization of wealth. With no bequests, the market is complete in the sense that Arrow Bonds or Arrow Annuities exist for every event—i.e. every event can be financed either with an Arrow Bond or an Arrow Annuity. Provided that the costs and expenses of annuities are less than their mortality credits, complete markets lead to full annuitization. All wealth is invested in Arrow Annuities. What happens when the set of Arrow Annuities, however, is not complete? The consumer can obtain future consumption under some states of nature only by purchasing an Arrow Bond—i.e. 100% annuitization will lead to zero consumption if the annuity market is not complete. Thus, the optimal consumption solution will include Arrow Bonds; but, where Arrow Annuity payouts exist, in those states of nature the solution will also include some annuitization.	wealth). At a 10% discount rate, if all wealth is again annuitized, the AEV = 19%. However, the optimal choice is 72% real annuity and 28% bonds. Not surprisingly, holding the discount rate at 3% and increasing risk aversion to 2, results in full annuitization with AEV = 56%. The second group models an individual with a large stock of initial wealth with which to finance the desired standard of living (100/5 = 20x). For this consumer with log utility at discount rate equal to the real interest rate, the utility gain equals 64% for a real annuity and 82% for the optimal payout path. At the higher discount rate, the choice is to put 99% of wealth in a real annuity. Note: contrast the 20x ratio with the 30x ratio in "Ruined Moments in Your Life: How Good Are the Approximations?" H. Huang, M. Milevsky and J. Wang [2004]. The third case requires large first-period consumption relative to initial wealth (100/50 = 2x). This path in inconsistent with a real annuity (consumption is front-end loaded) and annuities no longer provide high utility values. At the base case, the AEV falls to 36%; at a discount rate of 10%, the value falls to 3%; and at a risk aversion value of 2, complete real annuitization actually reduces utility (optimal mix = 60% annuity / 40% bond). [Note: the pattern of consumption reflects Fisher utility]. The authors stress that the simulation model retains the simplifying assumptions of no bequest motive, no risk other than longevity, no news regarding health status, and no liquidity concerns: "…dropping

		An example of "real world compound annuities" is the indexed Social Security payment annuity. Other types of annuities include fixed nominal annuities and market-indexed variable annuities. The ability to reinvest annuity and bond income will also impact the optimization solution. The model assumes that $R_{At\omega} > R_{Bt\omega}$ for $\forall$ $t\omega$. Thus, any consumption vector that may be purchased strictly through annuities is less expensive than when purchased with bonds. Additionally, full annuitization is not optimal if the pattern of annuitized consumption is worth changing by the purchase of a bond: $U_1(c_1, R_{a2}A, R_{a3}A) < R_{B2}U_2(c_1, R_{a2}A, R_{a3}A)$ Or, $U_1(c_1, R_{a2}A, R_{a3}A) < R_{B3}U_2(c_1, R_{a2}A, R_{a3}A)$ The authors also demonstrate (based on the fact that marginal utility gain at the point of zero consumption is infinite) that expenditure minimization implies a positive holding of at least one annuity: "If consumption is positive in every state of nature, then consumption is a linear combination of all strictly positive linear combinations of the Arrow bonds. But then since some strictly positive consumption plan can be financed by annuities…expenditures can be reduced holding consumption constant by a trade of some linear combination of the bonds for some combination of annuities with strictly positive payouts." This result, of course, relies on the assumption that the annuity payout is greater than the payout of the underlying assets. However, full annuitization may distort consumption (place an upper bound on feasible future consumption) and, therefore, may not be optimal. The article considers the Milevsky/Young argument regarding the time value of an option to delay annuitization. To do this, the model must be altered to allow trades beyond the start of period 1. Again, assuming that the Arrow Bond market is complete in period 1 and that Annuity trades are available in subsequent periods, households may modify	these assumptions…would be an important generalization, but obtaining results will require strong assumptions both on annuity returns and on the nature of bequest preferences and liquidity needs." However, without complete insurance markets, even a small amount of annuitization may be suboptimal. Additionally, incompleteness of annuity markets may render annuitization of a large fraction of wealth suboptimal. Annuitization may be imprudent because it locks consumption onto a fixed path over all states of nature. This is a reason to defer exercising the option to annuitize. Not only does annuitization put in an upper bound to the current beneficiary's income, it also is a risk to remaindermen. One possible solution is to annuitize all wealth immediately and retain a portion of each periodic payment to invest in a risky asset portfolio FBO the remaindermen. Generally, the authors' model suggests that the greater the wealth, the lower the demand to annuitize Note also that health shocks are seen as events that decrease future demands for liquid wealth insofar as they significantly reduce life expectancy.

		consumption by trading for annuities at a later period. Additionally, the possibility of reinvesting excess annuity payouts (Savings = Z) may still result in full annuitization as the optimal solution to the expense minimization problem. Allowing for savings, changes the minimization problem to: $$\min_{c_1,A,B,Z} : c_1 + B_2 + B_3 + A$$ s.t.: $U(c_1, R_{B2}B_2 + R_{A2}A - Z, R_{B3}B_3 + R_{A3}A + (R_{B3}/R_{B2})Z \geq \overline{U}$ where U 'bar' is required utility. There are non-negativity restrictions against dying in debt: $B_2, B_3, Z \geq 0$ $R_{B2}B_2 \geq 0$ and, $R_{B3}B_3 + (R_{B3} / R_{B2})Z \geq 0$ Given the model's assumptions regarding infinite disutility from a consumption of zero in any period, and given that any consumption plan that can be financed by annuities alone is financed more cheaply than through Arrow Bonds, it follows that A > 0. The consumer may partially undo annuitization if early period consumption is low by saving the excess annuity payments and reinvesting in bonds maturing at later dates. The prohibition against dying in debt, however, prevents the consumer from undoing annuitization by future borrowing. With Bonds liquid, the purchase of annuities imposes a strict upper-bound limit on consumption in each period (total consumption cannot exceed value of bond portfolio, cost of period one consumption, plus aggregate annuity payouts). Thus, in one sense, annuities are costly because they contribute to consumption constraints: "The welfare effects of larger increases in annuitization are more difficult to sign because they may constrain consumption." If consumption is less than the annuity payout, the savings difference can be used to purchase consumption at a later	

period, if consumption is greater than the annuity payout, than a bond maturing at time t must be purchased.”

The authors measure utility under standard assumptions (additively separable preferences over consumption, exponential discounting, actuarially fair annuity). They define $1 – m_t$ as the probability of surviving to period t. Thus, in addition to a bond valuation that involves interest rate only discounting, an annuity valuation involves the product of interest rate and mortality rate discounting:

$$R_A = \frac{1}{\sum_{t=2}^{T}(1-m_t)(1+r)^{1-t}}$$

Thus any change in utility from moving out of a bond portfolio to annuitization is measured by:

$$\sum_{t=1}^{T}\delta^{t-1}(1-m_t)u(c_t)$$

Thus, if $\delta(1+r) \geq 1$, complete initial optimization is optimal (where, δ is the investor’s discount / preference rate). For this not to be the case, there must exist for some period t a strategy where purchase of a bond maturing at time t provides greater utility than annuitization. However, the possibility of future trades in annuities can decrease demand for initial annuities and increase demand for later annuities. Purchase of annuities in period two corresponds to Milevsky’s thesis. When the Arrow Bond’s IRR is higher than the annuity’s IRR, it pays to delay annuitization.

The article identifies conditions for which incomplete annuitization of wealth may be optimal. If insurance is incomplete for liquidity-demanding events (medical expenses / nursing home costs), or if

		individuals receive news regarding decreased life expectancy, the demand for annuities will decrease. News regarding decreased life expectancy (e.g., zero probability for survival beyond period 2) will cause the consumer to cash in the bonds with period 3 maturities; however, period three payout annuities remain illiquid and will not contribute to consumer utility. Traditional approaches to annuity valuation assume that switching from a regime in which annuity markets are not available to one in which they become available results in a significant increase in consumer utility (e.g., AEW of +50%). The authors evaluate this hypothesis under the "industry standard" case—i.e. no bequest, additive separability of utility, etc., and under the assumption that past levels of consumption influence the utility of present consumption. The authors repeat their assertion at if $\delta(1+r) \geq 1$, then, under the usual assumptions: "valuation will increase in the patience parameter δ, which should push consumption later in life. Further, in cases where optimal consumption is decreasing over time, increased smoothing should increase valuation." The intuition is that consumption restraints imposed by annuitization will be balanced against a desire for consumption smoothing. However, it is often the case that utility is measured relative to the level of past consumption (standard of living). The utility of living in a studio apartment is different for a person who has lived in one throughout their life than for someone who lived in a mansion during previous years. Thus, the article contends that consumers trade off consumption between periods based not only on budget constraints, but also on standard of living ratios: $$U(c_1,c_2) = \sum_{t=1}^{T} \delta^{t-1}(1-m_t)u(\frac{c_t}{s_t})$$	

Where, $s_t = \dfrac{s_{t-1} + \alpha\, c_{t-1}}{1+\alpha}$

When α = 0, an individual's subjective standard of living is constant. Positive values of α mean that previous levels of consumption make current consumption less satisfactory.

The marginal utility of consumption in any period incorporates two effects not present in the additively separable utility model:

1. The effect of present standard of living on present marginal utility; and,
2. The effect of present consumption on future period utility through subsequent standards of living.

Under these assumptions, the marginal benefit of current consumption is:

$$\frac{\partial U}{\partial c_t} = \delta^{t-1} \frac{1}{s_t} u^{'}\left(\frac{c_t}{s_t}(1-m_t)\right) - \sum_{k>t} \delta^{k-1} \frac{\alpha}{(1+\alpha)^{k-t}} \frac{c_k}{s_k^2} u^{'}\left(\frac{c_k}{s_k}\right)(1-$$

Assuming u(c/s) = $\dfrac{\left(\frac{c}{s}\right)^{1-\gamma}}{1-\gamma}$ and that $\gamma \geq 1$, then:

$$\frac{\partial U}{\partial c_t} = \delta^{t-1} c_y^{-\gamma} s_t^{\gamma-1}(1-m_t) - \sum_{k>t} \delta^{k-1} \frac{\alpha}{(1+\alpha)^{k-t}}$$
$$c_k^{1-\gamma} s_k^{\gamma-2}(1-m_k)\,.$$

When γ > 1 effect #1 tends to push consumption towards later periods if standard of living is increasing over time (a higher standard of living increases the marginal utility of consumption). If the standard of living

		decreases over time and $\gamma \geq 2$, then effect #1 pushes consumption to earlier periods. For $\gamma < 2$, the effect is ambiguous. Effect #2 tends to push consumption towards later periods in life since later consumption raises the standard of living in fewer periods. "Hence, the result of complete annuitization when the discount rate is less than the interest rate, Result 7 [if $\delta(1+r) \geq 1$, complete initial optimization is optimal], continues to hold if s is constant or decreasing over the period of annuitization. This occurs if the initial value of s is small and the required level of utility, $\overline{U}$, is large. If the initial value s_1 is sufficiently large relative to the expenditures required to attain $\overline{U}$, then the smoothing implied by risk aversion may undo the result by rendering optimal consumption relatively decreasing over time. With the constraint that the only annuity available pays out a constant real amount, relative valuations are particularly difficult to calculate with standard-of-living effects, because the intertemporal effects compound the difficulty of the multiple positive wealth constraints. However, we conjecture that parameter changes that tend to defer optimal consumption will tend to increase valuation. Hence, simulated valuations should tend to be increasing in δ. Further, large s_1 should yield decreasing valuation and small s_1 increasing valuation, with both effects magnified by γ."	
2005	"Utility Evaluation of Risk in Retirement Saving Accounts," James M. Poterba, Joshua Rauh, Steven F. Venti and David A. Wise. Analysis in the Economics of Aging David A. Wise, ed. (University of Chicago Press, 2005)	This study compares evaluation of the empirical distribution of dollar wealth at retirement generated under three investment strategies to the expected utility value of wealth at retirement. Simply generating a distribution of dollar wealth "…does not adequately consider the potential cost to a retiree of the low levels of wealth at retirement that might emerge from the riskier, but higher-expected-return, strategy." By contrast, the utility-based evaluation of risk assumes "…that the value that the retiree assigns to the consumption stream after retirement can be parameterized using a simple utility function, in which utility is a function of the stock of wealth at retirement." The study compares the distribution of retirement wealth to the expected utility of retirement	The authors characterize Monte Carlo simulation as an "outcomes based" retirement planning methodology: "…'outcomes-based' financial planning software…enables clients to determine the probability of reaching retirement wealth goals. These software programs are based on Monte Carlo simulations of future wealth accumulations." Although the Monte Carlo approach provides valuable information, "Results that portray the 'picture' of retirement wealth risks provide no a priori way to describe how households or groups of

	pp. 13 – 58.	wealth for three strategies: 100% inflation indexed bonds, 100% S&P 500 and a 50-50 mix. The utility function is the commonly used CRRA function: $$U(W) = \frac{W^{1-a}}{1-a}$$ Where' a' is the coefficient of relative risk aversion. Expected utility is calculated based on the probability-weighted outcomes of simulated return histories. "The parametric utility function approach starts from the premise that a household's relative risk aversion can be characterized by a single parameter. Conditional on this parameter, it is straightforward to characterize the optimal portfolio strategy for the household. This approach assumes away the problems associated with eliciting a household's preferences with regard to risk, and it requires strong parametric assumptions about the form of the household's utility function…The parametric utility function approach can potentially provide some guidance on the extent to which observed portfolio choices can be reconciled with the optimizing choices of households that are trying to maximize their expected utility." [Note: expected utility is a weighted average. It is not always valid for investors exhibiting strong state preference utility]. However, as the authors acknowledge: "The states of nature in which [investment wealth is] low are likely to be states of nature in which other wealth balances are also low….virtually all of the balance sheet components may exhibit some covariance." At any level of risk tolerance, utility can be directly compared by translating the distribution of retirement wealth (Z) into certainty equivalent wealth levels: $$Z_{strategy} = [EU_{strategy}(1\text{-}a)]^{1/1\text{-}a}$$	households might evaluate…distributions and thereby decide which portfolio strategy to pursue. At the heart of this difficulty is the question of how households evaluate small probabilities of low retirement plan balances." Using a CRRA utility of wealth function presents a host of evaluation difficulties: "We are concerned more generally that choices predicted by the CRRA function may be a poor guide to actual behavior when the distribution of wealth outcomes includes values near zero."

		Thus expected utility calculated across the entire distribution is translated to a certainty equivalent wealth measure by using the CRRA utility of wealth function. Return sequences are simulated using a bootstrap methodology—sampling with replacement from the historical return distribution. The planning horizon for wealth accumulation is 35 years. At log utility [a = 1], the all equity portfolio exhibits a certainty equivalent value approximately three times as great as the 100% indexed bond portfolio. The certainty equivalent value of a 50-50 mix is between 80 and 85 percent more. However, as risk aversion increases, the certainty equivalent value for risky portfolios decreases" "At a risk aversion of four, the certainty equivalent of an all-stock portfolio allocation is only slightly greater than that of an all-index bond allocation, but the value of a fifty-fifty portfolio remains considerably larger in certainty equivalent terms." The authors observe that the cumulative density function for the distribution of utility values for a risk-neutral household [a = 0] is convex. "As risk aversion increases, the distribution of utility diverges more and more from the distribution of wealth, and it becomes clear that raising risk aversion puts more weight on the negative outcomes in the left tail of the potential retirement wealth distribution. The second derivative of the CDF rises as risk aversion increases. When a = 4, the CDF is highly concave, as the low retirement wealth outcomes generate very low utility outcomes." However, a 50-50 mix of stocks and index bonds is the preferred portfolio for most levels of risk aversion: "A value of a greater than eight is needed for a household to prefer all index bonds to a fifty-fifty index bond-stock mix. For a 2.75, a household prefers the fifty-fifty mix to an all-stock portfolio." If the expected return on stocks is decreased by 300 basis points, this has a profound effect on the utility of portfolio strategies for investors with high levels of risk aversion: "For a = 4, for example, the certainty equivalent of an all-stock allocation falls substantially *below* that of the all-index-bond portfolio…." However,	

		"…we still find that only highly risk-averse investors would choose not to hold corporate stocks."	
2005	Real Longevity Insurance with a Deductible: Introduction to Advanced-Life Delayed Annuities (ALDA) Moshe A. Milevsky North American Actuarial Journal vol. 9 no. 4 (2005) , pp. 109 – 122.	Thesis: Although there is almost universal agreement among economists regarding the financial benefits from immediate annuities, they have not proven popular with retirees. The paper discusses a "concept product" (which Milevsky suggested to insurance firms) called an advanced-life delayed annuity (ALDA). This is a product that would be purchased at an early age, funded with small periodic premiums, and designed to provide inflation-adjusted payouts beginning at age 80 or more. The product would have no non-forfeiture values (i.e. early cash out options) and would provide no survivor benefits (i.e. life insurance). The product is akin to an insurance policy with a large deductible. In this case, the policy is designed to mitigate a catastrophic loss of purchasing power for those who have the 'misfortune' of living too long. Milevsky reviews the arguments for the dominance of annuities over other risky assets by virtue of their "mortality credits" (the living inherit the assets of the dead, conditional on survival). Mathematically, if 'p' is the probability of survival, the return on an annuity is [((1+r)/p)-1] which is strictly greater than r (either risk free return or stochastic return). He also reviews the hypotheses designed to explain the "annuity puzzle" : bequest motives, high administrative loads and costs, pre-existing annuities (i.e. Social Security), and inflexible annuity payouts (payouts are fixed in either nominal or real terms, or they are linked to an index—thus forcing consumers to hold other marketable wealth). Milevsky favors the explanation that "people shun life annuities simply because they want to maintain control of their assets." This means that a sudden lump sum conversion of wealth to an annuity income stream will never be popular. Hence, the only workable alternative is gradual or slow annuitization wherein an annuity can be funded by small payments over working life. If $a_x(r\|T)$ is the factor of a real $1 per year life contingent annuity purchased at age x under a real interest rate of r, then the annuity will	The concept of an ALDA continues as a theme in the literature. See, for example, "Making Retirement Income Last a Lifetime," Stephen C. Sexauer, Michael W. Peskin, and Daniel Cassidy [2012], "Consumer Preferences for Annuities: Beyond NPV," Suzanne Shu, Robert Zeithammer, & John Payne [2012], and "Evaluating the Advanced Life Deferred Annuity—An annuity people might actually buy," Guan Gong and Anthony Webb [2012].

		pay \$1 per year increased by the CPI conditional on survival to age s: ($_sp_x$). In continuous time: $$a_x(r\|T) = \int_0^T e^{-rs}({}_sp_x)ds$$ In Milevsky's examples, the ALDA commencement age ranges from x=65 to 85 and the purchase age ranges from y=35 to 45. Thus, the net single premium for an actuarially fair annuity is: $$NSP = e^{-r(x-y)}a_x(r)({}_{x-y}p_y)$$ The real interest rate in the above equation covers two periods (the period of accumulation (x-y) and the period of payout (years after x). Thus, two different r's might be used; or, a real yield curve might be used for discounting. Milevsky fits a continuous function to the discrete mortality table (he uses the SOA 2000 annuity mortality table which is a loaded table used to reserve against anti-selection), assumes 3.25% real interest rate, and calculates the NSP of a unisex annuity. For example, the NSP for an annuity paying \$1 real income commencing at age 80 and purchased at age 40 equals \$1.665. In the transition from a NSP to a periodic premium payment during pre-retirement years, the calculated premiums are in real dollars (i.e. a variable nominal dollar premium). This eliminates the danger of a long-term asset / liability mismatch. The periodic premium calculation divides the NSP by the following denominator: $a_y(r\|x\text{-}y)$ which spreads the NSP over the x-y premium paying period. The annuity factor (a) in the numerator is subscripted by x (the payment start date) while the annuity factor in the denominator is subscripted by y (purchase date). However, in both cases, the interest rate (r) is the real rate. The NSP of \$1.665 in the above example, becomes a real yearly periodic	

		premium (RPP) of $0.0779 (.0779 x 12.8 = $1.00) or the annuity pricing factor at age 40 equals 12.8. Milevsky makes additional observations regarding the likelihood that purchasers will irrationally lapse the ALDA prior to the payout date. The lapsation curve functions like an interest rate to reduce the premiums required to fund the annuity for non-lapsing buyers. Lapsation reduces the NSP numerator (fewer people using the benefit) and will reduce the denominator (fewer people paying into the ALDA). The net effect is a total reduction in the NSP. Additional pricing variations might include stopping premium payments at some age prior to the start of payments (i.e. benefits start at age 80 but payments stop at age 65). In any event, the ALDA is a vehicle to acquire a late-in-life real income based on a dollar cost averaging strategy.	
2006	"Asset Allocation with Annuities for Retirement Income Management" Paul Kaplan The Journal of Wealth Management vol. 8 no. 4 (Spring 2006) pp. 27- 40.	The author uses simulation to determine the relationships among (1) asset allocation, (2) withdrawal rate and (3) time horizon for portfolios of cash, bonds & stocks. Each year within the horizon is weighted by the probability of death within that year. This yields a weighted average success probability for each withdrawal rate. As the withdrawal rate increases, the allocation to equity must increase if the portfolio is to maximize the probability of success (i.e., not running out of money). Thus, the author maps out an efficient frontier between withdrawal rates and success probability assuming the simulation-based risk model identifies the best allocation for each withdrawal rate. Adding annuities to the mix increases the success probability. However, the article assumes a high $7.50 nominal-dollar payout at male age 65 per $100 single premium. Excess income in early years is invested in risky asset portfolio. In later years, the portfolio makes up the difference due to decreases in the annuity's purchasing power.	The author uses three models of inflation—constant, 1period lagged, and 2period lagged autocorrelated models. Success probabilities differ under each model. "…the role that annuities play in an optimized strategy depends largely on how we choose to model inflation." The author uses normal distribution assumption for simulation model investment return parameters.
2006	"Monte Carlo Mania," Robert D. Curtis, Chapter 8 Retirement Income	Monte Carlo approaches to retirement income planning emerged "…to address the problem of returns-sequence risk. Instead of guessing at any particular return sequence, Monte Carlo explores all possible sequences." The outcomes are compared to client-specified goals or	The author asserts: "The assumptions in every financial program reflect the opinions and biases of the designers." He recommends the following (somewhat tongue-in-cheek) disclosure statement:

	Redesigned: Master Plans for Distribution Eds. Harold Evensky and Deena B. Katz (Bloomberg Press, New York) 2006, pp. 117 - 140	targets to determine the probability of success for a particular asset management strategy or election. However, the Monte Carlo approach has several problems: The Black Box: Return distribution usually assumes a specific pattern / historical volatility is used to project future volatility. The Meaning of the Projection: Usually a Monte Carlo analysis produces results that are "all-or-nothing. Strategies are on autopilot for long planning horizons, no spending adjustments are considered, and projections often mail to consider the magnitude of shortfalls: "…a shortfall of $1 is counted the same as a shortfall of $100,000." Hidden Risks: These include health care expenses, concentrated stock risks, and life-expectancy risk: "What possible sense does it make to tell your client that she can spend more money now because you're assuming in some of the Monte Carlo iterations that she'll die early? How does a person di 'some of the time'?" To address the above-listed weaknesses, the author suggests ranking goals in order of importance to clients and to sequentially model the goal hierarchy to see which goals can be met at a high order of probability. If all goals are met, then the surplus acts as a safety margin. An alternative to Monte Carlo is stress testing in which the model deliberately creates "bad timing" situations in order to determine the economic consequences of poor market performance. Additionally, "…we can stress test their plan by extending life expectancy by five years."	"Projections are based on many unprovable assumptions for future returns and do not reflect the risks of long-term care, premature death, tax increases, changes in Social Security, longevity, concentrated stock positions, or changes in your goals."
2006	"Annuities: Now, Later, Never?" Benjamin Goodman and Michael Heller TIAA-CREF Institute Trends and Issues (October 2006), pp. 1 – 19.	The article begins by setting up a model comparing a systematic withdrawal plan to a fixed annuity where the amount withdrawn equals the payout from an annuity. If the retiree withdraws an amount equal to the annuity payment, there is a greater than 50% chance that the portfolio will be depleted assuming a 23-year life-expectancy-based planning horizon. The authors then extend their excel spreadsheet model to address the question of the impact of delaying annuitization. Assuming that an age 65 annuitant elects to withdraw the equivalent of the annuity income stream from the investment account (assuming the account earns the	Focuses on the "costs" of delaying annuitization. Under certain static interest rate assumptions, the increase in mortality credits will not produce an increase in annuity income because the non-annuitized fund shrinks due to the necessity to fund the initial annuity income stream through a self-annuitization program. Fund shrinkage offsets increases in the annuity's mortality premium. Under a regime of rising interest rates, however,

		same rate of interest as the annuity), a delay of 5 years will deplete a $100,000 account to $86,982 which will purchase an annuity income of $7,025. This compares to the annuity income of $7,390 which was available at age 65. Thus the decision to delay results in an approximately 5% decrease in income assuming a static interest rate environment. A ten-year delay shows a projected income loss of over 15%. However, if interest rates rise, the model indicates a much different outcome. Assuming a 25 basis point increase in interest rates over a 5 year period, the decision to delay annuitization at age 65 is an increase in yearly income of more than 7%. However the ten-year delay strategy will result in lower payments: "If one firmly believes that interest rates are almost certain to increase significantly in the very near future, there may be good reason to postpone annuitization for at least a few years."	the opposite result may hold. These observations highlight the importance of a portfolio monitoring and surveillance system that enables the retired investor to condition asset management decisions on the current economic environment rather than on autopilot rules.
2006	"Immediate Annuity pricing in the Presence of Unobserved Heterogeneity," Kim Balls North American Actuarial Journal vol. 10 no. 4 (2006), pp. 103 - 116.	The author, an actuary, begins by noting that annuities are generally not underwritten: "insurance companies pad annuity mortality tables expecting that only people with optimistic views of their life expectancy will purchase payout annuities." Likewise, "the available annuities may be unattractive to those potential policyholders with the lowest survival expectations." The author notes: "This antiselection by policyholders, together with the insurance company response, could combine to eliminate the payout annuity market." To further explore the possibility of a viable future market for annuities, the author develops an 'annuitant mortality model': an "optimal consumption decision is derived for the case where no annuity market exists. Using the same utility framework, we derive the threshold price for an immediate annuity." The model focuses primarily on the health state of the potential annuitant as opposed to chronological age. The health state in any period can improve or deteriorate according to a transition probability [Markov transition matrix]. The transition probabilities are calibrated by maximizing a log likelihood function from the data in the U.S. Census population mortality tables for the years 1900 – 1990. The paper incorporates the health transition process into an economic model in which the utility of consumption in each period is influenced by the	The utility model is a variation on the Fisher utility model that allows for increased discounting of contingent future consumption. Ball's model is also closely aligned with the Gordon Pye [2012] model which develops an age-based / health-adjusted consumption rule for retirees. The author notes: "The newly retired policyholder finds for the first time is his or her life that good health is expensive, at least in the sense that it lengthens the expected cost of providing for retirement." Article provides a good mathematical expression of "brevity risk."

		health state of the annuitant. Bequests are not considered. Per-period utility, conditional on health state j is defined as: $$U_j(c_t) = K_c + (\alpha+1)^{-1}(\theta_j c_t)^{(\alpha+1)}$$ Where c is consumption, θ is a health-state consumption modifier, $\alpha \leq 0$ is the risk-aversion parameter, and K is a utility constant whose sole purpose is to make utility positive. The author acknowledges that the choice of the utility function (constant relative risk aversion—the risk aversion coefficient is independent of the level of wealth) "…is based on mathematical simplicity…Relative risk aversion (risk aversion divided by consumption or wealth) is constant." "The discounted net present expected value of total lifetime utility is a function of the health state and the current level of assets. The budget constraint is that net assets must always have a positive or zero value. The utility expectation over lifetime with respect to the health state and current consumption requires a set of Bellman equations. In a market without access to annuities, the results indicate that "…the optimal consumption is a set percentage of remaining assets, with the percentage varying by health state." The withdrawal percentage is changing with respect to both wealth and health: "…the optimal behavior is to consume a very small fraction of assets in the healthy states. Note that the policyholder in this model cannot experience financial ruin. Unlike the self-funded retirement model of Milevsky and Robinson (2000), consumption is reoptimized each period as a function of current wealth. For each health state, the policyholder consumes a constant proportion of his or her wealth in each time period. Although continuing good health may result in continually decreasing consumption, assets are never exhausted." In a market where the consumer has access to annuities, "The value of wealth is simply the utility value of the annuity purchased with the assets." The model solves for the payout rate that the annuity must provide so that the utility of annuitized wealth exactly equals the expected utility of lifetime consumption given the investor's current health state, risk aversion coefficient, and time preference discount rate.	

		For example, an investor in the best health state, with a risk aversion coefficient of -1.50 and a 2% time discount preferencing rate requires an annuity payout rate of 5.41% per year when the expected rate of return on invested assets is 5%. For a person in the worst living health state (health states range from 10 to 0 with 0 representing the state of death and 1 representing the worst living state) is indifferent to maintaining the portfolio with an expected return of 5% or annuitizing wealth with a payout rate of 11.88% Changing the risk aversion coefficient to reflect greater risk aversion [-0.75] changes the indifference rates to 5.54% and 13.79% respectively: "…the higher the risk-aversion parameter, the greater the policyholder's premium for retirement risk protection." Changing the health-state consumption modifier [θ] to a state-dependent variable indicates that the constant annuity is no longer optimal. The optimal annuity has extra payments as lower health states emerge. The model confirms that there "…are advantages to delaying annuitization, particularly when market returns available to the policyholder are superior to those available in the form of an annuity. However, the effect here is heterogeneous, depending also on the expected longevity of the policyholder." "The major finding of this paper is that the annuity puzzle may be explained by a heterogeneous population with relatively minor risk aversion, facing a product offering that is slightly uncompetitive."	
2006	"The Management of Decumulation Risks in a Defined Contribution Pension Plan," Russell Gerrard, Steven Haberman and Elena Vigna North American Actuarial Journal vol. 10 no. 1 (2006), pp. 84 – 110.	This publication continues the 2004 research wherein the authors used dynamic programming to solve the HJB equation for which the control variable was the portfolio's asset allocation. In this work, the control variable is the consumption path. This differs from many previous studies in which post-retirement consumption is "…the exact amount that a level annuity bought at retirement would provide." They contrast their study to the 2004 Milevsky and Young paper ["Annuitization and Asset Allocation"] which found the optimal consumption path and optimal annuitization time under a power utility function. The M-Y paper, however, did not include a bequest motive and did not follow a target-based approach to the decision-making problem. In the instant	A good case for the importance of ongoing monitoring and dynamic asset allocation. The work assumes that current wealth is insufficient to purchase an annuity at the desired level of consumption and that the retiree elects to invest in risky assets with the hope of achieving a more favorable future income stream. The problem is a risk-ruin-to-achieve-future-wealth-goal where the objective is to maximize the probability of attaining the goal while minimizing the probability of bankruptcy. The question is whether this is a prudent strategy to follow for trust-owned

		paper, however, a target exists for both income and bequest The model of wealth evolution follows the Gerrard-Haberman-Vigna 2004 paper where wealth follows a standard Brownian motion process in continuous time. At retirement, "we assume that the reasons that push the retiree to choose the option of deferring annuitization are both the hope of being able to purchase in the future an annuity higher than the pension income provided by immediate annuitization at retirement and the ability to bequeath wealth in the case of death before annuitization. It seems reasonable to assume that the individual has a certain target in mind, which is pursued during the drawdown phase. In particular, we shall assume that the pensioner has two different kinds of targets: a target for the size of the fund and a desired level of income to be consumed. Deviations from the targets will result in a loss." Loss is defined as "disutility." Disutility arises whenever the income is below the ideal target. Furthermore, disutility also arises if the income drawdown exceeds the ideal target because (1) removing the excess amount from the portfolio decreases the chance of being able to generate a fund sufficient to buy an annuity for future target income and (2) increases the chances of ultimate financial ruin. Disutility also arises when the fund is below target wealth level *and* when it is above. If the fund exceeds the target wealth level this is evidence that the investor has exposed wealth to unnecessary risk in previous periods. By this same reasoning, disutility arises if the annuity purchased at a future date is either above or below the target income. Positive utility arises if death occurs prior to annuitization because of the ability to fulfill bequest objectives. Where b_0 is the target level of income prior to the optimal time of annuitization and b_1 is the target level of income post annuitization—where $b_1 > b_0$—the loss function is expressed as: $L(t,x) = e^{-\rho t}[u(F(t) - x)^2 + v(b^0 - b(t))^2]$ Where upsilon and nu are positive constants that are "interpretable as weights given to the desire to monitor the growth of the fund and the daily consumption; and rho is the subjective intertemporal discount	portfolios. Once a target income is selected, deviations either above or below the target create disutility. Compare to argument that an income stream should be acquired at the lowest possible cost. Scott, Sharpe & Watson "The 4% Rule—At What Price?" [2008] Authors argue for constant monitoring of fund size over time relative to its ability to support performance targets. This mirrors the "hurdle race" asset management approach. Risky asset positions are maintained during times of a shortfall in wealth. The allocation is the Merton optimum. Thus, over time, the shortfall is "cured" by continued exposure to risk. Contrast with Milevsky & Robinson [2005]. Note, also, the conclusion of Sid Brown that the risk of the position increases as the time available to correct a shortfall decreases. Extreme leverage may be required if the time horizon becomes short and the shortfall amount remains large. Indeed, under these conditions, the optimal allocation to the risky asset can become significantly greater than 100%. "The Risk and Rewards of Minimizing Shortfall Probability" [1999]. The work also illustrates the necessity to consider asset allocation and retirement spending jointly rather than separately. Note: Compare with "Optimal Asset Allocation and

		factor. Nu is the relative importance given to ongoing consumption; upsilon is the relative importance given to achieving the wealth target. The ratio of v/u, which is reflective of the preferences of a specific investor, is an important factor in optimizing the controls. Note: the v/u ratio is the Utility of Consumption / Utility of Wealth ratio. The authors discuss the seemingly anomalous fact that the quadratic loss function penalizes performance above the target. They justify use of the function as follows: "…the choice of trying to achieve a target and no more than this has the effect of a natural limitation on the overall level or risk for the portfolio: once the target is reached, there is no reason for further exposure to risk, and therefore any surplus becomes undesirable." Note: This parallels the argument of Sharpe—the existence of a surplus suggests that the optimal income target was not funded in an efficient manner ["The 4% Rule—At What Price?"]. Contrast this approach with that of attempting to optimize surplus—positive growth with minimal variance. Finally, compare to the "retirement benchmark" concept discussed in "Making Retirement Income Last a Lifetime," Stephen C. Sexauer, Michael W. Peskin, and Daniel Cassidy [2012]. A disutility function that is quadratic in the level of wealth and in consumption is a variant of problems commonly found in the literature of stochastic optimal control problems: "…with a proper and not unreasonable choice of the target, the fund never exceeds the target, and the optimal running consumption never exceeds the targeted consumption." In the event that the investor survives to age T—the optimal age to annuitize—the "cost" function [K(x)] becomes: $K(x) = we^{-\rho T}(b_1-kx)^2$ And, in the event that the investor does not survive to age T, the utility of bequest [M(t,x)] is expressed as:	Ruin-Minimization Annuitization Strategies: The Fixed Consumption Case," Moshe A. Milevsky, Kristen S. Moore & Virginia R. Young. This paper also assumes that an investor lacks sufficient current wealth to buy an annuity to fund a desired income target. The preferred strategy is the one that minimizes the probability of lifetime ruin where ruin is a function of the Wealth/Consumption ratio. When wealth is sufficient to purchase an annuity, annuitization is immediate—an optimal stopping time problem. When wealth is insufficient, a positive allocation is made to the risky asset within the investor's risk tolerance as measured by the probability of ruin. Also compare to Korn & Krekel paper "Optimal Portfolios with Fixed Consumption or Income Streams" [2002].

		$$M(t,x) = e^{-\rho t}nx$$ Where k is the amount of annuity that can be purchased per 1 unit of capital, w measures the importance of achieving the target annuity income and n measures the weight put on the ability to make a bequest. The authors argue that although generating an amount for bequests and generating an amount sufficient to purchase the future annuity income target operate in tandem, nevertheless they are not redundant terms: "…we think that the constant monitoring of the fund size over time has an importance by itself, in that pensioners can check the performance of the fund against predetermined targets…." Furthermore, the utility of bequest is linear in wealth where the disutility terms for the wealth level and consumption are both quadratic. The objective is "…to minimize over all possible investment and consumption choices the expected discounted future loss from retirement…." This is done by finding the optimal value function in terms of the allocation to the risky asset (y*) and the consumption choices (b*). The solution takes the form of a quadratic equation: $$V(t,x) = e^{-\rho t}(A(t)x^2 + B(t)x + C(t)).$$ Where A, B and C satisfy a system of differential equations with the following boundary conditions: $A(T) = wk^2$; $B(T) = -2wkb_1$; and, $C(T) = wb_1^2$. The authors define a "Natural" Target Function for the fund. This consists "…of precisely the amount of money required to fund consumption at the fixed level until the time of compulsory annuitization and then to achieve the final target pursued." The interpretation of the natural target F(t) is as follows: "If a sum G(t) were invested at time t in the risk-free asset, then the interest payments would cover consumption at rate b_0 until the age of compulsory annuitization, and thereafter would permit the purchase of an annuity paying the required amount b_1 per	

		unit time. Therefore, the level G(t) can be considered to be a sort of "safety level" for the personal needs of the pensioner...." NOTE: annuitization is compulsory in England at an age no greater than 75. It may be optimal to annuitize prior to this age, however. Given than b_1 is assumed higher than b_0, the shortfall of the fund from G(t) is always strictly positive prior to the time of optimal annuitization, the optimal consumption never exceeds b_0 and the amount invested in the risky asset is always positive. The optimal amount to allocate to the risky asset is the Merton Optimum times the ratio of the shortfall to total wealth [X*(t)] in a portfolio operating under optimal controls: [G(t) – X*(t)] ÷ X*(t): "We notice that the optimal amount invested in the risky asset…is proportional to the shortfall S(t), which is the difference between the safety level and the fund level. This result is similar to a result found by Brown (1999); solving two 'survival problems' (maximizing the probability of reaching a 'safe region' before occurrence of ruin and minimizing the discounted penalty paid upon going bankrupt) he finds that in both problems the optimal policy implies investing in the risky asset a proportion of the (positive) difference between the amount needed for being in the safe region and the fund level." A simulation study tests the model under various market conditions where consumption occurs weekly. The authors find that the ratio of periodic consumption to the evolving status of the fund [v/u] is important "…both for the optimal controls and for the distribution of the final annuity." Increasing the weight given to consumption results in increased consumption and in an increase in allocation to the risky asset. At low values for v, the optimal policy is to consume little at the beginning of retirement. Likewise, increasing the final target annuity (b_1) relative to the annuity able to be purchased at time b_0 also produces a riskier optimal investment strategy. At some parameter values, initial consumption becomes negative which would "…be undesirable to many pensioners…." The simulation analysis reveals the frequency at which a variety of undesirable events occur. These include running out of money prior to time T and failing to achieve a wealth level sufficient to purchase the	

		target future annuity. Whereas the option to self-annuitize involves investments in risky assets, the distribution of final annuity results is heavily dependent on the choice of the Sharpe Ratio. The authors stress that the optimal strategies are not rules of thumb but rather depend on the circumstances and preferences of the investor. One investor may have income sources outside of the find and may opt for low consumption and a high future annuity target. Another may require a regular income from the fund and has no other sources of income. Various bequest motives and liquidity preferences are also important.	
2006	"A Problem with Monte Carlo Simulations," J. Harold Bell & Craig L. Israelsen. Paper Presented at the Academy of Financial Services Annual Meeting, October 2006, Salt Lake City, Utah.	Monte Carlo simulations are based on software programs developed by mathematicians: "The software calculates the interactions between the asset lifetime probabilities and the client lifetime probabilities to determine how the client will be impacted by the unsuccessful outcomes associated with the asset probability curve. This effect is integrated over the life of the analysis to calculate a success coefficient for the analysis. The success coefficient is defined as the probability that both members of the couple will live out their expected lifetimes without running out of money. In addition to answering the obvious question that every client asks, this parameter provides a means to rank and compare, on an apples-to-apples basis, the relative effectiveness of different retirement scenarios." The authors claim that most commercial software packages require input of an "annualized" return parameter for the mean. However, the output is expressed as an average return not a geometric return. This difference can result in substantial underestimation of the likelihood of success.	Failure rate = (portfolio depletion) Π (longevity).
2006	The Calculus of Retirement Income [Chapters 9, 10 & 12] Moshe A. Milevsky (Cambridge University Press), 2006.	Chapter 9: This chapter develops the topic of "Sustainable Spending at Retirement." It asserts that the three most important factors in planning for a sustainable retirement income are: (1) spending rate, (2) asset allocation, and (3) mortality considerations. The "probability of ruin" is an appropriate risk metric if the investor is interested in understanding the relative importance of these factors as well as the tradeoffs between	Milevsky poses the question: "Should a 65-year-old plan for the 75^{th} percentile, the 95^{th} percentile, or the end of the mortality table?" Note: The stochastic present value of the retirement income as measured by the cost of a single premium immediate annuity constitutes the

		them. The Stochastic Present Value of targeted retirement income is a key concept for determining the likelihood of ruin—the probability that retirement income is not sustainable. For a perpetual life investor (e.g., endowment), the present value of sustainable income is 1/R where R is an assumed constant earnings rate. When the planning horizon is also fixed, the PV of the target income stream is: PV = $\sum_{i=1}^{T}\frac{\$1}{(1+R)^{i}} = \frac{1-(1+R)^{-T}}{R}$ However, when both the rate of return and the planning horizon are uncertain, the liability must be expressed in the form of a Stochastic Present Value equation which is a sum-product function: SPV = $\$1\sum_{i=1}^{\tilde{T}}\prod_{j=1}^{i}(1+\widetilde{R_J})^{-1}$ As the return interval becomes smaller, the summation operator becomes an integral and the stochastic present value becomes a probability function in continuous time. If the stochastic present value of the targeted retirement income stream is greater than the present value of the retirement portfolio, it is not likely that the lifetime spending target can be sustained. Assuming portfolio returns are lognormally distributed and that the force of mortality [λ] is exponentially distributed, Milevsky's thesis is that the probability of ruin is best determined by fitting the parameters to a Reciprocal Gamma Distribution [RGD]. The exponential mortality rate [Pr (T,s) = $e^{-\lambda s}$] gives the expected value of the remaining lifetime random variable as E[T] = 1/λ; and the median expected lifetime as ln(2)/λ. Milevsky describes the characteristics of the RGD: it is a two-parameter distribution with a domain from zero to infinity. The probability that the SPV of the retirement income stream is greater than or equal to initial wealth—i.e., the probability of ruin—is the RGD: Pr[SPV ≥w] = GammaDist$\left(\frac{2\mu+4\lambda}{\sigma^{2}+\lambda}-1,\frac{\sigma^{2}+\lambda}{2}\mid\frac{1}{w}\right)$	feasibility condition. Mathematically, this is a free boundary problem. The risk metric of interest is not shortfall risk as projected by Monte Carlo, bootstrap or other risk models. Rather it is a current observable. A portfolio with a current asset value less than the annuity cost is technically insolvent. In this essay, Milevsky uses an analytical formula [Gamma Distribution] as a tool to measure the probability for portfolio "insolvency." However, the analytical tool returns a probability measure, and is less precise than an annuity cost benchmark. The sustainable spending rate calculated at one point in time can differ significantly from that calculated at another—by virtue of aging or change in health. [Note the similarity to the model presented in "Immediate Annuity pricing in the Presence of Unobserved Heterogeneity," Kim Balls]. Additionally, the SPV inputs are mean returns. If the initial sequence of realized returns are above the mean—during the period when many dollars remain in the portfolio—the recalculated sustainable rate could be much higher. The reverse is true for initial returns below the mean [or, volatility above the mean]. "…the first decade of retirement is the most crucial one in determining whether your retirement plan will be successful. Intuitively, a poor performance from the market when you have a lot of wealth at stake has a more detrimental impact overall." This may be an argument for beginning a retirement allocation conservatively and then increasing risk if the initial return sequence is favorable. Later papers will assert that sequence risk is present throughout

		Where the first term in the RHS bracket is the alpha parameter and the second term is the Beta parameter—i.e., this is a two-parameter distribution. The expected value of the SPV is $(\mu - \sigma^2 + \lambda)^{-1}$. For example, assume a median life span for a 65-year-old of 18.9 years. Given the equation $e^{-18.9\lambda} = 0.5$, the value for $\lambda = \ln(2) \div 18.9 = 0.0367$. [ln(.5) = -0.6931, ln(2) = 0.6931, and 0.6931 ÷ 18.9 = 0.0367] If μ = 7% and σ = 20%, then the expected value of the SPV is 1/(0.07 – 0.04 + 0.0367) which is an average of $15 required for each dollar of yearly consumption. $15 gives 50-50 odds of lifetime income sustainability. This gives the upper bound for retirement spending in that it tells the retiree the amount of spending which "on average" can be sustained over a lifetime. Inverting the equation for the Pr[SPV ≥ w] allows one to solve for the sustainable spending rate at a given probability of ruin—e.g., 5%, 10%, etc. <u>Chapter 10</u>: This chapter develops the topic of "Longevity Insurance Revisited" with special attention given to the election to delay annuitization. Under what circumstances should the investor annuitize? When should the investor opt to delay annuitization? The analysis begins with the mathematics of a single period, "renewable" tontine. Assuming (1) that the tontine funds are invested at a possibly stochastic rate of return [X] for one year, and (2) that the surviving tontine participants divide total wealth after one year, the expected value of arrangement is: $(1 + E[X] / {}_1p_x) - 1$. Whereas ${}_1p_x$ is always less than 1, the rate of return from a tontine arrangement is always enhanced by a mortality credit. However, the standard deviation of the return on the arrangement also increases by $SD[X] / {}_1p_x$. The chapter attempts to answer the question: when is the extra risk worth taking? Investments provide an alternate method for generating retirement cash flow. Therefore, the answer to the question largely depends on the extent to which the mortality credits influence the investment (asset allocation) decision.	retirement [e.g., 2011--"An Aged-Based, Three Dimensional, Universal Distribution Model Incorporating Sequence Risk," Larry R. Frank Sr., John B. Mitchell, David M. Blanchett] An annuity is defined as a "…stream of income [that] consists of three parts: the return of principal, the interest, and other people's money (the mortality credits)." $\gamma = -w \frac{-\gamma w^{-(\gamma+1)}}{w^{-\gamma}} = -\frac{U''(w)}{U'(w)}$ = The Arrow Pratt measure of relative risk aversion. Note that, by the division law of exponents, you can subtract –γ from –(γ+1) to yield an exponent of 1. Simply cancel the 'w's to arrive at the expression γ = γ QED.

		The author provides the example of an investor, with initial wealth W_0, wishing to select an asset allocation that earns the highest amount of return subject to a low probability of earning a return below a pre-specified threshold: Max $E[W_1]$ by varying the allocation to the risky asset (Θ) and to the risk free asset (1 – Θ), subject to the constraint that $Pr[W_1 \leq W_0] \leq \varepsilon$. This is a shortfall probability metric. Assuming returns are log normally distributed, this is equivalent to the probability that a standard normal random variable will take on a value less than or equal to c. For example, assuming an expected return of ν and a standard deviation of σ, the probability that the portfolio in time period 1 is less than its initial value is: $$\Phi\left(\frac{W_0 - W_0(\theta(1+\nu) + (1-\theta)(1+R))}{W_0\theta_\sigma}\right)$$ To find the largest value of Θ, or the largest risky asset allocation that will produce a shortfall probability no greater than ε, requires that we evaluate the inverse of the function at the selected value of ε—the 'z-score' value. Thus, a 1% confidence value for the left tail of the inverse distribution $\Theta^{-1}(\varepsilon) = -2.326$. Solving for Θ algebraically yields the equation: $$\theta^* = \frac{R}{-\sigma\theta^{-1(\varepsilon)-}\,(\nu-R)}.$$ However, when the R variable is replaced by a tontine arrangement, the optimal allocation to the risky asset (Θ**) is: $$\theta^{**} = \frac{R+(1-(1\mathrm{Px}))}{-\sigma\theta^{-1(\varepsilon)-}\,(\nu-R)}$$	

		The numerator contains an additional ${}_1p_x$ term that increases in value as the one-year survival probability decreases. Comparing the two equations for optimal Θ reveals that the survival probability factor in equation two increases the allocation to the risky asset; and, equally important, increases the proportion of the portfolio invested in the risky asset as the age of the investor increases: "Remember that the older age implies a lower probability of survival (${}_1p_x$) and hence a higher investment return value of $(1+v)/({}_1p_x) - 1$, even if this is at the expense of a higher standard deviation $\sigma/({}_1p_x)$." Each retiree has the option to delay annuitization in favor of electing an investment program—self-annuitization. Election to delay makes sense if, given a portfolio withdrawal equal to the cash flow provided by an annuity purchased at time zero, the remainder of assets can be invested so that an annuity providing equal or greater cash flow can be purchased in the future. If a_x is the pricing factor that determines the cost of an annuity, and G is the growth rate of investments net of withdrawal, the a one period decision to delay annuitization makes sense if: $a_x(1+G) -1 \geq a_{x+1}$ Milevsky argues that a measure called Implied Longevity Yield [ILY] is a good measure of the value of the option to delay. He notes, assuming constant annuity pricing factors, that the cost of providing a dollar of lifetime income at older ages is less than the cost at younger ages. The ILY is the investment return needed to withdraw an equivalent annuity cash flow at age y *and* to have a sufficient portfolio value to purchase, at an older age, an equal or greater lifetime annuity cash flow. Of course, the risk of the delay strategy is either (1) a lower-than-anticipated investment return fails to produce sufficient wealth to purchase an equivalent income stream, or (2) that the future annuity cost increases because of changes in interest rates or longevity expectations. Any self-	

		annuitization plan must produce an earnings rate at least λ above the insurer's annuity pricing rate r in order to make the self-annuitization plan reasonable. The investor must beat the insurer's pricing benchmark (e.g., cost of capital, bond return, etc.) plus the mortality credits (λ). Solving for the ILY provides insight into the feasibility of a successful deferral—how likely is it that the investor can earn the ILY or better at a level of risk that is comfortable? <u>Chapter 12</u>: This chapter focuses on the topic of "The Utility of Annuitization." Annuities provide the guarantee of a lifetime income and Milevsky explores what such a guarantee is worth. He distinguishes between 'value'—what is the guarantee worth to the owner, 'cost'—how much does it cost to manufacture, and 'price'—what do you pay for the guarantee. If the consumer values the guarantee and finds that its market price is below his or her subjective valuation, then the consumer can enhance welfare by purchasing the guarantee. Welfare enhancement—i.e., Utility—is the satisfaction or comfort that the guarantee provides. The traditional method of quantifying utility is: 1. Model the potential magnitude of loss absent the guarantee 2. Model the probability of loss 3. Combine the results in a "mathematical representation" called a utility function. A commonly used utility function is constant relative risk aversion: $U(w) = \frac{1}{1-\gamma} w^{1-\gamma}$ Where gamma (γ) is the investor's personal level of economic risk aversion. Plugging in a gamma value of 0.5, for example, delivers a utility value of 2 x (square root of wealth). For other gamma values—2, 3, 4…—the utility function takes on the form: $U(w) = \frac{1}{-2w^2} \quad \frac{1}{-3w^3}$ etc.	

Another way to view gamma is the investor's sensitivity to changes in wealth. The first derivative of a given level of wealth is $U'(w) = w^{-\gamma}$, while the second derivative (less than zero for a concave function) is $U''(w) = -\gamma w^{-(\gamma+1)}$. Thus:

$\gamma = -w\frac{-\gamma w^{-(\gamma+1)}}{w^{-\gamma}} = -w\frac{U''(w)}{U'(w)}$ = The Arrow Pratt measure of relative risk aversion.

The subjective value (utility) of an insurance guarantee is the maximum amount that the investor is willing to pay: I_γ. This amount is the point at which the investor is indifferent between having or not having the insurance guarantee. For any loss L:

$$U(w - I_\gamma) = pU(w-L) + (1-p)U(w).$$

Additionally, for any $\gamma > 0$, the investor is willing to pay more than the actuarial fair value of the insurance guarantee contract.

Milevsky uses this basic explanation of utility to launch a discussion of the utility of consumption in the face of an uncertain lifetime. He provides the following example—assume that you wish to spend a $1 by consuming a portion of it at the end of period 1 and the remainder at the end of period 2. The consumption amounts are, therefore, C_1 and C_2. In order to spend C_1 you must survive the first period. The probability of survival is p_1. Likewise, the probability of surviving the second period is p_2 with $p_1 \geq p_2$. The one-period interest rate is R. The investor's dilemma is to avoid spending too much in period one lest consumption in period two is inadequate; or, spending too little in period one lest there be unspent money because he failed to survive until the end of period two. Assuming logarithmic utility, the objective function is:

$$\text{Max } E[U] = \frac{p_1}{1+\rho}\ln[C_1] + \frac{p_2}{(1+\rho)^2}\ln[C_2]$$

		Subject to the budget constraint: $$\$1 = \frac{C_1}{1+R} + \frac{C_2}{(1+R)^2}$$ Where 'rho' is the subjective discount rate—the higher the value, the more early consumption is valued over later consumption. Note: this is also a model for Fisher Utility where consumer impatience leads to a "front-loaded" retirement portfolio distribution policy. Incorporating the Lagrangian of the budget constraint into the objective function and setting the partial derivatives of C_1, C_2 and λ to zero, yields the optimal consumption for a log-utility investor in each period: Optimal $C_1 = \frac{p_{1(\rho R+R+\rho+1)}}{p_2+ p_1\rho+ p_1}$ and Optimal $C_2 = \frac{p_2(1+2R+R^2)}{p_2+p_1\rho+p_1}$ However, when annuities are added to the mix, the budget constraint is relaxed because survival probabilities now appear in the numerator: $$\$1 = \frac{\rho_1 C_1}{1+R} + \frac{\rho_2 c_2}{(1+R)^2}$$ Consumption is now C_1/ρ_1 and C_2/ρ_2 which allows for more consumption in each period. All else equal, the presence of an annuity enhances investor utility. Milevsky points out that "classical arguments" for annuitization treat the option of the timing of annuitization as a boundary problem. The optimum time for exercising the option to annuitize is as soon as it is feasible to do so. However, "this framework assumes de facto that the budget constraint will not improve over time." In reality, "…by taking a chance in the risky asset, the future budget constraint may improve. In other words, it might be worth waiting, since tomorrow's budget constraint may allow for a larger annuity flow and greater utility." The	

		fundamental contrast in approaches, according to Milevsky's formulation, is a boundary problem v. an option valuation problem: "…uncertainty about future interest rates, mortality, insurance loads, and product design all increase the value of the option to delay." "…my main argument is that retirees should refrain from annuitizing today, because they may get an even better deal tomorrow." [Note: may not be true for utility which is not expressed as a power function]. Milevsky prices the real option to defer annuitization [RODA] as the amount of additional wealth the investor would require today if the RODA was not available. If the price is negative, then immediate annuitization is optimal; if the price is positive, it may pay to delay. The value of the option is a function of the investors risk aversion—the higher the risk aversion, the more inclined the investor becomes to select the option to annuitize wealth: "…as long as the risk-adjusted odds of a favorable change in the budget constraint are high enough, the option to wait has value."	
2006	"Life-Cycle Asset Allocation with Annuity Markets: Is Longevity Insurance a Good Deal?" Wolfram Horneff, Raimond Maurer, Michael Stamos Working Paper 146, Michigan Retirement Research Center (University of Michigan) 2006.	The authors examine the optimal lifetime asset allocation between stocks, bonds & annuities in the presence of labor income, capital market risk and mortality risk. They provide a brief survey of conclusions from "asset allocation" literature: 1. Time-varying investment opportunities create hedging demands ("buffer stock" of wealth") 2. Human capital is a non-tradable asset that is a close substitute for bonds. Younger workers are, therefore, "overinvested in bonds" and have a demand to hold buffer wealth in stocks. As the value of human capital decreases, the demand to hold stocks also decreases. 3. Annuities offer a mortality credit (bond yield + mortality credit). Investment in annuities, however, comes at the "cost" of illiquidity (irreversibility) and of foregoing the expected equity risk premium. Annuity purchases subject the buyer to "brevity risk." The article considers the option to annuitize over an investor's complete	The authors' claim that "annuities define an asset class with certain age dependent return characteristics because payments are conditional on survival." Annuities help investors make risk substitutions: 'brevity risk' is traded for 'longevity risk.' Investing is a process of a prudent exchange of risk.

		life cycle. This contrasts with Milevsky and Young (2006) who focus on purchase options only during retirement (a continuous time barrier control type problem that involves a numerical solution of a variational inequality [exercise annuity purchase option if mortality credit > benchmark bond yield]; or, with Cairns, Blake & Dowd (2006) where the retiree has a one-time option to annuitize all wealth at the beginning of retirement. The election to annuitize problem is addressed within the Hamilton, Jacobi, Bellman equation structure. Assuming a CRRA function [Epstein - Zin utility] they observe that low relative risk aversion leads to stock allocations. However, low elasticity of intertemporal substitution does not lead to increased annuity purchases. Investors with low EIS are more concerned about short-term consumption smoothing as opposed to hedging long-term longevity risk. The authors' model optimizes the value function with respect to Consumption and Bequest preferences given the investor's survival probability, relative risk aversion, EIS, bequest motive, and time-preference discount factor. Optimization of asset allocation is a function of four state variables: Age, Income (follows an exponential function of "permanent income," time, and income shocks); Income from previously purchased annuities, and wealth-on-hand [(a function of amount in Bonds, Stocks, annuity payouts and labor/retirement pension income) less consumption]. This is a dynamic asset allocation problem that cannot be solved analytically. Normalizing the state variables as ratios of income reduces the problem to a three-dimensional state space. The optimization is across a grid built from the three discretized variables (normalized wealth, normalized annuity income from past annuity purchases, and time). For each grid area, they calculate the optimal allocation and the corresponding value function where the calculations (numerical constrained maximizations) assume a log-normal distribution of the multivariate probability density function. In brief, annuities tend to first crowd out bonds and then stocks over	

		time. The annuitization decision shifts according to bequest motives, annuity costs, pension income, and wealth level. Welfare gains from annuities can be substantial even in the face of bequest and costs (an 8.01 percent equivalent increase in financial wealth at age 60). The higher the EIS, the higher the utility gains. The increase in welfare is directly attributable to the extra consumption financed by the mortality credit.	
2006	"Decision Rules and Maximum Initial Withdrawal Rates," Jonathan T. Guyton & William J. Klinger Journal of Financial Planning vol. 19 no. 3 (2006), pp. 48 – 58.	Authors argue that implementation of withdrawal management rules allows for safe initial withdrawal rates substantially higher than the 4 to 5 percent recommended by many practitioners. Long lists of rather convoluted decision rules emerge from studying the outputs of a simulation analysis which, unfortunately, is based on highly unrealistic assumptions [fixed planning horizons, lognormal asset return distributions, static co-variance matrix, etc.]. The gist of the article is that, if a retiree is willing to be flexible with respect to the percentage amount of yearly portfolio withdrawals, two benefits are forthcoming: 1. Reduced likelihood of portfolio depletion during the planning horizon (Capital Preservation Rules) 2. Prudent method to calculate the amount of increase in portfolio withdrawals during periods of surplus (Prosperity Rules). Through a judicious application of the rules, retirees can effectively manage a variety of risks (probability of ruin, decrease in purchasing power, taking too little out of the portfolio for fear of future depletion, and large swings in their income stream). The initial benefit is to increase the initial withdrawal rate while concurrently decreasing long-term risks by application of portfolio management rules: "For retirees seeking a virtually bullet-proof withdrawal plan, choose an initial withdrawal rate where the probability of success and the median purchasing power maintained are both at least *the 99 percent confidence standard*." Given an allocation of at least 65% to equities, the initial withdrawal rate can range between 5.2 and 5.6 percent over a 40-year period.	Authors specifically address the prudence/probability issue (p. 51): "But a stochastic approach is not without its limitations, either. In particular, one specific question cannot be avoided: How high a probability of success is high enough?" In one sense, this oft-cited article is an elaborate exercise in data mining (developing a decision rule for every portfolio scenario). Many of the decision rules, although designed to mitigate the long-term probability of ruin, have little or no relationship to the maximization of investor utility over the entire range of the return distribution. Consider: "No withdrawals are taken from any equity following a year with a negative return if cash or fixed income assets are sufficient to fund the required withdrawal" This means that an investor is willing to incur greater relative equity risk during down market periods. This is an extremely uncommon risk tolerance function. Authors end up maximizing the probability of a "safety-first" portfolio. The Guyton/Klinger approach can be compared to Gordon Pye's approach in his book: The Retrenchment Rule[2012].

2006	"Asset Allocation and the Transition to Income: The Importance of Product Allocation in the Retirement Risk Zone," Moshe A. Milevsky and Thomas S. Salisbury IFID Research Paper (September 27, 2006).	The retirement risk zone is "the 5 to 10 years before and after the onset of cash-flow generation—a.k.a. retirement…." The authors emphasize that success or failure in sustaining retirement income over a lifespan is highly dependent on the sequence of returns—sequence risk. Thus, "…retirees face a unique risk that has to be managed in a very different way." At the time of retirement, wealth tends to be at its highest level. Poor market performance at this time is "…more devastating, all else being equal, when you have a lot of wealth at stake." The authors express the retiree's quandary as follows: If the asset allocation at retirement is conservative, the portfolio forfeits the opportunity for growth and increases the risk that it will not be able to keep pace with long-term inflation. However, if the allocation is aggressive, there is a probability that a bear market will force a permanent reduction in spending. The way out of this dilemma, in the authors' opinion, is to collar the portfolio by purchase of out of the money puts financed by sale of out of the money calls. They acknowledge that the portfolio collar strategy also has a downside: "The strategy reduces the probability of retirement ruin by limiting the magnitude and frequency of (large) negative returns, but this comes at the expense of reducing the upside potential of the portfolio." Additionally, the dynamic hedging strategy involves constant monitoring and substantial transaction costs. As a fallback position, the authors suggest that the retiree consider a variable annuity offering a guaranteed minimum withdrawal benefit.	The authors take the position that there is no objective, bright line, standard for prudent retirement decision making: "…it is important for us to emphasize that we do not advocate or promote a particular retirement spending rate—either 4%, 5% or even 7% or 8%--as being optimal or sub-optimal under any set of circumstances. Obviously, the lower the spending rate the higher the chance of sustaining this standard of living during retirement. But, at the same time, a lower spending rate means precisely that: less spending. Some retiree might accept a 10% or 20% chance of running out of money as a chance worth taking, in exchange for a higher initial spending rate. Others might want a lower 5% to 10% risk of ruin, but they will have to survive on lower levels of retirement income and withdrawals. This is clearly a personal risk tolerance issue."
2006	"Hatching a Nest Egg," John Nersesian Financial Planning vol. 36 no. 2 (February, 2006), pp. 95 – 97.	This short article addresses a financial planner audience. It cautions that advice appropriate for clients in the wealth accumulation stage of investment, may not be appropriate for clients in the portfolio distribution stage. Planners should be aware of four factors: 1. Sequence of Returns, 2. Return Volatility (consistency of returns) 3. Withdrawal rate policy (fixed or percentage of corpus) and, 4. Method of distribution (from taxable accounts, tax-favored accounts, or a mix).	Good basic introduction to return sequence risk: "…it is not the average rate of return that determines the success or failure of a plan, but the way in which those returns are produced." Also discusses the concepts that are known as "Variance Drain:" "…the true importance of diversification is that it can lead to lower volatility, which can leads [sic] to a more consistent investment experience."

2006	"DIESEL: A System for Generating Cash Flow During Retirement," Stephan Q. Cassaday Journal of Financial Planning vol. 19 no. 9 (September 2006), pp. 60 – 65.	The author claims that a DIESEL process (Dividends, Interest, and Equity Select Liquidations] can improve the sustainability of portfolios making periodic distributions. A key piece of the process is to communicate to clients the difference between income—dividends and interest—and cash flow. Cash flow makes no distinctions regarding the origin or character of distributed funds. Retirees should be interested in cash flow—"retirement paychecks"—rather than interest from fixed income investments. The author claims that an optimal asset allocation historically would have permitted withdrawal of 7% of the initial portfolio value adjusted for a fixed 3% per year increase in the distribution amount ("an annual inflation adjustment"). Furthermore, the article argues that bonds are not safe because they produce lower rates of return and because, other than TIPS, they do not offer inflation protection. Additionally, longevity is an important consideration in planning a retirement portfolio. The DIESEL withdrawal method assumes quarterly withdrawals and quarterly portfolio rebalancing to the asset allocation target. The case-example portfolio is back-tested over the period 1972 through July 2005. The author claims that the DIESEL allocation is superior when compared to other portfolio allocations: a 100% bond allocation, a 50-50 stock/bond allocation and a 33-33-33 stock/bond/cash allocation. However, the DIESEL allocation is the only one that is broadly diversified (7 asset classes). For example, the author compares the "randomized results" over the 33 years (a bootstrap?) of the DIESEL portfolio with a 100% DFA Small Cap portfolio and concludes that the DIESEL portfolio is preferred because it has a lower failure rate (9.10% vs. 13.30%). The 100% bond allocation had the highest failure rate (65.20%).	A good discussion of the difference between accounting income and retirement cash flow. Note: The author claims that the DIESEL system is superior but fails to disentangle the effects of asset allocation and withdrawal/rebalancing strategies. Results are reported in nominal dollars rather than constant dollars giving the impression that the DIESEL system produces inflation-protected results. The net effect of the analysis is simply to demonstrate that a diversified portfolio is usually better over the long-term than a concentrated portfolio.
2006	"A Note on Parameter Elasticities in Monte Carlo Retirement	The authors note that the literature on a sustainable portfolio withdrawal rate is usually based on an initial withdrawal rate that is a percentage value of the portfolio at the beginning of retirement. Thereafter, periodic withdrawals either continue the nominal dollar	The article cites Rory Terry's article "The Relation Between Portfolio Composition and Sustainable Withdrawal Rates, Journal of Financial Planning, May 2003, 64-71 which suggests that a failure rate

	Planning Simulations," Woerheide, Walt and Taylor, Don. Paper Presented at the Academy of Financial Services Annual Meeting, October 2006, Salt Lake City, Utah. Copies may be obtained from Walt Woerheide at The American College or Don Taylor at Penn State - Brandywine.	amount of the initial withdrawal, or inflation-adjust the original withdrawal amount to provide a real annuity lifetime income. The key factors in deciding the initial withdrawal percentage are: (1) expected ROR, (2) volatility, (3) expected inflation, (4) client attitude towards risk, and (5) role of the investment portfolio income in the total retirement budget. A commonly used criterion for "safety" is the requirement that the portfolio should last for at least 30 years. Safety is defined as an acceptably low failure rate. In terms of the probability of failure, the article examines the elasticity of the failure rate to the portfolio Rate Of Return, Standard Deviation and the withdrawal rate. The authors run simulations assuming mean rates of return in the range of 4 to 12% in one-percent increments; SDs in the range of 10 to 20% in one-percent increments; and withdrawal rates ranging from 3 to 8% in one percent increments. After producing the simulation outputs (2,000 trials) the failure rate dependent variable is regressed on the three independent variables. As a follow up, the dependent variable is regressed on each independent variable in isolation, and finally, on the spread between ROR and withdrawal rate. Both ROR changes and withdrawal rate changes have greater impact on the failure rate than changes in portfolio standard deviation. As a final step, the authors calculate elasticities by multiplying the factor Beta (the regression coefficient on the independent variable) by the mean value of the independent variable. This product is then divided by the mean of the dependent (failure rate) variable. The elasticities are -1.43% failure rate with respect to ROR; +0.44 failure rate with respect to SD; and +1.50 failure rate with respect to initial withdrawal rate. The study concludes: "In general, if one can increase the expected rate of return on a portfolio by one-hundred basis points and hold the increase in the standard deviation to less than 600 basis points, then the client is likely to be better off in terms of the probability of portfolio failure."	of less than 1% is what most clients find acceptable. Also notes Milevsky and Abaimova (2006) observation that different commercially-available Monte Carol simulation programs produce different solutions even when given the same inputs.
2006	"Dynamic retirement withdrawal planning." R. Gene Stout & John B.	The authors opine that "The withdrawal phase of retirement planning may well require more professional guidance and expertise than the accumulation phase." The essay builds on the literature of "adaptive withdrawals" and cites Bengen [2001] and Pye [2001] as examples of	This article continues the research seeking to develop rules for distribution planning. See, for example, "Decision Rules and Maximum Initial Withdrawal Rates," Jonathan T. Guyton & William J.

	Mitchell Financial Services Review vol. 15 no. 2 (Summer 2006) pp. 117 – 131.	articles studying the interaction between flexible withdrawal rules and portfolio sustainability. The essay tests the economic impact of control limits on retirement income and portfolio values: "The control limits signal withdrawal rate decreases necessary to reduce or delay impending financial ruin and affordable withdrawal rate increases to avoid excess accumulation." The authors simulate a portfolio allocated 65% to U. S. Large Cap stocks and 35% to U.S. intermediate term government bonds. They use historical real—constant dollar--returns from 1926 through 2004. Taxes and fees are not considered. Although they do not describe their simulation algorithm in detail, they note: "There is no serial correlation in historic large-cap stock returns, but intermediate-term government bond returns show a first-order autocorrelation coefficient of 0.23. While the cross-correlation between the asset classes and the serial correlation within intermediate-term bond returns are not statistically significant, they are maintained in the Monte Carlo simulations." The article presents results from three models each of which builds more complexity into the simulation analysis. Model One assumes the investor withdraws a fixed fraction of the initial portfolio during each year over a 30-year planning horizon. This is essentially an autopilot distribution formula that does not incorporate a factor for mortality. At a real withdrawal rate of 4%, there is a 7.53% probability of ruin and an average terminal portfolio value equal to 3.20 times the initial value. Model Two introduces mortality into the analysis. The force of mortality makes the age at which an investor elects to retire a significant factor for determining the probability for retirement success. For example, a real withdrawal rate of 4% generates the following results: <table><tr><td>Retirement Age</td><td>Probability of Ruin</td><td>Average terminal value as % of initial value</td></tr><tr><td>55</td><td>6.35%</td><td>3.26</td></tr></table>	Klinger [2006]. However, intelligent portfolio monitoring calls for application of judgment more than for application of rules—especially when the rules are developed by data mining. The Stout & Mitchell article illustrates a monitoring process based on annuity-like principles—amortization of a fund over the expected life span. The monitoring is not based on the annuity pricing factor. Note: the 2001 Pye article ["Adjusting Withdrawal Rates for Taxes and Expenses"] argues that withdrawals should be adjusted to recognize the potentially significant impact of expenses and taxes on portfolio sustainability. For example, if expenses are assumed to be 1 percent of portfolio value, withdrawing 3.5 percent of the initial value of the portfolio has about the same sustainability as a 4 percent withdrawal with no expenses. The concept of "…an absolute limit on the maximum and minimum withdrawal rates. In the event of continuing poor market performance, the minimum withdrawal rate is necessary to prohibit the remaining portfolio from being amortized over the remaining expected life by a withdrawal rate below a minimally acceptable lifestyle. Retirees would be advised to set their personal minimum withdrawal rate at the lowest minimum they can tolerate as the alternative is ruin" stands in direct contrast to Pye [2012].

60	4.18%	2.56
65	2.47%	2.07
70	1.19%	1.70

Model Three incorporates both mortality and the portfolio withdrawal controls. "The controls are necessary to avoid overreactions to unanticipated market gains, which may subsequently threaten the survivability of the portfolio. A well-designed set of withdrawal change controls, adjusted for the retiree's risk tolerance in pursuit of higher real withdrawal rates, would reduce or delay financial ruin and would allow affordable increases in the withdrawal rate, as well." Model Three has three types of controls: (1) portfolio deviation thresholds, (2) withdrawal adjustment rates, and (3) absolute withdrawal rate limits. The portfolio deviation threshold is a precondition to making a withdrawal rate adjustment. An upward deviation establishes the minimum portfolio balance for a withdrawal rate increase; a downward deviation establishes the maximum balance necessary for a withdrawal rate decrease. Thus, depending on the deviation thresholds, no change in portfolio withdrawal is permitted "…until the portfolio reveals signs of significant over or under performance." The withdrawal rate adjustment factor "…limit(s) the extent of withdrawal rate changes in response to a signal that the portfolio value has broken a threshold of deviation….Withdrawal adjustment rates reduce the chance of overreacting to improved portfolio performance that is subsequently reversed, and they might also be used to smooth the income effects of poor portfolio performance." Finally, the absolute withdrawal rate control constrains the range of permissible withdrawal rates.

The portfolio deviation threshold is the first control variable computed. The required amount for the threshold is the amount required to sustain the existing withdrawal amount over the retiree's remaining life. This is the economic equivalent of the present value of an annuity due for the expected life span [note: "expected remaining life" is an average derived

		from a mortality table. 50% of retirees are expected to have a longer actual life span]. The annuity interest rate factor is not a function of the current interest rate environment as is the case for a commercial annuity underwritten by an insurance company. Rather it is the portfolio's rate of return that has been historically realized, on average, for the overlapping periods of time in the empirical data sample where the applicable time period is equal to the retiree's remaining expected life. Thus the upper and lower deviation thresholds are dynamic and depend on the amount by which the current portfolio value exceeds the discounted value of the lifetime annuity income stream. Assuming an increase in portfolio value, "no increase in the real withdrawal rate is permitted unless the portfolio balance exceeds…" the upper threshold. An upward threshold of 1.0 is the equivalent of a cushion of 100%. Once the portfolio deviation threshold requirements are satisfied, the second control—rate of withdrawal adjustment—comes into play. The third control is "…an absolute limit on the maximum and minimum withdrawal rates. In the event of continuing poor market performance, the minimum withdrawal rate is necessary to prohibit the remaining portfolio from being amortized over the remaining expect life by a withdrawal rate below a minimally acceptable lifestyle. Retirees would be advised to set their personal minimum withdrawal rate at the lowest minimum they can tolerate as the alternative is ruin." Applying the controls to an age 60 retiree, results in a 4.43% probability of ruin, a 6.63% average withdrawal rate and an average ending portfolio value of 1.07 times the initial amount. The authors claim that their results indicate that models with a fixed long-term horizon may overstate the possibility of portfolio depletion. Limiting withdrawals to fixed initial rates constrains the investor holding an over-performing portfolio while jeopardizing an investor holding an under-performing portfolio. "Integration of dynamic withdrawal management and mortality provides a flexible withdrawal management strategy."	
2007	"Money in Motion: Dynamic Portfolio	Although this study considers the management of financial wealth through variable annuities, it nevertheless provides interesting insights	Argues that annuities are a separate asset class—mortality contingent claims.

	Choice in Retirement," Wolfram J. Horneff, Raimond H. Maurer, Olivia S. Mitchell, and Michael Z. Stamos Working Paper #243 Center For Financial Studies Goethe Universitat (2007).	into dynamic utility maximization. The investor must make asset location decisions (invest financial wealth in a variable annuity or in liquid form—individual stocks, bonds or funds) as well as asset allocation decisions. They examine a strategy of optimal gradual annuitization and dynamic asset management over the retirement period. Unlike many previous studies that assume (1) a one-time decision to purchase annuities at the date of retirement, and (2) a fixed asset allocation throughout the planning horizon, their model allows the retiree to rebalance the asset mix optimally according to state variables. The study compares the pricing of fixed and variable annuities. Although the payout of variable annuities is uncertain, they offer a "mortality credit" because participants in the annuity pool forfeit the 'annuity unit' assets at death. The mortality credit cannot be replicated through direct investments. [Note: this suggests that annuities represent a distinct asset class]. The price of a variable annuity is a function of the number of annuity units required to support an expected future income stream. Each unit is discounted for an assumed rate of return and the probability of survivorship—discounted, probability-weighted payout. In the case of a variable annuity, the assumed interest rate refers to the "shrinkage rate" over time in the number of annuity units. If the risk-free rate is 2% and the assumed interest rate (AIR) is 4%, then the number of fund units will evolve exponentially (in this case, decline). Thus the AIR replaces the risk-free rate in the discounting factor. Following a "standard approach," they assume a probability distribution for annuity payouts assuming that the price of each annuity unit follows a log-normal i.i.d. process: $\text{Payout} = \frac{number\ of\ annuity\ units}{(1+AIR)} \Pi(Returns)$ Where expected payout is the mean of the log-normal distribution: $\text{E[Payout]} = \frac{n_1}{(1+AIR)^{t-1}} e^{\mu t+1/2(\sigma)^2 t}$	Note: to maintain an effective overall constant stock/bond mix means that a retired investor will decrease the percentage of stocks in the portfolio as he ages. The desired bond exposure depends on the ratio of financial wealth to pension wealth—the higher financial wealth relative to pension wealth (where a pension is a proxy for a bond), the higher the demand for bonds in the private portfolio. Thus a Constant Mix approach is fully compatible with a decrease in stock ownership as a function of age.

		Setting the AIR low compared to the expected annuity fund returns produces low initial payments with a likelihood of higher benefits in the future. The higher the AIR, the higher the initial payments and the lower the later payments. If an investor has initial wealth S_0, then the dynamic portfolio choice problem is to maximize utility as a function of initial wealth, risk aversion (1 ÷ coefficient of risk aversion = willingness to engage in intertemporal consumption substitution), time preference discount factor, bequest motives, and consumption goals. The investor, with possible existing pension income and annuity income, faces a dynamic optimization problem as the value of the stochastic input variables change. There are no analytical solutions to this type of problem, and the authors use a “backward induction” method assuming CRRA preferences. The CRRA preference results in a dynamic rebalancing of stock/bond positions either within the annuity or outside of it. At any given time, the optimal rebalance strategy is “countercyclical”—stocks are purchased after a bear market and sold after a bull market. This is a constant-mix asset management approach. However, if pension payments have a bond-like character, as the investor ages, the present value of the pension wealth decreases.	
2007	“Spending Buckets and Financial Placebos,” Moshe A. Milevsky Research Magazine (June 1, 2007)	A short piece designed to refute the wisdom of the retirement planning strategy that calls for a retiree to (1) fund the initial two to three years income from a fully funded cash account and (2) to invest the remainder aggressively. “…some commentators have expressed the view that placing a few years’ worth of retirement spending needs into safe investments—and then planning on not touching the remaining funds in the event of a bear market—can somehow avoid the ruinous impact of a poor sequence of investments [sic] returns. A fringe element of this sect believes that if markets decline retirees should simply be counseled to take only income from their bond allocation and then ‘wait for the stock allocation to recover’ and thus avoid selling at a loss. I believe these strategies are an optical illusion at best and create a potential for grave	An alternative view on the ‘two-fund’ solution.

		disappointment at worst. If you are unlucky enough to earn a poor sequence of initial returns, so-called bucketing of your retirement income is *not* a guaranteed bailout." [see, for example, "Decision Rules and Maximum Initial Withdrawal Rates," Jonathan T. Guyton & William J. Klinger 2006] Milevsky provides a counterexample of two investors with equal initial wealth. Each investor employs a strategy with the same expected return. One, however, invests in a balanced mutual fund while the other segregates assets into a cash account and an all equity account. Because the standard deviation of the all-equity account is higher than that of the balanced mutual fund, a prolonged bear market in which money is withdrawn only from the cash account means that an increasing percentage of retirement wealth is exposed to equity risk. The end result of this example is that the mutual fund investor achieves greater downside protection in a bear market; while the cash-all equity account investor achieves better results in a bull market. Of course, "…if neither of them withdraws any money whenever markets are down they will have immunized themselves against sequence-of-returns risk." Milevsky concludes: "Don't confuse your cash-flow-generation strategy with your asset-allocation policy. If you decide to adopt the so-called buckets approach to retirement income planning, then be aware that your total asset allocation and implicit exposure to equity will fluctuate unpredictably over time."	
2007	"Prudence—From Fuzzy to Precise," Steven M. Fast, Christiana N. Gianopulos & Leiha Macauley Representing Estate and Trust Beneficiaries and Fiduciaries ALI-ABA Course of Study	This is largely an exploration of the legal meanings and practical implications of the trustee duty of impartiality. Where the governing trust instrument is silent on how trust resources are to be used for the benefit of beneficiary classes, the trustee must not favor the interests of one class over another. Indeed the Uniform Principal and Income Adjustment Act [UPAIA] gives the trustee the power to adjust when a total-return investment strategy unduly skews results in favor of one beneficiary class: "An adjustment is required if the trustee applies the UPAIA's general rules for treating receipts as income or principal and determines that the result would not otherwise be impartial, based on what is 'fair and reasonable to all of the beneficiaries." §103 This would	Is portfolio bifurcation, actually a form of "asset segregation?" If so, the result is suboptimal in that the aggregation of each half (income half + remainder half) yield a total portfolio allocation that is inefficient. One reason for inefficiency is that the correlation between each half is difficult to estimate. In addition to paying the current beneficiary a sum of money sufficient to discharge reasonable expectations a trustee is faced with the task of

	(2007), pp. 171 – 190.	happen "if the income component of a portfolio's total return is too small or too large because of investment decisions made by the trustee under the prudent investor rule." §104. The problem is not determining why to use the adjustment power to avoid shortchanging a beneficiary class, but how to use it. The article reviews both court decisions addressing the duty of impartiality and legal commentaries thereon. The consensus seems to be that fair treatment is not necessarily equal treatment: "Rather the trustee must treat the beneficiaries equitably in light of the purposes and terms of the trust. However, the UPAIA does not provide a method for determining what an equitable payout should be. One straightforward method is to bifurcate the portfolio as if one-half was invested for the benefit of the income beneficiary and one-half for the benefit of the remaindermen. The total accounting income from both halves is aggregated and divided by 2. This is the "fair" net income payout to the current beneficiary. The authors contend that this method is administratively simple and has many advantages over creating a phantom traditional net-income portfolio [e.g., 40% stock / 60% bonds] from which an income payout benchmark can be calculated. However, even if an impartial payout formula can be established, the trustee still has the challenge of prudent investing to fund the payout. Should prudent investing preserve only nominal values or should it preserve inflation-adjusted values? A further issue is portfolio sustainability given the distribution target or distribution formula. One tool for assessing the risk of portfolio ruin is Monte Carlo simulations. If the trustee has the power to make discretionary distributions, then this duty is independent of the duty to invest prudently. However, many trust documents are silent on these matters. What is the trustee's accountability? The authors note that the initial characterization of investment prudence as a process seems to be giving way to judgments arising through a careful probability analysis. This in turn gives rise to the question of what is an acceptable rate for a successful financial outcome for each beneficiary class: "…all things	comparing two distributions: the distribution of aggregate lifetime income to the current beneficiary (the current beneficiary's "share" of enjoyment from initial trust principal) and the terminal wealth of the trust portfolio (the remaindermens' "share "of enjoyment from initial trust principal). Distributions can be compared by calculating mean squared differences over percentile ranges or by more formal tests such as Kolmogorov-Smirnov test and the Anderson-Darling test.

		being equal, the trustee's duty to the remainder interests is to preserve original principal value adjusted for inflation. That dictates the total return target. The investment strategy must then have a reasonable probability of realizing that target. If it does not, the trustee's initial determination of the amount to be paid out to income beneficiaries is by definition too high and needs to be scaled back to where there is a reasonable probability of both paying it out and preserving inflation-adjusted principal value."	
2007	Lifetime Financial Advice: Human Capital, Asset Allocation, and Insurance, Roger Ibbotson, M. Milevsky, Peng Chen, Kevin Zhu The Research Foundation of the CFA Institute (2007).	Wealth consists of human + financial capital. Human capital initially faces mortality risk, but as its importance diminishes, this risk is replaced by longevity risk to accumulated financial capital. Longevity risk is "uncompensated risk," and investors should be willing to pay a premium to avoid it. Annuities provide longevity insurance but have several potential drawbacks: (1) nominal payments have decreased purchasing power, (2) purchase decision is irrevocable, (3) contracts do not provide liquidity, and (4) an annuity locks in a payment reflecting the interest rate environment at the time of purchase. The authors develop a retirement model to solve for optimal allocation to fixed & variable annuities, a risky asset, and a risk-free asset. For retirees with great wealth, and for retirees that wish to maximize bequests, there is a zero allocation to annuities across all [CRRA] risk-aversion parameters. For investors wishing to maximize consumption, all wealth is allocated to annuities. Intermediate cases evidence a blend of assets depending on the value of the risk-aversion parameter. The interesting finding is that the higher the investor's risk aversion, the lower the allocation to annuities. This is because of the tradeoff between the probability of early death and the lack of any utility from forfeiting bequests. The decision on when to annuitize is a function of the available mortality credit. At any age, the credit is approximated by the following formula: $Credit = q_{\mathrm{x}}(1 + \mathrm{q}_{\mathrm{x}}) + Rq_x(1 + \mathrm{q}_{\mathrm{x}})$	Third restatement of Trusts: The Prudent Investor Rule defines prudent investing largely in terms of either avoiding uncompensated risk or developing a rationale indicating why the portfolio will benefit from retaining it. If an annuity program can eliminate longevity risk and a life insurance purchase can eliminate bequest risk, what is the cost of a pure actuarial solution? How do the costs and risks of the solution compare to investment alternatives? A critical issue is trustee duty to avoid inappropriate or unwarranted costs. Contrast the actuarial risk-free solution to the investment risk-free solution offered in Scott, Sharpe & Watson "The 4% Rule—At What Price?" [2008] The article suggests that increased wealth means decreased demand for annuities. But, it also suggests that increased risk aversion decreases the demand for annuitization. Highly risk-averse investors value liquid wealth to protect against economics shocks of illness and other emergencies.

		This credit must be higher than the annuity costs (profits, commissions, administrative, opportunity cost of setting aside reserves, etc.). In the authors' opinion, there is little justification for annuitization prior to ages 65-70. This rule may be modified in the case of extremely high long-term rates. "Annuitization is akin to purchasing a bond with amortized principal and souped-up coupons." Finally, life insurance can mitigate bequest risk.	
2007	"Measuring and Controlling Shortfall Risk in Retirement" Gary Smith & Donald Gould The Journal of Investing vol. 16 no. 1 (Spring 2007), pp. 82 – 95.	The authors argue that the probability of outliving wealth depends on the interaction between spending policy and asset allocation to investments with uncertain returns. When considering asset allocation, high equity exposures increase median terminal values at the cost of increasing shortfall risk. However, low equity exposure both decreases median terminal value and increases shortfall risk. The authors' model uses joint life mortality with fixed withdrawal rates of 3%, 4% and 5% of initial wealth. They simulate a two-asset portfolio assuming a lognormal distribution with historical parameter values. They discuss several measures of risk: (1) the standard deviation of terminal wealth, (2) semi-variance of periodic returns, (3) percentile measures of terminal wealth, (4) probability of a zero value portfolio prior to death, and (5) probability of a "disaster" outcome [Roy's safety first strategy to minimize the probability of falling below the lower bound level]. Graphs illustrate the tradeoff between shortfall probability and median terminal wealth. This is an "efficient frontier" where the "minimum variance" portfolio is the "safety-first" portfolio. Equity exposure mitigates shortfall risk until increased portfolio variability outweighs increased expected return. The authors discuss the differences between unlucky draws from a stable probability distribution vs. a substantial change (for the worse) in the probability distribution itself. They simulate such a permanent change (by decreasing expected return by 3%) and by applying the lower expected returns over random 15-year segments of the historical data.	An important discussion of changes in the probability distribution and benefits of eschewing an autopilot withdrawal rate (e.g., fixed annuity amount adjusted for inflation). High equity loading increases terminal value but also increases shortfall risk [Note: theme of the risk of increasing risk]; low equity loading both decreases median terminal value and increases shortfall risk. Equity exposure mitigates shortfall risk until increased portfolio variability outweighs increased expected return. Compare with Milevsky & Robinson [2005] and with the efficient frontier with annuities developed in Kaplan [2006]. Flexibility in distributions (elasticity of consumption) is an important factor in any discussion regarding portfolio sustainability. Is retirement spending likely to be driven by strict adherence to a spending policy or by actual needs?

		The changes fail to produce a substantial difference in the weighting of equities in the minimum variance portfolio. Finally, they test the affect of a flexible distribution policy on shortfall probability. The model assumes a 50% elasticity of spending—Δ10% wealth relative to initial wealth generates Δ5% spending relative to initial spending. The flexible spending policy dominates the fixed distribution rule (smaller shortfall risk and higher terminal wealth across all allocations). Within the distribution ranges, the minimum risk portfolio generally has a 50 to 70 percent equity weighting.	
2007	"Modern Portfolio Decumulation: A New Strategy for Managing Retirement Income," Richard K. Fullmer Journal of Financial Planning vol. 20 no. 8 (August 2007), pp. 40 – 51.	The author asserts that a systematic withdrawal plan reduces future spending following bear market returns. "This amounts to managing longevity risk through spending management. This approach sacrifices the investor's standard of living in the event of poor market returns." Unlike the modern portfolio theory approach to accumulation where the investor is concerned with terminal wealth and where standard deviation of wealth is an appropriate risk metric, this is not the case for decumulation portfolios. Rather, a more appropriate risk measure is the sustainability of a threshold standard of living. Shortfall risk, relative to this threshold standard, is more meaningful to the investor. However, a probability measure of the likelihood of achieving a sustainable threshold is not an ideal measure of risk because it fails to take into account the magnitude of failure should the threshold be breached. Likewise, a frequency approach to probability—i.e., the number of unsatisfactory trials divided by the total trials—is also not entirely satisfactory: "Constructing a portfolio based on numerous phantom lives is problematic because although longevity risk management is possible using multiple real lives, it is not possible using phantom lives." Given these insights, the author concludes that the best strategy for managing retirement income risk is to annuitize when you must—but not before. The key to this strategy is to evaluate continuously the option to annuitize financial assets. By exercising the option only when it is necessary to ensure a threshold standard of living, the investor takes full advantage of the time value of the annuitization	A nice expression of the logic underlying a top-down approach where the option to annuitize is a "last-resort' safety measure. The barrier control problem is solved in terms of decreasing wealth rather than in terms of wealth hitting the first feasible time to annuitize. Under this risk-management approach, the investor monitors the cost of buying an annuity to fund the target income and compares this cost to the market value of portfolio assets. The decision becomes how much of the investment "surplus" to put at risk before exercising the option—but this is a variation on surplus optimization. The author also provides a refreshing counterpoint to the conventional wisdom that suggests a retiree must take more equity risk to increase the probability that financial losses will recover from bear markets.

		option. "The key for leveraging this optionality is setting the projected cost to annuitize the investor's desired lifetime income stream as a *wealth* goal in the objective function. Doing so effectively transforms longevity risk into investment risk, because now it is the portfolio's job to preserve the ability to annuitize the desired lifetime income stream. "By monitoring the investor's wealth relative to the current cost of annuitization, the decision to invest or annuitize can be continually evaluated by a financial advisor." Surplus optimization management requires the investor to take wealth changes into account because "…movements in the markets have a major impact on the amount of shortfall risk present in the portfolio over time. It makes sense, then, to alter the asset allocation over time to manage this risk. A 10 percent decline in value may increase shortfall risk substantially, perhaps necessitating a more conservative portfolio to reduce the risk level. A 10 percent increase in value may reduce shortfall risk, allowing for a more aggressive portfolio to serve the objective of growing wealth." This logic leads directly to a recommendation for "…a dynamic allocation strategy." The author asserts: "…when substantial cash-flow risk is present, the objective function begins to take on more of the characteristics of a cash-flow matching model." The risk management approach mirrors the hurdle race problem in which the 'provision' must exceed the cost of securing the threshold living standard through annuitization. The author terms this an "annuitization hurdle." This threshold wealth standard is a more appropriate benchmark against which to measure the risk of shortfall than complete portfolio depletion—which measures the risk of ruin: "Portfolio values below zero represent financial ruin, while values below the annuitization hurdle represent an inability to fund the desired annuity."	

2007	"Annuitization and asset allocation," Moshe A. Milevsky and Virginia R. Young Journal of Economic Dynamics and Control vol. 31 no. 9 (2007), pp. 3138 – 3177.	The authors characterize a life annuity as a product "…akin to a coupon-bearing bond that defaults upon death of the holder and for which there is no secondary market." Unlike the all-or-nothing decision faced by retirees in a defined benefit pension plan, the individual investor not faced with this institutional constraint "…will initially annuitize a lump sum—if they do not already have this minimum level in pre-existing DB pensions—and then buy additional life annuities in order to keep wealth to one side of a separating *ray* in wealth-annuity space. This is a type of barrier control result that is common in the literature on asset allocation with transaction costs." When the investor is free to annuitize a fraction of wealth, "…we find that the individual annuitizes a lump sum as soon as possible (the amount may be zero) and then acquires more annuities depending on the performance of her stochastic wealth process. If her wealth subsequently increases in value, she purchases more annuities by annuitizing additional wealth; otherwise, she refrains from additional purchases and consumes from her originally-purchased annuities, as well as from liquidating investments in her portfolio." The paper solves for the optimal annuity purchasing strategy for such an investor and develops a metric to quantify the loss from implementing a suboptimal strategy. The irreversibility of the decision to annuitize creates "…an incentive to delay that can be heuristically viewed as an option." The authors assume a two-asset economy in which the risk-free asset grows at a deterministic interest rate (r) and the risky portfolio (π) evolves according to a standard Brownian motion. Consumption is subtracted in each period and is restricted to be non-negative. The model allows for both subjective conditional survival probabilities and objective probabilities. The actuarial present value of a life annuity paying $1 per year continuously to an investor age x is written $\bar{a}_x$ and is expressed as: $$\bar{a}_x = \$1\int_0^\infty e^{-rt}\,{}_tp_x dt.$$ Defining A_s as the non-negative annuity income rate at time s following	The authors claim that Canadian annuity data indicates that the pure actuarial pricing factor is loaded on the order of 1 to 5 percent for profits, fees and commissions. This is an annuitize-as–soon-as-possible argument—a barrier control approach. It differs from Milevsky's earlier emphasis on the option valuation approach to annuitization—an optimal stopping time approach. Under the optimal stopping time approach the value of the option to annuitize is independent of wealth given a CRRA assumption. Delaying annuitization incurs risk. The decision to annuitize is, therefore, a variation on the classic risk/return tradeoff. As volatility increases the demand to annuitize increases. Contrast the bottom-up annuitize at first chance strategy to Fullmer's [2007] top down annuitize when wealth decreases strategy. The level of current interest rates relative to the magnitude of the expected equity risk premium is an important factor.

annuity purchases at that time, the investors wealth process evolves as follows:

$$dW_s = [rW_s + (\mu\text{-}r)\pi_s - c_s + A_s]ds + \sigma\pi_s dB_s - \bar{a}^{O}_{x+s}dA_s$$

where B_s is the Brownian motion process and the last term [$\bar{a}^{O}_{x+s}dA_s$] is the cost of annuity purchase at time s.

When faced with an all-or-nothing option to annuitize, the investor will annuitize all wealth at time T if the volatility of investment return is zero and if the objective hazard rate λ^{O}_{x+t} is increasing. The optimal time T for annuitization under these assumptions is when the expected return from the non-annuitized wealth portfolio equals the risk-free rate plus the hazard rate—i.e., mortality credits. Henceforth, the individual optimally consumes the annuity income.

The assumed utility function for consumption and bequest exhibits constant relative risk aversion. Under a CRRA assumption, "…the optimal time to annuitize one's wealth is independent of wealth and is, therefore, deterministic." When the value of the option to annuitize equals the expected value of the payoff from the underling portfolio, then exercise the option. This is a boundary-value problem. The change in utility with respect to the change in time is proportional to $r + (\frac{\gamma}{2})(\frac{\mu-r}{\sigma})^2 - (r + \lambda_{x+T})$. This is the difference between risk-adjusted investment return and the annuity return which consists of the risk-free rate plus mortality credits. When the risk-adjusted investment return is less than the annuity return, it is optimal for a CRRA investor to annuitize immediately. Higher levels of risk aversion and higher levels of volatility tend to lower the age for optimal exercise of the option to annuitize. Higher levels of risk aversion and higher levels of volatility tend to lower the age for optimal exercise of the option to annuitize. Finally, the authors define a metric for measuring the loss in value from annuitizing early. This is the amount of additional wealth required to compensate the investor for forcing early annuitization—i.e., "…the least amount of

		money h that when added to current wealth w makes the person indifferent between annuitizing now (with the extra wealth) and annuitizing at time T (without the extra wealth).” The option to delay runs the risk of consuming less in the future than if one exercised the option to annuitize immediately. The authors term this: “the probability of deferral failure.” The probability of deferral failure must be weighed against the probability of ending up with wealth sufficient to purchase a higher future annuity income. The upside potential decreases—because of increasing mortality credits—as one approaches the optimal annuitization age. For an individual with the capability of annuitizing a fraction of wealth at various time intervals, “…the individual’s optimal annuity purchasing is given by a barrier policy in that she will annuitize just enough of her wealth to stay on one side of the barrier in wealth-annuity space.” If wealth is spent to purchase an annuity, periodic income increases. There is a barrier in wealth-annuity space. The location of the barrier is where the marginal utility of annuity income equals the marginal cost of spending down wealth to secure it. When the benefit of the income exceeds the benefit of retaining wealth, the investor will exercise the option to annuitize a fraction of wealth to restore equilibrium. In the case of CRRA, the barrier is a ray emanating from the origin in wealth-annuity space: “…we are interested in the critical value z_O above which an individual spends a lump sum to purchase more annuity income.” The value of z_O changes with the value of risk aversion (γ) where z_O equals the ratio of wealth to annuity income above which the investor spends a lump sum of wealth to increase annuity income. Not surprisingly, as investment volatility increases, the amount annuitized increases at all levels of risk aversion. The authors indicate that their model suggests: “when we move towards an open institutional system in which annuitization can take place in small portions and at anytime, we find that utility-maximizing investors	

		should acquire a base amount of annuity income (i.e., Social Security or a DB pension) and then annuitize additional amounts if and when their wealth-to-income ratio exceeds a certain level. In this case…individuals annuitize a fraction of wealth as soon as they have opportunity to do so—i.e., they do *not* wait—and they then purchase more annuities as they become wealthier." Additionally, "…we find that individuals prior to age 70 *should* have a minimal amount of annuity income and should immediately annuitize a fraction of wealth to create this base level of lifetime income….We reiterate that individual should always hold some annuities, even in the presence of a bequest motive, as long as $z_O(t)$ is less than infinity." They conclude by discussing some of the limitations of their model. For example, both the risk-free rate and the market risk premium are constants. Thus, their model cannot provide direct insight into the wisdom of recommending that investors refrain from purchasing annuities in low interest rate environments. Additionally, the model assumes fully determined mortality rates and annuity pricing equations throughout the planning horizon. They point out, however, "…there is a growing body of empirical and theoretical literature that argues that mortality risk is priced in equilibrium. In the extreme, this would imply that if one delays annuitization, one runs the risk that annuity prices will actually increase, even though the individual has aged."	
2007	"Integrating optimal annuity planning with consumption-investment selections in retirement planning," Aparna Gupta and Zhisheng Li. Insurance: Mathematics and Economics vol. 41,	The authors assert that the paper "…is the first effort to investigate retirement planning with a full consideration given to the dynamics and interrelation of a planner's consumption, investment and annuity purchase decisions. Our investigation is in a multi-period setting, while the existing literature has mostly studied the problem as a single-period analysis. We develop a multi-period wealth evolution model and study the issues in a multi-period setting, where the relationships among annuitization time and an individual's wealth status, income status, life expectancy and degree of risk aversion are addressed." To this end, the authors create an annuity price and benefit payment	The model assumes that investors cope only with the uncertainty of return from the risky asset because the planning horizon is fixed at a few years beyond life expectancy. The model's solution requires the method of sequential quadratic programming. The model is an important contribution to the annuity-as-safety-net approach to optimal asset allocation. High levels of wealth diminish the demand for annuitization of financial assets.

	no. 1 (2007), pp. 96 – 110.	model where the benefit amount is determined by: (the cumulative premiums paid into the annuity contract, earnings on the fund supporting the annuity, and subtraction of contact expenses) ÷ (annuity pricing factors determined by mortality and interest rates). Variable annuities pay a benefit based on the earnings of the supporting investment fund, fixed annuities deliver a fixed benefit—which is equivalent to assuming that the return of the supporting fund stays constant for all future periods. The authors use inputs of risk-free rates equal to 3%, 5%, and 7% in their model. The article notes that "labor income is the most important source of wealth and a central determinant of consumption." The authors adopt a labor income model: $$I_t = I_{t-1}e^{G_t}$$ Where I_t denotes income in period t, and Gt is a fitted curve to empirical data. The wealth evolution model assumes that the investor allocates total wealth among consumption, a risky asset, a risk free asset, and an annuity. At any time t+1, Wealth [W_t] increases by income [I_t], decreases by consumption [C_t], decreases by the amount paid towards the annuity premium [P_t], increases by annuity benefits received [B_t], increases by the return of the assets allocated to the risk-free asset [R_f], and increases or decreases by the return on the risky asset [S_t]. The wealth evolution model assumes a binomial lattice pathway for the risky asset in which, for any period, can increase in value with probability p or decrease in value with probability 1-p. Assuming that X is the proportion of liquid wealth allocated to the risky asset, and 1-X is the proportion allocated to the risk-free asset, the wealth evolution model is: $$W_{t+1} = [W_t + I_t – C_t –P_t + B_t] \cdot [X_t \cdot S_t + (1-X_t) \cdot R_f]$$ In this model. C_t equals the minimum consumption level required to	The model suggests that there is an upper bound for the age of annuitization because of 'brevity risk.' This means that the length of the expected planning horizon is a factor in the decision to annuitize. A sudden change in health may make an annuitization decision counterproductive. In a trust context, annuitizing shortly before a negative health shock decreases utility for both current and remaindermen beneficiaries. The authors' research is significant in that the model takes a minimum consumption level into account and solves for the optimal time to annuitize based on various relationships among annuity pricing factors and capital market returns. For example, the model suggests that the optimal time for a CRRA investor to elect annuitization—i.e., an option valuation approach—depends on *both* a measure of Annuity Equivalent Wealth [AEW] and on the level of the investor's wealth. This is a utility-based analysis under a shortfall constraint. The model tests how systematically increasing the level of wealth, changes the decision to annuitize—no annuitization at high wealth levels; and, how systematically increasing risk aversion influences the decision to annuitize—annuitization is valued for "stable consumption."

		support a basic standard of living in period t. The life-cycle investor trades off between current consumption and growth of wealth. Thus, the objective function to be optimized is the sum, over remaining years of life, of the discounted value of the expected utility of consumption plus the discounted value of the expected utility of wealth. The model "fixes" the planning horizon as a function of current age plus "a few years beyond life expectancy" at that age. Thus, in addition to the asset allocation, annuity premium contribution, and consumption amount, the planning horizon—the length of time for which an individual wishes to plan—is also a control variable. Because of the large number of control variables operating over many nodes in the binomial lattice, "…computation of the objective function gradients is very awkward and expensive." The authors describe the problem as "…a large-scale mixed integer nonlinear programming problem…." A 30-year lattice, for example, has 1,073,741,823 decision nodes. Such a problem is best solved by finding all local optimal solutions for each time period. This requires the method of sequential quadratic programming. The model's input parameters are calibrated to match the mean and standard deviation of the US stock market: 'up' stock price equals 1.25, 'down' stock price equals 0.90 at 50-50 probability. The risk-free return is set to 1.05. Utility is based on the CRRA power function. The first optimization occurs for an investor aged 50 with an assumed remaining life span of 30 years. The initial optimization assumes (1) a zero allocation to the risky asset, and (2) a low risk-averse investor. Under this solution, the investor keeps all wealth in the risk-free asset until the time of optimal annuitization. At this point—age 58—all wealth is committed to a single-premium annuity. However, by age 66, the investor realizes no utility from annuitizing any fraction of wealth. This is a function of the number of expected annuity payments with age 80 as the fixed annuity termination date: "…the planning horizon is also a critical determinant to judge whether an annuity is beneficial or not." Under the model, "if	

		the life span of the individual is smaller than 78 years (i.e., planning horizon is 28 years) the annuity is unprofitable and will not be purchased. For planning horizon greater than 78 years of age, the optimal age of annuitization increases as the life span increases, but stabilizes at age 65 if the life span is 85 or larger. With an increase in life span, the age duration when the annuity stays beneficial expands, hence delaying the optimal annuitization time." Another key factor in determining the utility of annuitization is the investor's wealth level: "A delayed optimal annuitization time also ties to the wealth level at a specific annuitization time, since the individual can obtain greater advantage from an annuity if he invests more money in it. Our framework assumes that an individual retires at age 65 and does not derive labor income beyond that age, hence the wealth level decreases rapidly after the age of 65. This explains why the optimal age of annuitization doesn't exceed 65." The optimal age of annuitization depends on (1) the gap between the present value of the annuity premium and the present value of the total future benefits for a fixed benefit annuity, and (2) the wealth level at a specific age. "An increase in the initial wealth or in the initial income level will only change the wealth factor, but not the gap factor. If we increase the initial wealth level, an early annuitization is preferred to avoid depletion of wealth by consumption. However, an increase in the initial (and as a consequence the subsequent) income level may help build up the wealth and provide high wealth levels in the later years. As a result, a late annuitization is preferred." The existence of a risky asset, however, produces the expectation for a higher return and, therefore, delays the time of optimal annuitization to between age 60 and 61. The annuity becomes unprofitable between 64 and 67 years old. Examination of the investor's allocation of total wealth at the optimal time of annuitization (allocation among consumption, risk-free asset, risky asset, and the annuity) depends on the level of wealth and risk aversion. For the low risk-averse investor, allocation is determined over	

		eight levels of wealth. At every level, no wealth is invested in the risk-free asset because the individual elects to annuitize some or all of his available wealth. At the lowest wealth level, wealth is allocated exclusively to consumption and the annuity. As the wealth level increases, the proportion of wealth invested in the annuity decreases while consumption and risky asset allocations increase. At higher levels of wealth, there is no annuitization. The outcomes differ for investors with high risk aversion. In this case the optimal allocation strategy consists of consumption and annuitization for all but the highest level of wealth: "high risk aversion results in a preference for more stable consumption level than low risk aversion….However, individuals with different degrees of risk aversion make use of the annuity in different ways. Lower risk aversion results in using the annuity to achieve as high consumption as possible for each period, while high risk aversion utilizes an annuity to achieve more stable consumption."	
2007	"A Gentle Introduction to the Calculus of Sustainable Income: What is Your Retirement RisQuotient?" Moshe A. Milevsky Journal of Financial Service Professionals vol. 61 no. 4 (July 2007), pp. 51 – 62.	The literature on safe withdrawals and portfolio sustainability uses models that incorporate "…historical, simulated, and scrambled returns to quantify the sustainability of various *ad hoc* spending policies and consumption rates for retired individuals. However, most studies "…provide little pedagogical intuition on the financial trade-off between retirement risk and return." Milevsky seeks to implement a model that combines asset allocation, spending policy and longevity risk with the goal of ascertaining "…an analytic probability of retirement ruin." The author describes his model for generating the probability that a fixed spending plan will not be sustainable in two steps: Step One: Define Retirement "alpha" as: $\alpha = \frac{2(\mu+2\lambda)}{\sigma^2+\lambda} - 1$ where μ is *inflation-adjusted* expected return, σ^2 is *inflation-adjusted* variance, and λ is the investor's mortality rate.	Back-tested spending rules and asset allocations offer little insight into fundamental interactions between critical variables. Rules based on return realizations over one period may be wildly ineffective in future periods. Compare to "Decision Rules and Maximum Initial Withdrawal Rates," Jonathan T. Guyton & William J. Klinger 2006. The article characterizes annuities as a "product class."

		Define Retirement "beta" as: $\beta = \sigma^2 + \lambda/2$ Step Two: Retirement Ruin is calculated via the gamma distribution function in Excel: Ruin = Gamma Dist (S/β, α, 1, TRUE), where S = inflation adjusted spending rate as a percentage of the initial portfolio value. The fraction S/β equals 2S / ($\sigma^2 + \lambda$) because $\beta = \sigma^2 + \lambda/2$ and is moved to the denominator. Milevsky states that the formula is a good first approximation to the actual ruin probability. The formula assumes that returns are distributed log-normally and that the force of mortality (death rate) is exponential. The hazard rate (λ) is calculated as follows: Determine the median remaining life expectancy—MRL. This is the expectancy that says a retiree has a 50-50 chance of surviving—probability of survival = 0.5. Equating the two gives the formula: $e^{-MRL(\lambda)} = 0.5$. The answer is the hazard rate input to the initial formulae. Purchase of an annuity creates a mortality subsidy that can improve portfolio sustainability. Milevsky characterizes an annuity as a "product class." In contrast to an investment-only program, the addition of the mortality subsidy means that wealth increases as follows: $\text{Weath}_1 = \text{Wealth}_0(\mu + \lambda) - c$ where c is a fixed spending rate. For example, a 3% hazard rate combined with a 7% expected rate of return adjusts the μ input upwards to 10%. Annuity loads and expenses would, of course, act as a decrement to expected return. Milevsky states that the gamma distribution calculation is a good way for advisors to arrive at an approximate probability of success/failure	

		without the necessity of running simulation programs.	
2007	"Optimal Retirement Asset Decumulation Strategies: The Impact of Housing Wealth," Wei Sun, Robert K. Triest and Anthony Webb Public Policy Discussion Papers Federal Reserve Bank of Boston No, 07-2 (January 20, 2007).	Home ownership is equivalent to a lifetime annuity indexed to housing costs. Absent ownership, the occupant must pay for the cost of "housing services"—e.g., rent—a cost that could be substantial in order to secure a right to occupy an equivalent property. The authors define the value of a house in excess of the flow of services for the remaining lives of the home owners as "the reversionary interest." A reverse annuity mortgage makes a large portion of the reversionary interest available for non-housing consumption. Terms of a reverse annuity mortgage [Home Equity Conversion Mortgage, or HECM] permit homeowners to access the available reversionary interest amount either as a lump sum, a lifetime income, or a line of credit. The goal of the paper is to investigate three topics: 1. The optimal age to take a reverse annuity mortgage; 2. The optimal form of a reverse annuity mortgage; and, 3. The optimal portfolio allocation when a HECM is included. The authors use a VAR model to arrive at estimates regarding asset returns. They proceed to simulate returns to the year 2040. The model assumes a husband and wife both age 65. The portfolio is managed on a constant-mix, annually-rebalanced basis. Consumption equals 7.2 percent of the current year's value of the stock/bond/housing portfolio. Assets are allocated to maximize expected utility over an uncertain lifespan at a 3% rate of time preference. The model lacks any bequest motive. Expenses amount to 43 basis points for stocks and 25 basis points for bonds annually. Estimated amounts for HECM closing costs are included. The study considers the impact of delaying a HECM until ae 70,75,80,85 or until it has exhausted its financial wealth. The base case strategy is to take a reverse annuity mortgage at age 65 and invest the proceeds in financial assets. The utility value of alternate strategies is compared to the base case, compare the reverse annuity base case utility value to competing strategies, and determine an equivalent wealth value—the amount by which current wealth would	The article points out that the "capital return" on the price index compiled by the Office of Federal Housing Enterprise and Oversight during the period 1975 to 2005 has a 1.9% annual mean with a 3.7% standard deviation. By contrast, the age-65 household reversionary interest exhibits a mean of 16.0% with a standard deviation of 40.6%. The large discrepancy is due to a number of factors including the amount that can be borrowed as a function of age and the dramatic long-term decrease in interest rates over the period in question. "Schiller (2006) calculates that there has been little increase in real house prices over the period 1890-2000." One planning alternative is to take a lump sum HECM and purchase a lifetime income via a single-premium immediate annuity issued by an insurance carrier. Commercial annuity contracts can, under certain circumstances, offer more favorable payouts because of the "mortality credits" that arise through risk pooling. This is especially true at older ages.

		have to be increased under the base case formula to provide an equivalent utility value of an alternative. A value greater than 1.00 indicates that a strategy other than the base case strategy should be adopted. The study concludes: "…over a wide variety of assumptions about asset returns, the optimal strategy for all but the most risk-tolerant households is to take a reverse mortgage in the form of a lifetime income." Additionally, "We also find that including the reversionary interest in the household's portfolio results in an increase in the optimal allocation of financial assets to stocks, regardless of the strategy adopted for decumulating the reversionary interest. For example, at a coefficient of risk aversion of 5, the optimal allocation of financial assets when reverse mortgages are unavailable is 55 percent in stocks. But the same household will optimally invest 100 percent of its financial wealth in stocks if it plans to take a reverse mortgage in the form of a lifetime income when its financial wealth is exhausted." "Regardless of the value of the coefficient of risk aversion, taking a lump sum at age 65 is always preferable to taking a lump sum at a later time. The strategy of taking a reverse mortgage in the form of a line of credit once financial wealth is exhausted, which the National Reverse Mortgage Lenders Association tells us is most frequently chosen, performs particularly badly…. At high levels of risk aversion, taking a reverse mortgage in the form of a lifetime income, either at retirement or when financial wealth is exhausted, is preferable to taking a lump sum at age 65." Given the model's assumptions as applied to the base case, the authors assert: "…when a reverse mortgage is taken at older ages the strategy of applying the proceeds to the purchase of an immediate annuity yields a substantially higher income. Therefore, the dominant strategy for households is probably to spend down their financial wealth, take a reverse mortgage, and use the proceeds to buy an annuity." When the model assumptions are varied, the optimal strategy generally remains to either take a lifetime income at age 65 or when financial wealth is depleted. Delay in taking a reverse annuity mortgage becomes more attractive when housing returns are increased and vice versa.	

2007	Jeffrey E. Horvitz and Jarrod Wilcox, "Back to Markowitz: The Problems of Portfolio Compartmentalization," The Journal of Wealth Management, (Summer, 2007), pp. 43 – 53.	The authors briefly review several articles that use the tenets of Behavioral Finance [Asset Segregation, Framing, Loss Aversion, etc. for designing and implementing investment programs that appeal to private investors. However, in the authors' opinion, designing portfolios by consciously incorporating common cognitive biases identified by behavioral finance, often leads to a "bucketing" approach. A bucketing approach aims to match specific investor goals to investment vehicles by creating sub-portfolios for each investment objective. This is a significant departure from the portfolio optimization approach, first developed by Markowitz, in which investments are not considered piecemeal, but are evaluated in terms of their contribution to total portfolio risk and expected return. A primary goal of the paper is to argue that the resulting compartmentalization of investment portfolios often results in a sub-optimal investment program; and, that a return to Mean/Variance Optimization [MVO] can remedy such defects: "The main argument of these pragmatists is that individuals are simply too resistant to investment best practices, and that the variation in individual circumstances, including tax and estate issues, requires approaches that are intuitively attractive and resonate with clients. We, on the other hand, believe that there are workable solutions that incorporate the portfolio-wide optimal in asset allocation." The authors summarize the arguments in favor of incorporating behavioral finance into the wealth management process as follows: "Most of the commentators who would like to adopt a behavioral finance framework in lieu of Markowitz's model of optimal diversification reason that : 1. Most individual investors are not qualified to understand and interpret even basic statistics, e.g., mean and variance, so they do not find mean-variance portfolio optimization useful. 2. Asset allocation should accommodate what people prefer to do, and how they prefer to think of investments, rather than what they should do if they were perfectly rational in their own self-interest.	This paper is largely a response to a series of papers based "behavioral finance" insights. Generally, the papers suggest that investors are better served by an asset allocation process conforming more closely to an investor's decision making tendencies. Such an approach often leads to segregating portfolios so that each sub-portfolio is matched to an identifiable goal. This is sometimes termed "Goal-Based Investing." Goal-Based Investing plays an increasingly large role in discussions about how to fund an adequate and sustainable lifetime income stream for retired investors. Goals-Based investing is a more granular form of Liability-Driven Investing. During the early to mid-1990s, a lively debate played out in articles published in The Journal of Wealth Management. Much of the debate concerns general approaches to asset allocation and wealth management, as opposed to a focus on specific issues relevant to retirement income planning. Some authors stake out the position that quantitative approaches developed by financial economists using methods grounded in statistical analysis and operations research—a la Markowitz—offer a superior way to design portfolios and manage wealth. Other authors take the opposite side of the argument. They point out that investors seldom, if ever, make decisions based primarily on quantitative-based decision criteria. However, if investment planning recommendations are to have a useful effect, they must resonate with an investor's actual decision making heuristics. Finally, a third group of authors attempt to synthesize or to reconcile the contrasting positions. The Horvitz/Wilcox strategy represents, in large

		3. Individual investors are incapable of escaping the various cognitive biases identified in behavioral finance, so professional advice needs to accept and incorporate those biases." Their short answers to these assertions are: 1. We don't have to understand details underlying optimization in order to use it effectively—"…we all drive automobiles and most of us could not construct one." 2. Behavioral finance identifies flaws in thinking that lead to systematic mistakes in judgement; it does not offer a rationale to either ignore these flaws or to incorporate them into financial decision making. 3. Investment advisors have a fiduciary responsibility. "What if doctors treated patients based on what they were most comfortable with, e.g., herbal tea instead of antibiotics?" The authors observe that the Markowitz model is prescriptive, i.e., investors should diversify in order to achieve suitable risk/return tradeoffs, while behavioral finance describes what investors usually end up doing instead. The remainder of the paper is a polemic against recommendations made by A.B. Chhabra in his 2005 paper "Beyond Markowitz: A Comprehensive Wealth Allocation Framework for Individual Investors" [The Journal of Wealth Management Vol. 7, No. 4 (2005), pp. 8 – 34.] They decompose, in great detail, Chhabra's bucketing system framework: Personal Risk—"Do not Jeopardize Basic Standard of Living" Market Risk—"Maintain Lifestyle" Aspirational Risk—"Enhance Lifestyle" For the purposes of this bibliography, a detailed summary of their critiques is not necessary. They conclude: "For all practical purposes what Chhabra is saying is to just let the cognitive biases, i.e., mistakes, govern the asset allocations." More to the point is their concluding section which advocates using the "discretionary wealth" approach. This	part, a top-down approach to securing a threshold amount of retirement income. Immediate "bucketing," "flooring," "immunization," "annuitization" and so forth can, in their opinion, carry significant opportunity costs. This, of course, runs counter to advice recommending an annuitize-as-soon-as-possible strategy. The underlying rationale is that a failure to lock in a minimum income threshold places an intolerable risk burden on the investor. Sensitivity to downside risk can morph into genuine fear in down markets; but this emotion may be mitigated in the presence of adequate downside (principal and/or income) protection. It is noteworthy that complex modeling by financial economists aimed at exploring implications of barrier control approaches to life-cycle investing, can, in some cases, be linked—either correctly or not—to a behavioral finance approach. In any event, there continues a lively debate on the proper role of behavioral finance in shaping asset/liability investment management strategies.

		approach encompasses all assets, including human capital, and liabilities either contractual or aspirational. "Discretionary wealth is what remains after subtracting the present value of personal spending requirements from assets. We are not compartmentalizing the assets on the left side of the balance sheet, but simply calculating the residual 'equity' on the right side of the balance sheet. Consequently, *there is no implication that we need to exactly match our commitments at any one point in time with 'safe' assets*. That is generally suboptimal." Bucketing approaches, in the authors' opinion, "…create parts that do not 'communicate' with each other to the peril, or at least serious inefficiency, of the investor."	
2007	Moshe A. Milevsky, "Lesson 5: Spending Buckets and Financial Placebos," Research Magazine (June, 2007).	The analysis begins with the recognition that a sequence of poor returns occurring at the beginning of an investor's retirement can deplete the portfolio to the point where it would be difficult to sustain target future withdrawal amounts. This is known as 'sequence of returns risk.' Milevsky notes that some commentators suggest putting funds in liquid, low risk fixed income instruments sufficient to fund the initial 'x' years of retirement expenses. Withdrawals from the risky-asset portfolio would not occur until the initial low-risk (cash) portion is spent. If retirement begins in a bear market environment, the stock portion of the portfolio would remain untouched; and, presumably, because the investor avoids selling assets at a loss, would have ample opportunity to recover its value. Milevsky, however, is blunt in his disavowal of this strategy: "I believe these strategies are an optimal illusion at best and create a potential for grave disappointment at worst. If you are unlucky enough to earn a poor sequence of initial returns, so-called bucketing of your retirement income is not a guaranteed bailout." The article offers the example of two similarly situated investors each owning a $100,000 portfolio from which they intend to spend $750 per month--$9,000 per year. The first investor (Stephanie) invests $100,000 in a constantly rebalanced mutual fund allocated 70% to stock and 30% to cash. Expected fund return is 7% per year. She liquidates fund shares each month to achieve the $750 per month target.	This article is a good counterpoint to some aspects of a 'bucketing approach' or a 'time segmentation' strategy for improving retirement income sustainability. Such strategies are based on the belief that, although the short-run risk of equities is high, long-run stock risk is low—perhaps because of mean reversion in stock returns. Therefore, in order to avoid a precipitous drop in portfolio value at the start of retirement, the strategy calls for segmenting the portfolio into several parts—a cash portion designed to fund expenses during the initial retirement years; a bond-oriented portfolio designed to cover subsequent expenses; and a stock portfolio designed to provide retirement income only upon depletion of the cash/bond assets. That is to say, spend cash first, then bonds, and finally stocks. Milevsky's article is a precursor to the article by Walter Woerheide and David Nanigian ["Sustainable Withdrawal Rates: The Historical Evidence on Buffer Zone Strategies," Journal of Financial Planning (2012). [Available at SSRN: https://ssrn.com/abstract=1969021]. This article is

		The second investor (Brett), wishing to reserve funds to pay for the initial 36 months spending, places $25,400 in a cash account yielding 4% per year [PV of $750 monthly cash flow for 36 months at 4% = $25,400]. The remaining $74,600 is used to buy stocks with an 8% expected yearly return. The weighted combination of cash earning 4% and stocks earning and expected 8% exactly matches the expected 7% return on Stephanie's portfolio. Over the three-year period under consideration, the return on cash is constant and guaranteed. In any given year, the return on stocks takes on one of three values: +35%, +8%, or -19% for an expected return of 8%. The standard deviation of the equity portfolio is approximately 21.9%. At the time of retirement, each investor owns a $100,000 portfolio—earmarked to distribute $750 per month—with an expected return of 7% and a standard deviation of 16.3%. Over the next three years, there are 27 possible paths—in one year steps—for equity returns (3^3 = 27). In the least favorable equity return path, Brett's bucketing strategy leaves him with a portfolio valued at $41,996. This compares to the $45,105 value of Stephanie's portfolio. However, at the end of this period, Brett's allocation is 100% to risky stocks; Stephanie continues to maintain a 70% stock / 30% cash allocation: "Ergo, you have not protected yourself against a poor sequence of returns." Given the 27 possible paths, 16 favor Brett and 11 favor Stephanie. However, "…in just about all scenarios for which the market lost money in the first two or three years, Stephanie is better off than Brett." Milevsky concludes: "If you decide to adopt the so-called buckets approach to retirement income planning, then beware that your total asset allocation and implicit exposure to equity will fluctuate unpredictably over time….you have neither reduced nor mitigated financial risk but simply taken a bet on economic scenarios you believe will not happen. Safety is just a mirage."	discussed in the bibliography.
2007	"Rational Decumulation," David F. Babbel and Craig B. Merrill	The authors divide previous academic literature on the role of annuities into (1) "economic theory" prescriptions and (2) explanations for empirical observations that seem to differ from theoretical predictions. The root of the first group of literature is Yaari who demonstrated that	Incorporates risk of insurance company defaults into annuity purchase decisions. Good discussion of state guarantee funds, history of company defaults, economic consequences to annuitants and

	Wharton Financial Institutions Center Working Paper 06-14 (May 2007).	utility maximizing investors will annuitize all wealth in the absence of a bequest objective provided that the annuity was fairly priced. Davidoff, Brown and Diamond [DBD] relaxed the strict assumptions of the Yaari model and demonstrated, absent a bequest motive, investors would annuitize all wealth as long as annuities paid a higher return than assets of equal risk. Furthermore, DBD showed that annuitizing a substantial portion of wealth is optimal even in the face of incomplete markets and even if there is a strong bequest motive. [“Annuities and Individual Welfare,” Thomas Davidoff, Jeffrey Brown and Peter Diamond The American Economic Review vol. 95 no. 5 (December 2005) pp. 1573 – 1590.] A common characteristic of the prescriptive models is their use of continuous time finance theory. One result of this approach is that the annuity purchase decision is often modeled as a continuously renegotiated, instantaneous, buy/sell term insurance decision rather than in irrevocable choice to forgo future liquidity through an irreversible purchase of an annuity. Theoretically, the investor receives a periodic “insurance premium” (the monthly annuity income). The investor’s death gives rise to a claim on the assets of the decedent’s estate—i.e., principal is forfeited to the remaining annuitants in the risk pool. There follows a discussion of the explanations for the empirical “annuity puzzle.” The observed level of annuitization is far less than the amount considered optimal by economists. The authors present an algebraic model that attempts to capture more realistically the economic consequences of a decision to purchase an annuity. The important factors in the model include the possibility of default by the insurer as well as the irreversibility of the purchase decision. The model assumes HARA class utility functions. As a result, the investor will not pursue a strategy that leaves a positive probability of failing to support the threshold level of lifetime consumption. Penetrating this minimum produces infinite disutility and, given the assumption that utility is additive across all economic states, such a strategy is irrational. Thus, the model assumes that the investor allocates	techniques to minimize default risk by diversifying the annuity portfolio among several carriers. Authors employ a bottom-up approach: minimum standard of living locked in by government and private annuities, excess wealth allocated between risk free, risky investments and annuities. In general, annuities tend to crowd out bonds. Trustee has comparable issues—except that the process is top-down. As wealth decreases, what is the demand to fund current beneficiary income with annuities? If trust assets are sufficiently low, the risky asset allocation (remaindermen share) may be suboptimal. Does this put the duty to invest prudently in conflict with the duty of impartiality? It is irrational, in the authors’ opinion, to pursue a strategy with a positive probability of failing to support the minimum threshold level of lifetime consumption. Investors will annuitize up to the point where the marginal utility of an extra dollar of consumption equals the marginal disutility of spending down wealth to fund an annuity income. The article suggests a bottom-up approach where the investor buys and annuity to guarantee a consumption floor and invests the remainder of current wealth. The option to annuitize is the first planning step—not a final step which occurs only if investment returns to fund future consumption. The tradeoff is expressed as balancing the disutility of locking in a budget constraint via annuitization v. the utility of securing lifetime income.

		risk-free assets sufficient to support the minimum standard of living goal. In a multi-period context, the risk-free asset is an inflation-adjusted annuity like Social Security. If the minimum consumption target requires periodic income greater than that available through government or corporate pension benefits, the model assumes that an amount of current wealth will be annuitized in order to fund the deficit. Thus, the model seeks to provide insights into the optimal allocation of “excess” wealth, given full funding of the investors minimum standard of living level. The optimal allocation depends, of course, on the utility of consumption and the utility of bequest given an uncertain lifespan. The model uses stochastic dynamic programming to calculate “optimal controls” [consumption, bequest, and allocation to risky assets] based on the Hamilton-Jacobi-Bellman equation. However, the allocation to risky assets is also a function of available wealth given the amount of “excess” wealth that is to be annuitized. Additionally, if the amount of “excess” wealth allocated to the annuity is large, the investor may not have sufficient remaining funds to implement the optimal allocation to the risky asset portfolio. Assuming that the risky asset portfolio has a higher expected return than the annuity portfolio, the decrease in aggregate expected return (disutility) must be balanced against reduced uncertainty in future consumption (utility). The feedback loop plus the wealth constraint makes an analytic solution impossible. This requires the authors to simulate results. Simulation parameters include estimates of the real rate of interest, investor subjective discounting (time preference factor), expected equity risk premium, equity volatility, risk-aversion, the force of mortality, annuity loading factors, and the likelihood of an insurance company default (based on Altman’s “bond mortality table” for investment-grade ratings). The model yields expected results in that increases in equity volatility, interest rates, and risk aversion lead to increased annuitization of excess wealth; increases in the equity risk premium and insurer insolvency risk lead to decreased annuitization in the optimal allocation of excess wealth. For investors with average risk aversion parameters, annuity loads have minimal impact on the allocation decision provided that the	

		markups above the actuarially fair price are less than 30%. Generally, allocation of excess wealth is dominated by purchase of life annuities [c. 44 to 83%] even in the presence of a strong bequest motive.	
2008	"Efficient Post-Retirement Asset Allocation," Barry Freedman North American Actuarial Journal vol. 12 no. 3 (2008), pp. 228 – 241.	Author reviews classic Markowitz framework for evaluating investment choices. One advantage is that it permits analytical solutions by calculating an efficient frontier as a function of risk (standard deviation) and expected return. However, most articles define retirement risk in terms of the probability of an income shortfall. Likewise, the Markowitz model cannot easily accommodate discretionary spending, and does not adjust for asymmetries in the distribution of terminal wealth over a multiperiod horizon. Previous articles fall into several groups: 1. Highly complex mathematical studies of decision making designed to maximize a utility function. Although the articles are rigorous and use dynamic programming or stochastic optimal control approaches, they pose difficulties for practitioners. 2. A simpler version of utility maximization utilizing both numeric and analytical approaches that capture some of the important aspects of the retirement portfolio choice issue. The aim of these articles is often to produce general statements regarding the safety of withdrawal amounts, the use of annuity instruments, and so forth. The author uses a standard (Brownian motion / continuous time) model to develop an efficient frontier (ending wealth / SD ending wealth) illustrating tradeoffs between desired ending wealth and constant real withdrawal rates. His base case assumes 0% inflation. The base-case model's output suggests that at lower withdrawal percentages retirees will not find it advantageous to annuitize. However, at a 5% of initial wealth withdrawal rate, the retiree will annuitize 39% of initial assets. A final observation is that asset allocation strategies that are a function of age do not add efficiency; strategies that are a function of changes in wealth, however, can add efficiency.	Final Expenses enter the model as variable with fixed valuation: Freedman asserts: "…one might assume that a prudent retiree with wealth in retirement of $1,000,000 might desire to have at least $250,000 (real) dollars available for end of life expenses. This desire could be approximated by suggesting that the retiree would like $E[W_T]$ to be at least one standard deviation above $250,000." The model optimizes on terminal wealth—objective function is to minimize "bankruptcy" caused by an inability to fully fund last expenses. Tradeoff is between obtaining the desired terminal wealth and achieving an adequate level of consumption prior to death. All else equal, an increase in the withdrawal rate leads to an increase in the demand for annuitization of a fraction of initial wealth subject to attaining adequate ending wealth.
2008	"The Trajectory of	Although simple consumption life-cycle models generally predict a	This study contributes to the body of knowledge

	Wealth in Retirement," David A. Love, Michael G. Palumbo and Paul A. Smith Finance and Economics Discussion Series, Divisions of Research & Statistics and Monetary Affairs, Federal Reserve Board, Washington, D.C. (2008-13), pp. 1 – 37.	decrease in wealth during retirement as the consumer's life expectancy shortens, empirical data evidences the opposite—wealth tends to increase with age when measured according to the 'annualized comprehensive wealth' metric. This metric "…measures the amount of wealth that is available for each expected year of remaining life and for each person in a retired household." The authors claim that it is a 'comprehensive' measure because it includes financial assets, non-financial assets net of debt—e.g., housing equity, the value of Social Security benefits, defined-benefit pensions and transfer payments for lower-income retirees. The metric calculates "…the constant amount that a retired household could afford to spend, in expectation, every year until they die." Specifically, the definition of 'annualized comprehensive wealth' is "…the per-person annual payout from an actuarially fair, inflation-indexed, joint life annuity that a retired household could, in theory, purchase with its comprehensive wealth balance." An "annualizing factor" converts the wealth balance into "…dollars per person per expected year of remaining life." The annualizing factor includes an economy of scale for households and rises as a household's age increases. The particular focus of the study is to determine how well a more comprehensive life-cycle model incorporating factors for bequest motives and for uncertainty surrounding medical expenses and longevity can explain the data for older households found in the Health and Retirement Study [HRS] from 1998 to 2006. Based on responses from the HRS survey, the authors find "…that the median value of annualized comprehensive wealth for the cohort of households aged 70 to 75 years in 1998 rises significantly in retirement, from about $32,800 per person per year in 1998 to about $42,200 per person per year in 2006—a net increase of nearly 30 percent in just eight years. At the median, comprehensive wealth therefore tends to fall much more slowly than life expectancy shortens in old age. These results hold even when controlling for the large capital gains in the housing market during this period.	focused on investor spending strategies in the face of uncertain life expectancy and both temporary and permanent shocks to retirement income in the form of medical expenses. It Introduces the 'annualized comprehensive wealth' metric. Although this metric is calculated as a per-person per household constant-dollar, actuarially fair, joint annuity, the authors note "…in the U.S. at least, prices for actual fixed life annuities are far from actuarially fair." In simple life-cycle models, there is a close correspondence between optimal yearly consumption and annualized wealth. The permanent income hypothesis—consumption occurs at a level consistent with expected long-term average income—is an estimated level of spending that is both safe and feasible for the retired household. However, empirical data does not fit the model's predictions. Spending strategies reflect bequest and precautionary savings motives.

Given the HRS survey data, the authors investigate the expected behavior of annualized comprehensive wealth in “standard specifications of stochastic life cycle models.” The authors present a model in which the household maximizes expected discounted utility of consumption from their current age through a maximum age of 120 subject to a budget constraint and a non-negativity constraint for wealth at each age. The value function for a male is as follows:

$$V_{t,g}(X_t) = \max_{C_t}\{\upsilon_g(C_t) = \upsilon_g(C_t) + \beta\rho_{g,t}\mathbf{E_t}V_g(X_{t+1}) + \beta(1 - \rho_{g,t})\mathbf{E_t}B(X_{t+1})\}$$

Where X_{t+1} is the sum of the return on net savings from the previous period plus income (net of medical costs) received from all sources; and, Xt ≥ Consumption in period‘t’. Period utility is υ, the discount factor is β, the bequest Function is B, and the conditional survival probability is ρ. The model assumes relative risk aversion.

The model normalizes the variables by permanent income and then uses the normalized value function to obtain consumption decision rules for each point in the state space where the space is a function of cash-on-hand values and expectation integrals. Income is the product of a transitory shock [θ_t], a permanent shock [N_t], and a growth factor [G_t] that captures income trends over time. The model considers household age profiles with separate variance/covariance matrices based on the 1998 through 2006 HRS data. Coefficients are estimated through a series of regressions for each education group, age, marital status, gender, birth-year groupings and net retirement income. The estimated coefficients are used to construct the average growth rates for permanent income by groupings such as education and marital status.

Simplified life-cycle models suggest that retirees maximize utility by a spending pattern close to the amount calculated under the annualized comprehensive wealth metric. Of course, such a spending pattern would result in a significant decrease in this metric as a function of age—the reverse of what is seen in the data: “A simple specification of the life

		cycle model—one that allows for uncertainty only in survival and in which the discount rate is equal to the interest rate—would predict a *downward*-sloping trajectory of annualized wealth at older ages, due to the effects of survival discounting." The authors note, however, that models incorporating a bequest motivation "…drives a wedge between the levels of annualized wealth and optimal consumption as households age." Furthermore, adding both a precautionary motive and a bequest objective "…leads retirees to build up relatively large buffers of resources relative to life expectancy from age 65 to about 83, then to essentially maintain the large buffer in old age even as their rates of survival drop sharply." When the life cycle model incorporates a bequest motive and medical expense uncertainty (precautionary savings motive), the simulations produce results closer to those found in the HRS survey data.	
2008	"Generating Guaranteed Income: Understanding Income Annuities," John Ameriks & Liqian Ren Vanguard Investment Counseling & Research (2008)	The authors begin their article with several propositions: 1. "The longer one expects to live, the higher the implicit rate of return on an income annuity. Like all forms of insurance, annuities 'pay off' only if the risk they insure against is realized." 2. "Those in poor health generally should not consider annuitization." 3. "Income annuities are appropriate only for those who can afford to permanently lose access to some of their wealth" 4. "Compared with a program of systematic withdrawals from an investment portfolio, annuities can result in a significantly higher tax cost for investors in higher brackets." There is a brief discussion of mortality. The standard deviation of lifespan for a male aged 65 is between eight and nine years. Standard deviation, however, ignores the skew in the distribution: "…one must look at the full set of potential outcomes and their probability as a whole." The article reviews the basic annuity pricing formula [Sum of probability-adjusted annuity payments discounted by an appropriate interest rate factor]. They assert: "Often insurers will set annuity prices assuming a constant discount rate, although some will use a complete term	Proposition One counters the IRR argument that annuities are a poor investment because they simply represent a return of capital for many years. See, for example, David Maratta [2012] "The false promise of annuities and annuity calculators." The authors recommend using the SOA RP-2000 table because "…the SOA data more closely reflects the population of investors who possess significant retirement assets and might realistically consider income annuities." This is key because many studies use a general population [Social Security Administration] table to calculate longevity expectations. The article contributes to the theme of investing as a prudent exchange of risks. For example, "If spending is kept low enough, the risk that it will need to be cut even further late in life can be

		structure of interest rates in these calculations, effectively discounting payments over time at a date-specific rate….” Although the annuity contract guarantees a fixed level of periodic income, the authors indicate that the guarantee does not translate into the ability to maintain retirement lifestyle: “Given ongoing variability in both prices and spending needs, many people might do better by continuing to share in the risks and returns of a well-chosen and diversified set of investments as a resource for their cash flow. Compared with fixed payments, certain assets may be more volatile but also may provide a far better means to hedge spending risks such as inflation over long periods.” [Annuities fix the budget constraint]. The paper lists a variety of annuity costs. As a general rule, “…the fact that investment returns, administrative fees, and longevity forecasts interact to establish an annuity price can make it very hard to analyze the built-in costs.” The costs surrounding the loss of liquidity are especially difficult to estimate. They recommend the ‘Money’s worth’ approach to estimating total costs—both explicit and implicit. “A money’s-worth analysis measures total annuity costs by comparing the prospective return with what might be earned from a hypothetical cost-free alternative. It shows the annuity’s value as a fraction of the full present value of the costless alternative.” The article presents two sets of Money’s worth calculations—one for fixed income annuities purchased either in 1995 or in 1998, the other for liquidating a diversified mutual fund portfolio evenly over a 20-year period at various mutual fund expense ratios. The authors note: “…the expected cost of pursuing a systematic withdrawal strategy designed to exhaust the investment at life expectancy, with expenses roughly equivalent to what is charged by the average mutual fund, [1.25%] is similar to the cost of pursuing the annuity strategy.” Finally, in their general discussion of fixed income annuities, the authors note that the guarantee is not risk free: “The guarantees embedded in an annuity contract are conditional on the financial strength of the annuity provider.” They point out that “…after California regulators took control of Executive Life in 1991, 44,000 retirees received only 70	virtually eliminated. The cost of reducing longevity risk is simply spread throughout retirement via a reduction in spending.” See the studies of Amihud and Mendleson regarding quantifying the costs of illiquidity—especially their 1986 model. The Ameriks/Ren article leaves the issue hanging—“…the cost of illiquidity cannot be easily assessed, and will vary by investor and wealth level.” Ameriks & Ren discuss the subjective assessment of longevity risk by a prospective annuity buyer versus the objective assessment of risk by the insurer. They fail to point out that the fair cost of an annuity is determined as the point where the utility of the buyer’s offloading longevity risk matches the disutility of retaining the risk. See Milevsky’s discussion of utility and the cost of insurance. The simulation models used in this study incorporate the following items: • Taxes (given a pre-specified cost basis), • Longevity (given the SOA RP-2000 table) • Multiple Distribution policies • One time “health shock” resulting in a doubling of real spending.

		percent of their promised annuity payments for a period of 13 months." [Source is US General Accounting Office (1993)]. The remainder of the paper outlines results from a benchmark simulation model. The benchmark model assumes a $1million initial portfolio split between a taxable and a tax-deferred account. The cost basis for the taxable account is $400,000. The retiree is age 70 and the portfolio assets are allocated between a diversified investment account and annuities purchased both within the TDA and the taxable accounts. Overall portfolio allocation remains at 30% equity and 70% fixed income—there is a mix of fixed and variable annuity contracts. The model projects success rates for sustaining an annual inflation-adjusted income goal of $45,000. The base case model indicates a success rate at age 95 [20 years] of 89% with an annuity component to the portfolio versus a success rate of 64% without an annuity. Other risk metrics include (1) the average shortfall in cases where the spending goal is not met and (2) the estate values at various ages. The study concludes: "…spending shortfalls are significantly lower when the annuity is present." Variations of the benchmark or, base case, model are also considered. These include the purchase of only a fixed annuity providing a nominal benefit and a distribution policy decreasing by 1% annually for nine years followed by a doubling of real spending following a health emergency in year 10.	
2008	"Converting Retirement Savings into Income: Annuities and Periodic Withdrawals," Janemarie Mulvey & Patrick Purcell. Washington, DC: Congressional Research Service. http://digitalcommin	The paper provides an overview of strategies that can be used to provide retirement income. The authors note that most retirees do not voluntarily choose to annuitize income. They cite a Vanguard study indicating that only 40% of defined benefit plan participants who are offered a lump-sum option choose to annuitize. The study outlines some factors contributing to the "annuity puzzle:" "There are several reasons why the demand for annuities is low despite the aging of the population. Some potential purchasers may already feel they have a sufficient amount of annuitized income from Social Security, and about a third of people 65 and older also have annuity income from defined benefit pensions. Another reason may be the amount and non-transparency of fees and expenses charged by insurance companies.	The paper uses a general population mortality table. The simple risk model generates outputs suggesting that a withdrawal rate of 4% to 4.5% of the initial portfolio value for an age 65 retiree is relatively sustainable. The study provides a clear, non-technical explanation of simulation: "*Monte Carlo* analysis is a method of estimating the probable outcome of an event in which one or more of the variables affecting the outcome are random….The essence of a Monte Carlo estimation process is to simulate an

	s.ilr.cornell.edu/key_workplace/566/	Further, annuity contracts are not easily canceled, and many individuals fear that after purchasing an annuity they may later need a large sum of money to pay for unexpected expenses, such as long-term care or health expenses. Even among people who understand that it is important to insure against longevity risk, some fear that they will die before reaching their normal life expectancy, and will end up 'losing the bet' with the insurance company that sold the annuity. Finally, recent adverse publicity about deceptive sales practices in the annuity market has added to concerns among potential buyers of immediate annuities." The authors point out that the median age of purchase for an immediate annuity in 2003 was 70. The alternative to annuitization is the implementation of a withdrawal strategy from a personally owned portfolio of financial assets. However, they state: "…it can be difficult to choose a rate of withdrawal that can be sustained in the face of uncertain life expectancy and variable rates of return on investment." Furthermore, "There is no fixed standard for the minimum probability of success that a retiree should be willing to accept for the annual rate of withdrawal that he or she chooses." They summarize the risk/return tradeoffs faced by retirees attempting to design and implement an optimal withdrawal strategy: "A retiree who wishes to achieve a predictable annual income can take annual withdrawals that are equal in inflation-adjusted dollars. An individual who chooses a rate of withdrawal that is too high risks spending down the account too quickly, possibly leaving the person impoverished. An individual who chooses a rate of withdrawal that is too low risks spending down the account too slowly, unnecessarily reducing his or her consumption and leaving substantial assets unspent at death. On the other hand, the retiree can choose to take withdrawals that vary from year to year based on the current balance in the account and the retiree's remaining life expectancy. This strategy can result in highly variable annual income." In order to quantify the risks of withdrawal strategies, the study conducts a Monte Carlo simulation analysis of two asset allocations (65% S&P / 35% AAA Corporate Bonds, and 35% S&P / 65% AAA Corporate	event many times, allowing the random variable to vary according to its mathematical mean and variance. Each outcome is then ranked according to the likelihood of its occurrence." Note—like many studies, this one considers the tradeoff between budgetary certainty and variable cash flows during retirement. However, predictability of annuity income provides utility in that there is an early resolution of uncertainty regarding cash-flow sustainability. An early resolution to a problem may not generate calculable utility; nevertheless, it is a benefit despite the fact that the annuity may not be the optimal solution.

		Bonds) over both fixed time periods as well as over investor life spans. Investment data covers the 82-year period 1926 through 2007. The study also tests various withdrawal rates from 4% through 6% at intervals of 50 basis points. The initial withdrawal is a fixed percentage of the beginning portfolio value and is adjusted annually by the rate of inflation. Portfolios and rebalanced annually. Fees and taxes are not considered. The output indicates "The higher long-run average rate of return on common stocks compared to bonds acts as a form of longevity insurance for the retiree." For example. "In simulations representing retirement at age 65, withdrawal rates of 4.0% and 4.5% were successful in 95.0% or more of simulations, for both men and women under both investment portfolios at all ages up to 100." As the withdrawal rates climb beyond 4.5%, the need to load for equity becomes greater in order to prevent portfolio exhaustion assuming a life span greater than age 90. The conclusion suggests that a mix of portfolio withdrawal strategies might be one way to balance the risks faced by retirees. They speculate that separating the portfolio into two or more accounts and adopting different strategies for each account—or purchasing an annuity with an account—might yield significant benefits. The study, however, does not explore these planning options.	
2008	"Regulating Markets for Retirement Payouts: Solvency, Supervision, and Credibility," Phyllis C. Borzi and Martha Priddy Patterson Chapter Eight in Recalibrating Retirement Spending and Saving—Eds. John Ameriks and Olivia S. Mitchell. (Oxford Univ. Press,	A general survey article on regulation / consumer protections in Insurance, Annuity, Investment, and Banking Product markets. With respect to annuities, the authors point out that: "While established under state law, guaranty associations are not state agencies (GAO 1993); this implies that the states do not guarantee that the guaranty associations themselves will have sufficient funds to cover their obligations." In the event of insurer insolvency, the state looks first to the assets of the failed company. If these prove insufficient, "…each state guaranty association assesses the member insurers in its state a share of the amount required to meet the claims of resident policyholders…." The authors point out that the state guaranty agencies react only after the occurrence of losses: "It can be months or years before policyholders receive compensation…."	Provides further insight into (1) the risks of annuity ownership and (2) the degree of consumer protections from the perspective of the buyer. DOL standards for ERISA fiduciaries clarify the duty to invest with caution. These standards were issued on March 6, 1995 in Interpretive Bulletin No. 95-1. Purchase of an annuity by the sponsor of a qualified plan is a fiduciary decision. The standards have been termed: "the safest available" annuity standards. In 2008, the DOL issued regulations for offering an annuity option in a 401(k) participant directed plan. Rather than the "safest available" standard, the menu offering must meet ERISA's

	2008).	Additionally, there is a discussion of section 625 of the Pension Protection Act of 2006. The DOL's Interpretive Bulletin 95-1 states that any employer-purchased annuity contract must be 'the safest available' annuity. Section 625 directs the DOL to issue regulations clarifying that when a fiduciary offers an annuity "…as an optional form of distribution from an individual account (DC) plan covered by ERISA, that decision is not subject to the 'safest available annuity' requirement." However, the fiduciary must prudently select and monitor the annuity issuer and the performance of its contracts.	general prudence standards. If the 401(k) plan sponsor satisfies five requirements in the annuity selection process, the sponsor enjoys "safe harbor" protection.
2008	"Following the rules: Integrating asset allocation and annuitization in retirement portfolios," Wolfram J. Horneff, Raimond H. Maurer, Olivia S. Mitchell, Ivica Dus. Insurance: Mathematics and Economics vo. 42 no. 1 (February 2008), pp. 396 – 408.	The study examines several decision variables including distribution policy ("payout rules"), mortality, interest rate risk, and investment risk within a "utility framework." The study considers both the binary choice--to annuitize all wealth or to remain a participant in the capital markets—and blended or switching strategies that encompass both annuities and investment positions. "Our assessment indicates that permitting the consumer to switch to a fixed annuity sometime after retirement gives her the chance to benefit from the equity premium when younger, and exploit the mortality credit in late life by switching to life annuities between ages 80 to 85. Such an integrated strategy can enhance retirees' well-being by as much as 25%-50% for a moderately risk-averse retiree. Even if interest rates are stochastic, retirees will do well to wait until age 80 in the current low interest rate environment." The authors concentrate on fixed, nominal single premium immediate annuities. Their literature survey considers the "annuity puzzle" hypothesis and notes that "…annuities appear to be relatively expensive in a low interest rate environment, as compared to equity-based mutual fund investments." However, any plan that requires a periodic fixed amount distribution from a risky asset portfolio "involves a strictly positive probability of hitting zero before the retiree dies." The authors organize the literature according to the following research assumptions and methods: • Utility framework v. Shortfall framework • Risk neutral investor v. Risk averse	The authors contend that a "…constant payout life annuity constitutes an asset class with a unique return profile, as payments are conditional on the annuitant's survival." Assuming low current and continuing interest rates, the optimal time to annuitize is after age 80. Annuities tend to crowd out bonds and push investors towards a two-fund solution. The article uses an annuity 'benchmark' in the sense that the annuity payout must be matched by the financial asset portfolio's payout. This is a *distribution* benchmark which is common when comparing annuitization with a do-it-yourself investment program—a synthetic annuity. The article does not utilize an annuity cost benchmark which amount is then compared to the current market value of existing financial assets. This is the distinction, however, between a shortfall metric—distributions are compared—and a feasibility metric—costs of retirement are compared.

- Additive utility v. Habit formation
- Illiquid life annuities v. Term (tradable) life annuities
- Constant annuity v. Variable annuity
- No bequest motive v. Bequest utility
- Constant asset mix v. Dynamic asset mix
- Deterministic asset model v. Stochastic asset model
- Deterministic interest rates v. Stochastic interest rates
- No inflation risk v. Inflation risk
- Annuity and financial wealth only v. Human capital
- Exogenous annuitization strategy v. Endogenous annuitization strategy.

Among the noteworthy approaches to the optimization of utility (consumption and/or bequest) are the shortfall probability studies that calculate the likelihood of running out of money prior to a stochastic date of death using assumptions regarding age, sex, investment returns and the initial wealth-to-consumption ratio. Shortfall studies may consider both fixed and variable annuity payouts. Other approaches consider the optimal time for a retiree to switch all investible wealth to an annuity. Several studies fully endogenize the strategy by determining the timing and extent of annuity purchases. One contribution of the current paper is to address the merits of a mixed strategy where the retirement portfolio can include a combination of a life annuity and a phased withdrawal plan. The appropriate mix is a function of the investor's risk aversion and key assumptions regarding actuarial tables and capital market returns.

The authors begin by noting that an annuity's total return is a function of capital return on the asset plus a mortality credit. The actuarially fair value of any nominal fixed annuity is expressed in terms of an annuity cost factor:

$$\ddot{a}_y = \sum_{t=0}^{w-y}(1+\delta)\cdot {}_tP_y \cdot (1+r_t)^{-t}$$

		Where w is the last age in the applicable mortality table. The annuity payout becomes the benchmark against which other withdrawal plans ("self-annuitization") are compared. The study considers three withdrawal alternatives: 1. A fixed percentage rule—this is a unitrust rule where a constant fraction [B] is withdrawn each period from the remaining wealth [W]: $B_t / V_t = \omega_t$. 2. The 1/T rule where the withdrawal fraction rises with age: $B_t / V_t = \omega_t = 1 / T - t$. 3. The 1 / E(T) rule which considers the retiree's remaining life expectancy: $B_t / V_t = \omega_t = 1 / E[T\ (y+t)]$. The 1 / E(T) rule is akin to the IRC minimum distribution rule for certain qualified retirement accounts. The fixed percentage rule, by construction, equates the initial year withdrawal benefit to the annuity payout. Capital market returns are proxied by a continuously rebalanced portfolio invested 60% to US Large Cap Stocks and 40% to Long Term Bonds. Return evolutions are modeled as a one-dimensional Geometric Brownian Motion process with a term for drift and for diffusion. On average [statistically expected results], the fixed percentage rule matches the initial annuity payout and, thereafter, generates a rising payout benefit as the retiree ages because the gross rate of return on the retirement account exceeds the constant consumption to wealth ratio. By contrast, the 1/T rule generates lower expected benefits until age 74 and higher benefits thereafter. The 1/E(T) rule also begins with a lower benefit which exceeds the annuity payout at age 69. However, the benefits peak at age 88 and decline thereafter as less wealth remains in the account. The authors also report the first percentile ("worst case") results. The fixed benefit designed to match the fixed annuity payout becomes $0 at	

		age 80. In all cases, the variable benefit withdrawal rules produce a payout significantly less than the annuity during the first 20 years of retirement. Eventually, only the 1/T rule produces a higher benefit but the crossover point does not occur until age 85. The 1/E(T) rule never generates a payout that exceeds the annuity benefit. The study examines how retirees exhibiting differing risk aversion levels (low, moderate and very risk-averse) will assess the various payouts assuming a CRRA utility function. Utility is defined over lifetime consumption (with a time preference of 0.96) and over a bequest utility where the strength of the bequest motive differs across retirees. They compute the lifetime utility for each of the three withdrawal rules and then "…transform this utility level into an equivalent nominal annuity income stream for life." This is a certainty-equivalent income stream that can be directly compared to the benchmark annuity payout. At almost all levels of risk aversion, the fixed benefit payout—i.e, a fixed amount replicating the annuity benefit—is rejected "…as it exposes the retiree to the risk of outliving her assets." The fixed percentage rule (unitrust) dominates the 1/T rule for all levels of risk aversion, and "…it is more appealing than the annuity for low/moderate levels of risk aversion." The 1/E(T) rule appeals to low and moderate risk-averse investors but is not attractive for investors with high risk-aversion. Compared to the "optimal" withdrawal rule, only the very risk-averse will prefer the fixed annuity. For each payout program, the authors calculate the optimal asset allocation. The allocation remains the same for each of the three withdrawal rules. However, it changes with the level of investor risk aversion. For low risk-averse investors the optimal allocation is 100% equity; for moderate risk aversion, the allocation drops to 60%: It is interesting that the 60/40% stock/bond portfolio commonly recommended by financial advisers is appropriate only for those with risk aversion of around four, but the curve slopes slowly, so even very	

		risk-averse consumers will still optimally hold about 40% of their assets in equities.” The equity allocation never falls to zero. All withdrawal plans become more attractive if there is a positive bequest motivation. When the retiree, at the date of retirement, is given a one-time option to blend a withdrawal plan with an annuity, the demand to hold an annuity is zero for the risk-tolerant investor but increases to approximately 62.6% of wealth allocated to annuities for the moderately risk-averse investor. The remaining wealth is held 100% in equities suggesting that the demand to hold an annuity crowds out the bond allocation. However, when the retiree is given the option to defer annuitization until a later date—i.e., continue to participate in the capital markets—the problem becomes one of finding the optimal switching age. In this case, a retiree with moderate risk aversion defers annuitization of wealth until she turns age 80. Finally, the authors explore the sensitivity of their findings to changes in the interest rate environment. As a general rule, the higher the interest rate, the sooner the retiree elects to annuitize. “However, a more risk-loving retiree will also demand a higher short rate than her risk-averse counterpart. The switching frontier itself is concave because the mortality credit increases over time and replaces cost advantages formerly generated by the related short rate. Interestingly, the advice about when to switch to annuities depends on the current level of the short rate, as it relates to the level of mean reversion. The retiree would likely want to wait until age 80 to annuitize if the short rate is below the long-term mean and mean reversion is anticipated….The lower the risk aversion, the higher the short rate must be to induce the retiree to annuitize her assets.” The study concludes: “…annuities are attractive as a stand-alone product when the retiree has sufficiently high risk aversion and lacks a bequest motive. Withdrawal plans dominate annuities for low/moderate risk preferences, because the retiree can gain by investing in the capital	

		market and from betting on death.'"	
2008	"The Role of Government in Life-Cycle Saving and Investing" Alicia H. Munnell In: The Future of Life-Cycle Saving and Investing (Second edition) CFA Institute, eds. Zvi Bodie, Dennis McLeavey and Laurence B. Siegel (2008), pp. 107 – 141.	The author notes that investors "…face the risk of either spending too rapidly and outliving their resources or spending too conservatively and consuming too little." Although annuities are sometimes recommended to retirees, "annuity payments are highly sensitive to interest rates, which means that different cohorts of retirees who annuitize will likely end up with different monthly benefits for the same total accumulations. The variation is substantial. A $1,000 premium would have purchased a monthly income of $9.50 in 1989, when the yield on a 10-year Treasury note was about 8.5 percent; the monthly payment in March 2003 was about $6.69 per $1,000 because 10-year Treasury note yields fell below 4 percent." Additionally, annuity contracts are not risk free: "Two relatively large insurance companies ended up paying only 70 cents on the dollar after they got into trouble as a result of bad investments."	Discusses the concept of the term structure of annuities. Depending on the interest rate environment, the amount of monthly income variation is substantial. The distribution of annuity costs over time is an important factor in monitoring the cost of retirement.
2008	"Choosing the optimal annuitization time post retirement," Russell Gerrard, Bjarne Hojgaard & Elena Vigna Collegio Carlo Alberto Working Paper No. 76 (July 2008), www.carloalberto.org.	The authors consider annuitization strategies for participants in the UK retirement system which grants the retiree the option to withdraw funds from a financial asset portfolio—the drawdown option-- or annuitize the funds for a lifetime income. At age 75, by UK law, all remaining funds must be annuitized. Electing the drawdown option at retirement creates "annuity risk"—the risk that lower future interest rates will increase the cost of the annuity to the point where the periodic income is less than the current achievable pension annuity income. Assuming election of the drawdown option, there are three degrees of freedom: 1. The decision regarding how much to withdraw between retirement date and the sooner of an election to annuitize or attainment of the mandatory annuity purchase age; 2. The decision regarding asset allocation of the portfolio net of withdrawals; and, 3. The decision regarding when to annuitize. The first two choices "…represent a classical inter-temporal decision	Wealth falls into a region of "continuation" in which the investor does not annuitize or into a "stopping region." The model is a top-down, free-boundary type. It stresses the concept of "annuity risk" which is defined as the risk in delaying the option to annuitize. Note, however, that the 'optimal time' is not necessarily the time at which wealth is about to hit a lower bound dollar level. It might be helpful to elucidate a distinction between *optimal* stopping time and *necessary* time for annuitization.

		making problem, which can be dealt with using optimal control techniques in the typical Merton framework…, whereas the third choice can be tackled by defining an optimal stopping time problem.“ The basic model assumes that, at the time of retirement, the investor has a lump sum of money (x) that can be invested in a riskless bond paying the risk-free rate (r), or in a risky asset (y = the proportion invested in the risky asset) where the risky asset evolves as geometric Brownian Motion with fixed parameters for mean and variance. The force of morality (δ) is assumed to be constant. At any future time, the size of the annuity that can be purchased is kx where k > r because of the mortality credits used to price the annuity. The investor will derive utility from the amount withdrawn from the fund b(t) prior to annuitization [U_1(b)] and from the annuity income post annuitization [U_2(kx)]. Thus, the authors seek the solution to selecting the two optimal control variables y(t) and b(t) and a stopping time T in such a way as to maximize the utility of the two-period planning horizon. The model places no constraints on borrowing or short selling. As the portfolio evolves under the withdrawals selected by the pensioner, the value of the account will either fall into a “region of continuation” in which case the investor continues the drawdown option or into a “stopping region” in which case the investor annuitizes immediately. If the fund x remains in the continuation region, the optimal risky asset allocation (y*) is : $y^*(x) = -\frac{(\lambda - r)V'(x)}{\sigma^2 x V''(x)}$ which is the well known Merton optimum adjusted by the Arrow-Pratt risk aversion function. The optimal choice of the amount to withdraw from the fund is a function of the change in value of the fund from period to period. Therefore optimal b is:	

		$$b^*(x) = \arg\sup[U_1(b) - bV'(x)].$$ The maximum of the value function is attained when the first derivative equals zero and the sign of the second derivative is negative. Therefore, assuming a subjective preference rate of ρ: $$U_1(b^*(x)) - (\rho+\delta)V(x) - (b^*(x) - rx)V'(x) - 1/2\beta^2\frac{V'(x)^2}{V''(x)} = 0$$ Where β = The Sharpe Ratio. This equation is a quadratic equation which has a positive and a negative root. By defining an inverse function of V'—X(z), the roots of the equation can be expressed as the exponents of the complementary function $X(z) = C_1z^{a1} + C_2z^{a2}$ where a_1 and a_2 are the roots of the equation. Thus the first order equation can be rewritten: $$U_1(b^*(X(z)) - zb^*(X(z)) - (\rho+\delta)V(X(z)) - 1/2\beta^2z^2X'(z) = 0$$ And differentiating with respect to z gives the classic form $[ax^2 + bx + c]$ for a quadratic equation: $$P(\alpha) = 1/2\beta^2\alpha^2 + (\rho+\delta+1/2\beta^2-r)a - r$$ which can then be solved for the optimal withdrawal amount or the minimum of the quadratic utility function. Given the simplifying assumptions that lead to the expression of the problem as a quadratic equation, the authors explore the nature of the continuation region as well as the boundary of the region. The model indicates that the value of x cannot be less that b_1/k and that, upon reaching this point, the investor should annuitize. Given this lower bound, they continue to solve for the optimal time to annuitize. The model indicates that the optimal time is the smallest amount of time t such that $x(t) \in U^c$, where the continuation set U is given by U = [0,x*) ∪	

		$\left(\frac{b_1}{k}, \infty\right)$. The results of the model are highly sensitive to the value of the input parameters. A high β (Sharpe Ratio) makes the risky asset portfolio more attractive and, all else equal, leads to a delay in annuitization. If the weight placed on annuitization wealth is greater than that placed on withdrawals for current consumption, then the penalty for annuitization prior to attaining b_1/k is high. If b_1/b_0 is high, the investor has low risk aversion (willing to shoot for a high future annuity payout) and will be reluctant to abandon the opportunity to invest in the financial markets. Older retirees have high values of $\rho+\delta$ (they are less patient regarding future events with a high value of ρ) and should be more willing to annuitize than to continue investing. Finally, the authors run 1000 Monte Carlo simulations with selected input parameter values. The results indicate a high probability of annuitization within 15 years of retirement with a 4.66 year average optimal time of annuitize. The mean size of the annuity is 90.39 income units per initial 1,000 wealth units. This compares to 69.95 income units per 1,000 wealth units if the participant elected immediate annuitization and rejected the drawdown option" "This highlights the financial convenience for the retiree of deferment of annuitization until a more propitious time."	
2008	"Time to Rethink Portfolios and Distributions," Douglas Moore Wealth Management.Com (September 1, 2008), http://wealthmanagement.com/philanthropy/time-rethink-portfolios-and-	A general review article that cautions trustees to check their asset allocation assumptions in light of potentially challenging economic times: "…it's helpful for fiduciaries to run new Monte Carlo simulations periodically to update projections and to ensure their investment strategies and asset allocation models are on target." The author notes: "…faulty assumptions can cause well-intended plans to go awry. These problems are amplified when hard-to-value assets and non-traditional assets are included in a trust portfolio." Other observations: 1. High-net worth clients wish to maintain an ability to purchase	Asserts that a 3.5% unitrust distribution rate will preserve the initial value of a trust.

	distributions-0	luxury goods. The inflation rate for these goods is far higher than CPI. 2. Grantor trust arrangements enable a grantor facing imminent death to exchange high basis personal assets for low basis trust assets. At death, the low basis assets receive a step up in basis and the trust is left owning high basis assets. 3. Four common distribution methods are (1) unitrust, (2) annuity, (3) mandatory net income, and (4) discretionary net income or principal. 4. "The general rule of thumb is that a unitrust payout of 3.5 percent annually will sustain the initial value of a trust's asset base." 5. Diversification is important because "the probability of depletion before a trust term ends increases significantly when asset allocation is insufficiently diverse to safeguard against early downturns in the market."	
2008	E.A. Medova. J.K. Murphy, A.P. Owen and K. Rehman, "Individual asset liability management," <u>Quantitative Finance</u>, Vol. 8, No. 6 (September, 2008), pp. 547 - 560	The paper begins with a short review of economic papers devoted to the subject of optimizing household financial decision making. The seminal papers include the lifecycle hypothesis proposed by Modigliani and Brumberg, Samuelson's use of dynamic stochastic programming for portfolio selection, and Merton's continuous time analysis. These papers "…formulated relationships between consumption and portfolio allocation in terms of expected returns and volatilities in order to maximize total lifetime utility." By contrast, Kahneman and Tversky use a different type of utility function that reflects a behavioral finance orientation. Unfortunately, in the authors' opinion, practitioners do not base their advice to households on solid principals of economics and finance. The article quotes Samuelson's observation that the "…domain of common sense…is not the same thing as good sense." Fortunately, advances in computer power enable financial advisors to improve the quality of their advice. The article introduces a planning application [*i*ALM] developed by the authors as an example of possible improvements. Of special interest is the fact that *i*ALM "…emulates behavioural patterns. We believe that the optimization results using	This is a 'precursor' article to the longer paper by Dempster and Medova in the British Actuarial Journal: "Asset liability management for individual households," <u>British Actuarial Journal</u>, Vol. 16, Part 2, (2011), pp. 405 – 439. This latter paper is also an entry in the annotated bibliography. The article outlines technological advances in promoting an ALM portfolio design, implementation, monitoring, and management approach. It cites A.J. Berger and J.M. Mulvey's "The Home Account Advisor" as an early example of this technology [<u>Worldwide asset and liability management for individual investors</u>, ed. W.T. Ziemba and J.M. Mulvey (Cambridge University Press, 1998), pp. 634 – 665].

		*i*ALM support many empirical observations from behavioural finance." The authors describe the household's financial challenge as follows: "With liabilities arising at any time, investment, saving and other financial decisions must change across household lifetime as a response to changing life and market conditions. This is therefore a dynamic multi-stage problem. The stages correspond to major changes in personal circumstances, e.g. retirement date, big purchases such as real estate, and many others. The household decision maker must deal with mostly *stochastic cash flows*; both incomes and liabilities are linked to future economic fundamentals, which are uncertain." The article provides a brief introduction to the methodology of dynamic stochastic programming in which the family units stages form a series of nodes within a "…discrete-time, continuous-state, multi-dimensional stochastic *data process*." The process is represented by a decision tree where a node is located at each stage: "All decisions at intermediate nodes of the tree take into account the possible evolution of the stochastic data process from that point forward." The structure of their application is a series of nested optimization problems within each node's boundaries. As the investor negotiates a path through the decision tree, the application provides a forward-looking simulation of the relevant components of the data process (assets and liabilities), a solution to the optimization problem for that portion of the tree, and an output that facilitates analysis of the decision tradeoffs. Much of the article is devoted to a technical description of this three-step process. Optimization is based on a utility function "for each individual goal." The function is "…constructed for a range of spending between acceptable and desirable values, subject to existing and foreseen liabilities, and a minimum required spending." The function's slope value for each goal reflects the priority weighting that the investor assigns to the goal. This is the application's behavioral finance component. The authors compare the current year's allocation derived from the *i*ALM Algorithm, to the allocation that the Mean-Variance, asset-only Markowitz algorithm would recommend: "…the optimum portfolios	

		should be in close proximity to the Markowitz efficient frontier to assure users that *i*ALM does not contradict established economic concepts and practices." By inspection, the authors conclude that the *i*ALM produced portfolios "…are slightly below the Markowitz frontier due to various effects from liabilities, transaction costs, management fees and portfolio drawdown limitations as modelled in our multistage dynamic problem."	
2008	"The Longevity Annuity: An Annuity for Everyone?" Jason S. Scott Financial Analysts Journal vol. 64 no. 1 (January/February 2008), pp. 40 - 48.	Scott suggests that a new type of annuity contract—the longevity annuity—may make annuitization more attractive to retirees: "…a longevity annuity involves an up-front premium with payouts that begin in the future. For example, an age-85 longevity annuity can be purchased at age 65 with payouts commencing only when *and if* the purchaser reaches age 85." The longevity annuity must compete in the marketplace with zero-coupon bonds and immediate annuities. A portfolio of zeros maturing yearly over a deterministic planning horizon is priced as: $$\text{Price}_{\text{bond}} = \sum_0^T \frac{1}{(1+r)^t}$$ Where r is the yield curve interest rate corresponding to the desired bond maturity date. The Price of a portfolio of actuarially fair immediate annuities, by contrast, is the replicating zero coupon bond portfolio adjusted for the likelihood of survival: $$\text{Price}_{\text{annuity}} = \text{Price}_{\text{bond}} \sum({}_tP_x) \text{ for all t } \forall\, T.$$ Since $\sum({}_tP_x)$ is strictly less than one, the annuity income stream is cheaper to finance than the zero-coupon bond income stream. For a retiree aged 65, the initial few years of payments from an immediate annuity are not substantially discounted because the mortality rate remains low and the interest factor has little time for compounding. However, the longer one delays the start of the annuity income, the greater the discount. Substantial price reductions are possible for delaying the initial payment start date for 15 or 20 years. Most investors are not willing to annuitize a large portion of their liquid	A variation on the "hurdle race" problem outlined by S. Vanduffel, J. Dhaene, M. Goovaerts, and R. Kaas. In Scott's model the liability payments are purely deterministic and nominal. The advantage of the longevity annuity is that the initial "provision" is sufficiently small to reduce consumer reluctance for annuitization. Scott acknowledges that inflation is a problem and that there are no commercially available longevity annuities as of the article's publication date. The article generally ignores (1) the strategy of self-annuitization coupled with the option to annuitize approach of Milevsky, or (2) the delta-reserve approach of Vanduffel et al. The argument is that funding a contingent annuity—pays out only if alive—may reduce the present value of the cost of retirement.

		wealth. Scott argues that the optimum longevity annuity purchase strategy would be to fund the target income at the latest possible start date offered by a commercial annuity carrier—currently age 85. "The start age for the longevity annuity payments will continue to be reduced until the annuity allocation is exhausted. Surprisingly, only retirees interested in fully annuitizing their assets should select an immediate annuity. All other retirees should opt for the longevity annuity that exhausts their willingness to annuitize." Scott provides the following example: "Suppose the retiree with the $1 million funds spending prior to age 85 with a longevity annuity. Using the bond and annuity prices derived previously, he finds that an 11.5 percent allocation to an age-85 longevity annuity will generate annual payouts of $55,385 starting at age 85. Allocating the balance of the portfolio to zero-coupon bonds generates $55,385 in annual income prior to age 85. Thus, this combination has increased annual spending throughout retirement by 33.7 percent relative to using a bond-only portfolio. If an immediate annuity were used instead, the same 11.5 percent annuity allocation would increase spending only by 6.5 percent. "	
2008	"Portfolio choice and mortality-contingent claims: The general HARA case," Huaxiong Huang & Moshe A. Milevsky. Journal of Banking & Finance vol. 32 no. 11 (2008), pp. 2444 – 2452.	The authors create a stochastic model incorporating mortality-contingent claims—life insurance and annuities—labor income and financial capital in an attempt to solve a portfolio choice problem of allocation to life insurance/annuities, risk-free bonds and equities. The demand to hold life insurance is considered from the family unit and, therefore, there is no need to incorporate a separate utility of bequest into the model. Because the family unit must have a floor value of wealth/income, a CRRA utility function is inappropriate. Therefore, utility is derived from modeling general HARA preferences: "…we effectively impose the axiomatic condition that instantaneous consumption rate must be above a lower bound while the breadwinner is alive and after his or her death." The model considers three dates: (1) The applicable horizon for the family unit or household—50 to 100 years; (2) the time of the breadwinner's retirement which is the time when the wage/income process stops; and (3) the death of the breadwinner. The wage/income	The article concludes that the minimum value income needed by the family unit is the main driver of the demand for life insurance or annuities. The optimal insurance amount to hold long or short is not very sensitive to the family unit's risk aversion. This is a justification for using a dollar-level shortfall measure approach. Income—not marginal utility—is the appropriate measure of risk.

		process stops at the earlier of retirement or death. This creates a second discontinuity in the wealth process if a life insurance benefit is payable. The model uses a Poisson distribution to accommodate the jump process. The insurance program is term insurance in which the premium allocation to the insurance carrier is divided by the actuarially fair cost of insurance to determine the amount of death benefit payable. An annuity contract is considered the equivalent of being short a life insurance contract. The single payment for the annuity is divided by the actuarially fair cost to determine the (negative) premium cost which is the annuity payment per unit of time received by the annuitant. The incorporation of a wage process creates a "two state variable class of problems – wage income and asset prices…." This problem requires use of the Hamilton-Jacobi-Bellman [HJB] partial differential equation except in the case where wages and asset prices are perfectly (positively or negatively) correlated. The authors reduce the partial differential equation by applying a finite difference method. The first order conditions for maximum utility require setting the partial differential equation to zero and solving for the optimum with respect to three control variables: 1. Long/Short holdings of Life Insurance 2. Asset Allocation weighting of risky and risk-free asset 3. Consumption (subject to minimum floor). The consumption variable is calculated as the level that maximizes the family unit's utility of consumption (over all economic states). The authors acknowledge that a state-dependent utility function would be more realistic. However, they adopted a simpler HARA utility function in which utility is generated only by consumption amounts above the floor consumption targets before and after the death of the wage earner. Assuming that risky asset prices follow a Brownian motion process, the evolution of family wealth [marketable wealth follows a stochastic differential equation] evolves according to:	

		$dM_t = W_t dt - c_t dt - I_t dt + a_t M_t(\mu_m dt + \sigma_m dB_t^m) + (1 - a_t) r M_t dt.$ Or Change of Wealth is positive in Wages, negative in consumption, positive in the return of the risky asset (a) weight and positive in the risk-free asset (1 – a) return. At retirement, the Wage term drops out of this equation. Thus the family's objective function is to maximize utility of discounted consumption subject to the three control variables: $J = \max_{a_s I_s c_s} \left\{ E_t[\quad] \int_t^T e^{-\delta s} u(c_s) ds \right\}$ subject to the Filtration. The authors solve the HJB equation numerically under a variety of sample ages, minimum consumption floors, risk-aversion parameters and wage/investment return correlations. Depending on the number of years to retirement, there is a critical wage curve: "the critical wage curve is the point above which insurance is purchased and below which insurance is sold (i.e., pension annuities are purchased.) At a young age the critical wage curve is quite low and as the breadwinner approaches retirement, the curve increases and then asymptotes to infinity by retirement. Recall that when the instantaneous wage is above the critical wage curve insurance is purchased and therefore at a young age one is likely to have positive insurance holdings. But just prior to retirement the realized wage process is likely to fall under the critical wage curve, in which case the breadwinner starts to purchase pension annuity units	
2008	"Adaptive Withdrawals" John J. Spitzer, Jeffrey C. Strieter, and Sandeep Singh The Journal of Investing vol. 17 no. 2 (Summer 2008), pp. 104 – 113.	The authors revisit the work of Bengen and others by employing "more stringent testing." Whereas Bengen's methodology is based solely on the single historical sequence of returns over the period 1926 through 2003, this paper employs a bootstrap of annual returns to stocks (S&P 500) and bonds (long-term U.S. Treasury bonds) from 1926 through 2003. Rates of return are inflation adjusted. The bootstrap enables the authors to examine 10,000 return sequences to determine the extent to which sequence risk alters the rate at which portfolio sustainability fails. The paper is interested in the extent, if any, of "time period selection bias" in Bengen's results. Further, the paper extends previous research	A good example of a Bootstrap methodology for modeling retirement risk. Authors test various formulae—including "adaptive" withdrawal rules. Authors suggest that modest allocations to equity are most likely to produce the lowest failure rates over a fixed 30-year planning horizon.

		by introducing adaptive withdrawal policies under which the amount of annual withdrawal is a function of the previous year's investment results. They define the paper's goals as follows: (1) attempt to find a 'safe' inflation-adjusted withdrawal amount from a retirement portfolio, and the best stock/bond allocation that corresponds to it, and (2) determine if there is a 'safe' set of non-constant withdrawal amounts. In addition to testing a spectrum of constant-dollar withdrawals that are a fixed percentage of initial portfolio value, the authors test four variable withdrawal formulae: 1. If the real return increases over the previous year, set the withdrawal amount to 4.4% of initial portfolio value; if the real return falls, set the withdrawal amount to 3.6%--increase or decrease the withdrawal by 10% based on previous performance. 2. The opposite-- If real return increases, decrease withdrawals to 3.6%; if real return decreases, increase withdrawals to 4.4%--leave more money in the portfolio if it is trending upwards. 3. If the real portfolio value falls, reduced withdrawals to 3.6% of the original portfolio; if real portfolio value rises, withdraw the *larger* of 4% of the initial portfolio value or 4% of the current portfolio balance. 4. The opposite—decrease to 3.6% if the portfolio increases or to 4% of initial value or 4% of current value if portfolio value decreases. The authors test the formulae over 13 stock-to-bond allocations over a fixed 30-year planning horizon. No allocation/withdrawal rate combination evidenced a zero-percent failure rate. At a low withdrawal rate of 3% an allocation to 40% stocks resulted in the lowest failure rate of 1.07%. A 4% of initial portfolio value (adjusted for inflation) withdrawal rate had the lowest failure rate (7.72%) at a 50-50 stock-to-bond allocation. Among the variable withdrawal formulae, the most attractive results are achieved by formula	

		#3. This formula produces high average withdrawal amounts for all allocations and generates the lowest failure rate (5.76%) at a 55% allocation to stocks.	
2008	"Retirement Income Products: A Wish List," Joe Tomlinson Financial Planning vol. 38 no. 9 (September 2008), pp. 76 – 81.	The author, a former actuary for John Hancock, describes the challenge faced by the investor with modest resources who seeks to maintain an adequate standard of living: "One such strategy might be that the client withdraws 4% to 4 ½% of assets in the first year and increases withdrawals each year, based on inflation. An alternative might be that the client takes a higher percentage initially, but then varies each year's withdrawals depending on investment performance. These approaches can work well for a client who has built up sufficient assets (or has sufficient pensions or other income) that a withdrawal rate of 4% to 4 ½% can provide adequate cash flow. It preserves investment flexibility and liquidity, and, if the client dies early, it maximizes bequests. However, for clients of more modest means, the 4% to 4 ½% may not be enough to sustain their pre-retirement standard of living. Raising the withdrawal percentage increases the risk that assets will not be adequate for the full span of retirement. Among the risks are: living longer than expected, incurring significant health- or long-term-care costs and experiencing poor investment performance, particularly in the early years after retirement. Such clients may be able to increase retirement cash flow and guard against the risk of outliving assets by purchasing some kind of insurance, in the form of an annuity or other financial product. Purchasing such products gives up some liquidity and flexibility, along with spending funds that would otherwise be available for bequests. However, for many, it may be a necessary tradeoff." The author suggests that retirees lacking large nest eggs combine annuities with investment-oriented portfolios: "A workable solution might be to invest a significant portion of assets in an income annuity with fixed payments and then weight the remaining non-annuitized assets heavily toward equities."	A good description of the economic choices faced by middle-class investors and trustees of modest-sized family trusts. Makes the valuable point that retirement spending is driven primarily by spending *needs* rather than by abstract spending *rules*. The needs v. rules approach may result in maximum utility generated through a front-loaded retirement distribution policy—touches on some of the insights developed in greater detail by Gordon Pye in 2012. The income annuity / invest the difference approach is a variation on the safety first / performance seeking portfolio segregation approach. It is a variation on the two-fund approach. Also casts doubt on the workability of unitrust distribution formulae—in down markets, the income may not be sufficient to sustain a minimum standard of living.

		The author contributes his opinion regarding the "annuity puzzle." Reasons for low sales of immediate annuities may include: 1. Investor aversion to large, irreversible transactions—"like buying a house and making a lifetime commitment to live there." 2. Low sales compensation to insurance company salesmen. 3. Complexity of an annuity sale. 4. Clients' aversion to facing reality—"…individuals may be temporarily happier living with the vague hope that somehow their savings will earn enough to support their retirement." 5. Anti-selection pricing—"For the client in average health, an income annuity may seem like an expensive way to protect against longevity risk."	
2008	"Precautionary Saving, Prudence, and Introduction to Change in Risk" Carl Nelson ACE 501 Fall, 2008 ACE 501 Lecture Notes - 10/15/08	Risk aversion is a response to current risks. The author reviews (in his Johns Hopkins Ph.D. course) academic models designed to explain the precautionary savings motive which is a response to future uncertainties. Precautionary savings, although related to risk aversion, is also known as *prudence.* The concept of prudence was introduced by Haynes Leland in an optimization model where the decision maker with certain period one income and uncertain period two income optimizes the amount to save in period one—i.e., period one consumption is reduced to finance period two consumption. The expected utility of period two consumption incorporates a third derivative term around consumption variance: $\frac{1}{2}U'''(\bar{c}_2)E(\tilde{c}_2 - \bar{c}_2)^2$ It is this (third) term in the utility function that "explains" prudence. In terms of the traditional Pratt definition of utility, the coefficient of absolute prudence is: $\frac{1}{2}\sigma^2_{income}\frac{-v'''}{v''}$	"Prudence" is both a mathematical term (the third derivative of the utility function) and a legal term (requisite care, skill and caution taking into account the terms, purposes, distribution requirements and other circumstances of the trust). Query: What is the role of precautionary savings in prudent trust administration?

		Rothschild and Stiglitz extend research to incorporate various risk concepts including (1) increase in variance, (2) addition of "noise," (3) a mean preserving spread [probability mass shifts towards tails of the distribution], and (4) a change that "makes all risk averse agents worse off." Risks 2 and 4 are equivalent. Risk 1 may not make the investor worse off. A counterexample is two negatively skewed random variances (with most probability mass in the right tail). This points to a potential weakness of mean/variance optimization approaches that ignore higher moments of the distribution. Finally, risk 3 approaches risk 4, and both are best analyzed under the stochastic dominance approach.	
2008	"Immediate Annuities and Retirement Income Portfolios," Timothy E. Hill & Susan J. Sell Milliman Research Report (May 2008).	The authors, actuaries from Milliman, develop a model to test the optimal asset allocation between stocks, bonds, and single premium immediate annuities [SPIAs] for retirement income portfolios. The model considers three cases: (1) Affluent households ($2.5 million portfolio) seeking a target annual constant-dollar income of $111,000 for the joint life of husband & wife both age 65, (2) High Net-Worth households ($8 million portfolio) seeking $300,000 per year, and (3) Middle Market households ($500 thousand portfolio) seeking $20,000 per year. Inflation is an assumed constant 2.5%, fees are a constant 75 basis points. Mortality assumes a factor for improvement and is a stochastic variable (annuity 2000 Mortality Table Frasierized Joint Mortality). Equity returns are based on US Large Cap Stocks, Bond returns are based on U.S. Long-Term Corporate bonds. Given the use of SPIAs, bequests are measured by the ending value of the mutual fund stock/bond allocation. SPIA benefit is estimated by taking an average of current rates from the CANNEX SPIA database to determine the annual income benefit per $100,000 premium. The model calculates shortfall probability defined as occurring if the value of the mutual fund is insufficient to provide the target income net	

		of the SPIA benefit—the SPIA benefit is not indexed to inflation. In each of the three household cases, shortfall probability is first calculated by assuming that the market does not offer an SPIA. Simulations are rerun with an SPIA market to determine any improvements in the probability of successfully meeting the income goal. Thus, the study is a variation on the "welfare improvement" of annuities research topic. For the Affluent household, a 5% equity / 95% bond mix success probability was approximately 40%. The highest success rate was 69% with an 85/15 stock to bond mix. However, the addition of an SPIA generated additional improvement. At an allocation of 70% SPIA / 30% mutual fund (55% stock / 45% bond) the success rate increased to approximately 84%. "The success of the program was measured by target income lasting as long as at least one of the two joint annuitants is still alive." However, the addition of an SPIA tends to decrease the value of the average bequest. Therefore, the authors develop a weighted index for the purposes of comparing the overall benefits of income + bequest. The income component of the index (given a 50% weighting) calculates the present value of any income shortfalls [PVIS] over the projection period. The income component value is calculated as: $$(1 - \text{PVIS}) \div (5 * \text{initial target income})$$ The formula normalized results over the interval of 0 to 1 (no shortfalls generates a value of 1). The bequest component of the index is based on the following formula: $$1 - (e^{1-D} \times .375)$$ Where D = [2 * minimum (4*total initial retirement assets, mutual fund death benefit)] ÷ total initial assets. The formula also gives an index value constrained to a zero to one interval. The bequest component calculation is based on *average* death	

		benefits generated under each SPIA/mutual fund allocation. For the Affluent household case, the portfolio with the highest index value assuming a 50-50 weighting was 70% SPIA / 30% Mutual Fund (85/15 stock to bond mix). Under this solution, there is a 17% failure rate with respect to the income goal and an average bequest of $3.5 million. The model also determined the optimal allocation given the additional constraint that the investors wished to maintain a minimum liquidity target of $1 million for the first 20 years or retirement. In this case, the optimal solution was a 55% SPIA and 45% Mutual Fund allocation (80/20 stock to bond mix). There is a 22% chance of failing to meet the income target with an average bequest value of $3.6 million.	
2008	Prudence: What Are the Odds? 3 to 1? Donald P. DiCarlo, Jr. & Steven M. Fast Representing Estate and Trust Beneficiaries and Fiduciaries ALI-ABA Course of Studies (2008), pp. 479 – 485.	If "probability analysis" ("…the probability that the targeted outcome will in fact be the outcome") is a way to demonstrate Prudence, then what level of probability does a trustee require to justify an asset management course of action? The authors quote various commentators regarding the acceptable chance for success. Generally, opinions range from a 75% minimum to a high of 95%. They conclude: (1) trustees will find it difficult to avoid probability analysis, and (2) will find it difficult to justify decisions with less than a 75% chance of success.	Application of the concept of portfolio sustainability to trusts.
2008	"Stochastic optimization of retirement portfolio asset allocations and withdrawals" R. Gene Stout Financial Services Review vo. 17 no. 1 (Spring 2008), pp. 1 – 15.	Retirement planning often incorporates Monte Carlo simulations to model the sustainability of "fixed" (nominal or inflation adjusted) withdrawals from a portfolio given uncertain returns and planning horizon. More recently, Monte Carlo simulation has been used to model the effects of various withdrawal management "rules." Simulation outputs are expressed in terms of the probability of portfolio depletion, the distribution of terminal wealth, or the distribution of withdrawal amounts under various distribution rules. Stout's essay considers simulation outputs as random inputs for optimizing asset allocation—if you want to have $w distributions, $x terminal value over y years, operating under z distribution formula, here is the "best" allocation.	The optimal portfolio is the "minimum probability of ruin" portfolio—a safety first preferencing criteria. The essay emphasizes the importance of the "withdrawal management process." This emphasis also provides a strong rationale for a client-centered portfolio monitoring and supervision policy. From the perspective of a portfolio depletion risk metric, the optimal withdrawal formula incorporates a calculation based on remaining life

		Stout identifies the "withdrawal management process" as a key driver of long-term economic success. He is interested in demonstrating the advantages of managed vs. unmanaged withdrawals given optimized asset allocations. The simulation model uses annual returns of US large cap stocks and intermediate US gov't bonds over the years 1926 to 2006. The autocorrelation coefficient for bonds is 0.23. The model assumes that return relatives are log-normally distributed (a Kolmogorov-Smirnov goodness of fit test fails to indicate that simulated returns differ from historical returns). The Box complex method is used to search simulation outputs to determine the allocations that minimize the "mean probability of ruin." A second set of simulations determines the "minimum mean probability of portfolio ruin" for unmanaged and managed withdrawal strategies. An unmanaged strategy assumes a constant inflation-adjusted amount; a managed strategy reduces withdrawals when there is evidence that the portfolio will be unable to sustain the distribution, or increases withdrawals for better than anticipated experience. The managed strategy withdrawal equates a feasible withdrawal rate with the present value factor for an annuity due with a payout based on the retiree's remaining lifetime: Withdrawal = PVIFADue The algorithm is as follows: Withdrawal = [1 + Avg inflation-adjusted return – (1 + Avg inflation-adjusted return)$^{-L-1}$] / Avg inflation-adjusted return For an inflation-adjusted rate of return of 2% and a life expectancy of 30 years, the feasible withdrawal rate is: $\frac{1.02-\frac{1}{1.78}}{.02} = 22.91.$ For a $1 million portfolio, 1,000,000 ÷ 22.91 = $43,650. The withdrawal rate is initially modified downward to reflect the	expectancy.

		possibility of investment declines. If such declines occur, there is a greater likelihood that the portfolio will not be able to sustain a minimum standard of living threshold. If the portfolio increases in value, a portion of the surplus is considered to be a reserve against future declines. According to the author, this managed withdrawal process reduces the average probability of ruin by about 50%.	
2008	"Retirement withdrawals: an analysis of the benefits of periodic "midcourse" adjustments," John J. Spitzer, Financial Services Review vo. 17 no. 1 (Spring 2008), pp. 17 – 29.	Often, a constant withdrawal strategy with a low probability of ruin will leave a large amount of terminal wealth. A substantial mean terminal value offers the opportunity to establish flexibility in the withdrawal strategy such that various types of mid-course corrections might reduce unused ending wealth in favor of producing more income. The author notes a growing number of papers that provide sets of "rules" for distribution management: "Some rules encompass dynamically changing asset allocation, changing asset mixes, setting withdrawal caps and limits, present value analysis, eschewing inflation adjustments under some circumstances, and so forth." However, the author also notes that the complexity of the rules make it difficult for a retiree to implement them. The author tests three cases: 1. A benchmark case of a constant inflation-adjusted annual withdrawal that can be sustained over a period of 25 to 35 years; 2. A case of constant asset allocation with the withdrawal amount decision revisited every five years; and, 3. Dynamically reconsidering both the withdrawal amount decision and the asset allocation decision every five years. The first step is to bootstrap the sample data using 21 asset allocations (0% equity to 0% bonds in 5% intervals), 231 withdrawal amounts (2% to 25% of initial portfolio value) over 7 withdrawal periods (5 years to 35 years in 5-year steps). This process creates 33,957 trials. In a grand homage to granularity, the author sifts through this mass of data to find the largest withdrawal amount given the asset allocation and time horizon. The exercise helps the investor calibrate the answer to the following question: given my planning horizon, what is the optimal asset	Taxes & Transaction fees are ignored in this analysis. Calculations are based on real returns of stocks and bonds (only two assets in portfolio) from 1926 through 2005. Two-asset bootstrap preserves pair-wise correlations at the cost of unrealistic simplicity with respect to the asset allocation decision. Author points out that "rules" for distribution management are complex. It is futile to apply a set of "rules" to distribution policy in lieu of judgment. The rules-based approach is merely a pipedream in which a set of autopilot rules substitute for informed and prudent judgment. In any event, retirement spending is driven more by retirement needs rather than by ad hoc rules.

		allocation that will yield the maximum sustainable withdrawal amount at x% success. Once the optimized allocation/withdrawal rate strategy is located for each planning horizon, the author tests the proposition that case 2 and case 3 are both likely to provide greater total withdrawals than case 1. The author simulates results for various planning horizons in order to calculate the probability of shortfall at the 1, 5, and 10% risk levels [bequest objectives are considered to be of secondary importance]. The author finds that Cases B and C are less likely to run out of money early in a planning horizon. Furthermore, Cases B and C are more likely to provide larger total withdrawals. However, Case A is more likely to produce a higher estate value (terminal wealth amount).	
2008	Ruin Problem in Retirement Under Stochastic Return Rate and Mortality Rate and its Applications, Feng Li MS Thesis, Department of Statistics and Actuarial Science, Simon Fraser Univ. (Spring 2008).	Author defines "ruin probability in retirement" as the probability of running out of self-managed assets under a strategy of self-annuitization. Central to the measurement of ruin probability is the distribution of the present value of a whole life annuity under stochastic interest rates and mortality. Assuming that current wealth permits the purchase an annuity that provides sufficient periodic consumption [i.e., current wealth ≥ PV annuity], then electing a self-annuitization strategy carries a positive probability of ruin. Thus, the PV annuity is an appropriate standard for measuring the risks of self-annuitization. The thesis uses an Ornstein-Uhlenbeck [OU] process for generating stochastic rate of return evolutions. In actuarial calculations, OU is a mean reverting process where the instantaneous change in interest rate ["the force of interest"] is a differential equation with the following terms: a (1) coefficient of reversion, (2) magnitude of difference between current and long term average interest rates, and (3) a diffusion coefficient [σ] applied to a standard Brownian Motion process. An OU process allows for autocorrelation in both the drift and diffusion terms. In general finance, as the OU process unfolds over time, the rate of return accumulation function is the integration from time zero to time 't' of the investment returns. The return accumulation function [Y(t)] is normally distributed and the present value of the accumulation function	Argues the case that risk of ruin probabilities, absent a bequest objective and a liquidity need, should be measured by matching commercial annuity payouts to self-annuitization portfolio withdrawals. Model with Ornstein-Uhlenbeck process is an improvement in that it better replicates the dynamic evolution of asset returns. However, the model assumes fixed parameters and varies them according to pre-determined amounts. Asserts that high equity allocations are required to prevent portfolio depletion. Risking continuation of managing a portfolio of risky assets in the hope that an annuity generating higher future income can be purchased depends on the distribution of annuity values. The optimal stopping time occurs "If the PV of the asset allocation strategy (the "synthetic annuity") is less than the PV of the current annuity at equal consumption levels, then it may be more advantageous to self-annuitize."

		is lognormally distributed. Finally, the PV of the Whole Life Annuity function is the standard actuarial annuity pricing formula with its terms adjusted for uncertainty in life expectancy and in the interest rate. In continuous time finance, the expected price of a whole life annuity (the first moment) is $\int_0^\infty E(e^{-Y(t)})\ {}_tP_x\,dt$. The discrete time equivalent uses sums rather than the integral with the limiting upper bound as the oldest age in the mortality table. The thesis calculates the first four moments of the PV Annuity function. Using historical data, the OU model fits parameter values for various asset allocations from 100% equity to 100% (Canadian) T-Bills. The survival function fits a Gompertz law to Canadian Life Tables. At the time of writing, the annuity market price for a 65-year-old male was $14. The author examines three methods for calculating the distribution of the present value of an annuity: (1) a recursive formula (uses trapezoidal numerical integration), (2) fitting the moments to known distributions (best fit = reciprocal gamma distribution); and (3) simulation under an OU process. The simulation and recursion results are close; the gamma distribution has a poor left-tail fit especially for high volatility parameters. Note: It is important to keep in mind that the PV of an "annuity" also refers to the present value of the asset allocation strategy which is used to provide annuity-like income. In this case, it is the PV of a self-annuitization strategy. If the PV of the asset allocation strategy (the "synthetic annuity") is less than the PV of the current annuity at equal consumption levels, then it may be more advantageous to self-annuitize. Assuming a wealth-to-consumption ratio [w/k] that must finance constant consumption (k, where k matches the withdrawal amount provided by a commercial annuity) for life, the probability of ruin (portfolio depletion) is the likelihood that the present value of lifetime consumption exceeds the PV of the annuity alternative [$14 per $1 annual payout]. "Thus, the ruin probability is equal to the probability that the present value of a $1 whole life annuity, under the asset allocation strategy, will be greater	

		than $14." Assuming current interest rate environment equal to the long-term mean, for a 100% equity allocation, the CDF suggests that approximately 75% of self-annuitization results have a lower cost than the alternative of purchasing a commercial annuity. For an all T-Bill portfolio, approximately 50% of results have a lower cost. Thus, an all equity portfolio for a male age 65 that distributes lifetime consumption equal to that which could have been secured by an commercial annuity purchase, has a ruin probability of approx. 25%; an all T-Bill portfolio has a ruin probability of approx. 50%. The lowest probability of ruin, given the model's assumptions, occurs at an allocation of 80% equity / 20% long-term bond (risk of ruin = approx. 23%). The author concludes that the choice of asset allocation is the single most important factor impacting the probability of ruin under a measure that uses the annuity payout as the benchmark for the withdrawal amount. If the portfolio under evaluation is not the sole source of retirement income (Social Security, Pension benefits) or if there is a desire to leave a bequest, then the retiree may have limited "ruin tolerance." If a male retiree age 65 has a ruin tolerance greater than 25%, he should self-annuitize. Conversely, a retiree with only a 5% ruin tolerance would be willing to pay up to $24.10 for a $1 life annuity. This is a measure of the welfare benefit of annuities for highly risk averse investors.	
2008	"Retirement Income Redesigned: Master Plans for Distribution," Harold Evensky CFA Institute Conference Proceedings Quarterly vol. 25 no. 3 (September 2008), pp. 65 – 74.	"It is not uncommon for me to read that investors insist on a 90 percent probability of success. I consider that an absurd standard because it means that the investors are quite likely to underspend significantly and, therefore, have an unnecessarily substandard quality of living. As financial planners, we should not be trying to protect only the downside. We need to think about protecting the quality of life on the upside…." [p. 68]	Asserts that 90% probability of success is imprudent because it is too conservative--too much money is left on the table. Reverses usual arguments concerning maximizing safety in favor of maximizing standard of living throughout retirement.
2008	"The Impact of	The authors define the primary purpose of an annuity as follows: "…to	Paper provides a nice description of the intuition

	Health Status and Out-of-Pocket Medical Expenditures on Annuity Valuation," Cassio M. Turra and Olivia S. Mitchell Chapter 10 of <u>Recalibrating Retirement Spending and Saving</u> eds. John Ameriks and Olivia S. Mitchell (Oxford University Press, 2008), pp. 227 – 250.	protect people against the risk of outliving their financial resources in old age." The article considers two annuity valuation models: 1. The Yaari Life-Cycle model –with an uncertain life span--which assumes that annuity markets are complete; and, 2. A model that assumes health changes throughout the Life Cycle and the motivation for precautionary savings when the annuity market is not complete. Although the original Yaari model suggested that utility would be optimized by complete annuitization of wealth in the absence of a bequest motive, extensions of the Yaari model incorporate bequest motives, mortality assumptions, interest rate factors and so forth. The consumer choice problem is to determine how much of initial wealth should be annuitized. The answer to this question is determined by selecting the amount of an annuity that maximizes the discounted sum of expected future utility over both consumption and bequest. The paper provides a nice summary of the backward recursion approach to determining maximum utility. Assuming a terminal age—e.g., age 95—and no bequest motive, the authors provide the following guide to the HBJ value equation: "…the retiree would maximize utility while consuming all remaining wealth, W_t; the period t single immediate life annuity, A_t; and preexisting real annuity (e.g., Social Security benefits) S_t: $V_t(c_t) = \max[u(c_t)]$ Subject to the following constraints: s.t. W_0 given $W_t \geq 0 \;\; \forall\, t$ $W_{t+1} = (W_t - C_t + S_t + A_t)(1+r)$ Where r is the interest rate. Knowing the optimal consumption decision in period t allows one to find the optimal consumption decision that maximizes the value function in period t-1. The same logic is used subsequently in each previous period to choose the consumption that	underlying the solution to the HBJ equation—the backward recursion approach to maximizing utility. Examines the utility value of annuities in incomplete markets—i.e., where certain health risks generate liquidity needs that cannot be adequately funded by periodic annuity income. The value of AEW—annuity equivalent wealth—diminishes in the face of the possibility of health shocks.

		maximizes the Bellman equation: $$V_{t-1} = u(c_{t-1}) + \beta_1 p_{t-1}[V_t(c_t)]$$ Where β is the discount factor, and ${}_1p_{t-1}$ is the probability of surviving from period t-1 to t for an individual of health status j at the age at annuity purchase. The concept of Annuity Equivalent Wealth [AEW] is the difference in utility realized by individuals with complete access to an annuity market and utility realized by those lacking access to such a market. Although previous research suggested that a retiree would be willing to forgo approximately 40% of initial retirement wealth to secure a guaranteed lifetime income, the AEW in the face of health problems may be substantially lower. Even for healthy annuity purchases, "our stylized life cycle model with uncertain out-of-pocket medical expenses shows that annuities become less attractive to people facing such medical expenses." This observation raises the issue of the attractiveness of annuities in an incomplete market setting: "…when both adverse selection and uncertain medical expenses are accounted for and annuity markets are incomplete, we show that annuity equivalent wealth values are fairly low for people in poor health, and about 25 percent higher for people in good health."	
2008	"Longevity Risk Quantification and Management: A Review of Relevant Literature," Thomas Crawford, Richard de Haan, and Chad Runchey available at https://www.soa.org/Research/Research-Projects/Life-Insurance/research-	This study was sponsored by the Society of Actuaries. Among its observations are: • US Risk-Based Capital formulas omit longevity risk from the calculation of insurance risk. "Current longevity risk management techniques in the US market are limited…." "…insurers tend to back their annuity liabilities with a significant amount of corporate debt. These investments have inherent risk of default, which would leave the insurer with less assets than expected to provide for the annuity payments in situations where economic growth was slowed below expected levels, or the economy was in recession." • In Europe, "insurance companies have already declared	Annuities are not financial instruments with payoffs orthogonal to the capital markets. They are backed by bond portfolios and are guaranteed by corporations operating within the markets. An increase in bond default risk simultaneously increases the risk that an annuity carrier will fail to provide promised payments.

	long-risk-quant.aspx	significant losses as they have been forced to strengthen reserves for annuity portfolios." • "Adverse mortality experience, whether higher or lower than expected, has implications for reserving and for capital requirements if the ability of the life industry to raise capital becomes impaired based on the market's perception of the variability of life company debt. • A survey of morality improvement indicates that historically it was primarily limited to improvements in infant mortality. Now, however, there is substantial improvement of life expectancy at older ages. • In the US, less than 5% of retirees have voluntarily purchased an annuity. • "…the lives that purchase annuities can be very different to those who purchase life insurance. Thus, companies that use natural hedging to manage longevity risk are therefore exposed to basis risk." • "In the annuity market, and particularly for immediate annuities, the current standard of practice does not involve significant price differentiation for health status. This results in annuitants that are selecting against the insurance company, as they expect to live longer than the average individual."	
2008	"Annuity Valuation, Long-Term Care, and Bequest Motives," John Ameriks, Andrew Caplin, Steven Laufer, and Stijn Van Nieuwerburgh Chapter 11 of Recalibrating Retirement Spending and Saving eds. John	The article examines the demand to hold annuities at various levels of wealth when faced with a positive utility for bequests as well as precautionary motives when faced with long term care requirements. They cite a study by Sinclair and Smetters ["Health Shocks and the Demand for Annuities," CBO Technical Working Paper no, 2004-9: Congressional Budget Office 2004] which posits a decreasing demand for annuities as wealth diminishes. At low wealth levels, a serious medical shock would simultaneously deplete liquid assets, raise current expenses, and decrease the mortality-adjusted future value of annuity payments. The model presented in this article incorporates a strength of the bequest motive parameter ($\overline{\omega}$) and a measure of the extent to which	Demand to hold an annuity is, under the authors' model, decreasing as wealth diminishes—i.e., positive correlation between wealth and annuity demand. As trust resources are depleted, the ability to fund lifestyle expenses may diminish. At the same time, an annuity purchase solution may increase the current beneficiary's vulnerability to health shocks. An interesting expression of the trustee's dilemma. Model is an interesting example of a Markov

	Ameriks and Olivia S. Mitchell (Oxford University Press, 2008), pp. 251 – 275.	bequests exceed a threshold subsistence level—i.e., are a "luxury good" (φ). Given a CRRA investor with additive, time-separable utility, the optimal bequest will cover $\overline{\omega}$ years of spending at an excess expenditure rate of φ. The individual may be in one of four states in the model: (1) good health, (2) medical problems not requiring LTC, (3) requirement for LTC, and (4) death. Initially, the individual is endowed with positive wealth and is in the good-health state. The matrix is a Markov chain transition matrix with "an age-varying, one-period" transition probability. The Markov transition matrix confirms the results of the Sinclair/Smetters study: "…with greater wealth, the risk subsides of bankruptcy due to LTC costs or depletion of the intended bequest, making the individual more willing to pay for the annuity…..as wealth decreases, demand decreases as well, as annuitization exposes the less wealthy retirees to a greater risk of ending up on Medicaid or sacrificing their bequests." If, as the authors suggest, "retirement security…can be summed up simply as 'having the resources you need, when you need them,'" then standard annuities may be only a partial solution to security in the face of severe health shocks: "such products do little to deal with retirees' need for resources when emergencies arise, and they can even exacerbate financial distress in exigent situations."	transition matrix.
2008	"Will your Savings Last? What the Withdrawal Rate Studies Show," William Reichenstein AAII Journal vol. 30 no. 6 (July 2008), pp. 5 – 11.	Reichenstein advances the proposition that: "Withdrawal rate studies using four research methods reach similar conclusions…." This short article reviews two studies in detail: (1) "Retirement Savings: Choosing a Withdrawal Rate That Is Sustainable," by Cooley, Hubbard & Walz [1998]. This study uses historical returns on stocks, bonds and cash over rolling 30-year planning horizons for portfolios consisting of the S&P 500 and Corporate bonds. Taxes, fees and transactions costs are not considered. For a 50-50 stock-to-bond allocation and an inflation-adjusted 4% withdrawal policy, the portfolio survived the 30-year planning period 95% of the time—i.e., a 5% shortfall	Study considers the impact of fees, transaction costs and taxes on the sustainable withdrawal rate.

		risk. (2) (2) "Guidelines for Withdrawal Rates and Portfolio Safety During Retirement," by Spitzer, Strieter & Singh [2007]. This study uses the S&P 500 index and intermediate-term government bonds. Returns were simulated over a 30-year planning horizon. For a 50-50 allocation the failure rate (shortfall risk) is approximately 6%. Reichenstein concludes: "In short, a 4% withdrawal rate is a rule of thumb reasonably 'safe' withdrawal rate." "The selection of shortfall risk should vary with the retiree's ability and willingness to reduce withdrawals if returns prove disappointing." Reichenstein draws three lessons "related to the choice of asset allocation:" (1) The withdrawal rate literature suggests that "…the target asset allocation should include no less than 50% stocks….Based on historical returns, this literature indicates that portfolios with heavy stock exposures are more likely to survive a long retirement period than portfolios with heavy bond exposures." (2) "…sustainable withdrawal rates looking forward from portfolios that include more asset classes are not likely to be substantially higher than the sustainable withdrawal rates from portfolios containing only U.S. large-cap stocks and bonds. In short, don't look to asset allocation to help increase your withdrawal rate above the levels shown to be 'safe' in the studies reviewed here." (3) As your targeted withdrawal rate increases, so must the allocation to stocks: "Investors who select an aggressive withdrawal rate need a heavy stock exposure to provide the best chance of survival." Most studies, however, ignore fees, transaction costs and taxes. These frictions drive the net return below the projected future return. "For example, for a retiree with a 30-year withdrawal horizon, the sustainable withdrawal rate is estimated to be 0.99% lower [given 2% annual	

		expenses]; for a 20-year withdrawal horizon, the decrease is estimated at 0.79%; for a 10-year horizon, it is estimated at 0.54%." Reichenstein assets: "If lower gross returns and investment expenses cause the retiree's future net returns to fall 2% below historical gross returns, then the sustainable withdrawal rate for a retiree with a 30-year horizon is about 1% lower. The reduction in sustainable withdrawal rate is estimated at about 0.8% for a retiree with a 20-year horizon and 0.67% for a retiree with a 15-year horizon."	
2008	"Data Dependence and Sustainable Real Withdrawal Rates," David M. Blanchett and Brian C. Blanchett Journal of Financial Planning, vol. 21 no. 9 (September 2008), pp. 70 – 85.	The authors point out that the early history of research into sustainable portfolio withdrawals (e.g., Bengen, & Cooley, Hubbard and Walz) used historical data series. When the data series is limited to the previous 35 years, it may be misleading because of higher than average returns combined with lower than average SDs. A Monte Carlo analysis is used to examine the sensitivity of distribution rates to risk and return parameters. The analysis assumes distributional normality because "a normal distribution better enables practitioners to recreate this analysis." The study is not primarily interested in creating "rules regarding distribution rates (as an infinite number of combinations exist and the future is unknown"). The study concludes that the sustainability of long horizon portfolios is more sensitive to returns (errors in the mean) than to SDs.	"The acceptable probability of failure (or success) has varied in past distribution studies. Some authors use 90 percent, while others use 95 percent or even 99 percent. These levels of 'certainty' can provide retirees with a false sense of security should future market conditions differ from those in the past."
2008	"Aspiration Level, Probability of Success and Failure, and Expected Utility," Enrico Diecidue and Jeroen Van De Ven International Economic Review Vol. 49, No. 2 (May, 2008), pp. 683 – 700.	The authors create a decision making model that combines an aspiration level point with expected von Neumann and Morgenstern utility. Mathematically, the combined value function is equivalent to expected utility with one or more discontinuities at the aspiration level. Expected utility is additive over the spectrum of possible results. Once an aspiration level is defined, utility is positively weighted by a scalar [μ] for results in excess of the aspiration level and negatively weighted [λ] for results below the aspiration level. The mathematical expression for expected utility of a prospect [X] is the utility value for each possible outcome weighted by the probability of the outcome: $X \rightarrow \sum_{i=1}^{n} p_i \, u(x_i)$	An aspiration level refers to a rate of return or a level of terminal wealth that one seeks to attain in the future. The introduction of such a wealth level or return target simultaneously introduces the concept of shortfall risk—the probability of failing to attain the target. An aspiration level is not equivalent to the concept of free boundary—the point at which a fund is solvent. The free boundary concept is divorced from the concept of probabilities. It is a current empirical test. Likewise, a reference point may or may not be associated with the free boundary

		With an aspiration level, the model for decision making under risk becomes: $$X \rightarrow \sum_{i=1}^{n} p_i\, u(x_i) + \mu P(x^+) - \lambda P(x^-)$$ With $P(x^+)$ the overall probability of success and $P(x^-)$ the overall probability of failure. In the authors' model, the aspiration level is exogenously fixed and set to a value of zero. The model assumes that the decision maker is fully aware of the probabilities. The authors state that their model better accounts for decision making behaviors because (1) it incorporates the concepts of shortfall and success probabilities relative to the aspiration level; (2) it acknowledges that investors tend to use a heuristic incorporating an aspiration level primarily as a method to simplify complex decisions regarding preferences over the range of possible outcomes; and (3) the values to which the utility function converge at the aspiration point differ depending on whether convergence occurs from below x^- or from above x^+. The risk aversion function has a different slope for outcomes below the aspiration level than for outcomes above the level. There is a non-differentiable discontinuity at the aspiration level point. Although the model provides a behavioral basis to explain observed results in decision making experiments, the model suggests that decision making is not primarily related to a reference point. Hence, the authors do not over rely on explanations based on prospect theory. Rather the discontinuity in the value function is more closely related to loss aversion. The decision making model is thus an additive combination of classic expected utility theory and a behavioral-oriented jump at the aspiration level point. When convergence towards the aspiration level occurs from below, investors tend to exhibit risk-averse behavior above and risk-seeking below the level.	concept. A reference point may be determined subjectively—i.e. the highest historical periodic return or wealth level attained to date. Usually a reference point is fixed and unlike a free boundary, is not the outcome of a stochastic process. The authors stress: "We do not see the aspiration level as primarily part of the intrinsic values of outcomes, but as a consequence of a decision heuristic of decision making to simplify decisions." The aspiration level acts as a kind of shorthand signpost that sends out a signal that a portfolio is or is not on track with respect to an investor's goals. The behavioral explanation offered by Diecidue and Van De Ven stands in contrast to that offered in "The safety first expected utility model: Experimental evidence and economic implications," Haim Levy & Moshe Levy, [2009].

		Note: the above observation holds for two-outcome prospects but may not hold for duplex prospects where each prospect has the probability of multiple outcomes.	
2009	"A new strategy to guarantee retirement income using TIPS and longevity insurance," S. Gowri Shankar Financial Services Review vol. 18 no. 1 (2009), pp. 53 – 68.	Paper explores the potential benefits of a two-stage investment program that could be designed by institutions and offered to investors. Alternately, an individual could construct a personal program by a combination of TIPS and a deferred (contingent payout) annuity. The author contents that the payout amount could be both substantial and guaranteed. Some previous studies suggest that a static withdrawal rate the value of which is 4% of the initial portfolio value carries a positive probability of financial ruin. The probability depends on the assumptions underlying the retirement income risk model. Dynamic management of withdrawals, however, can diminish the risk of ruin likelihood and/or improve the starting withdrawal percentages. The only instrument that totally eliminates the possibility of ruin is a life annuity. The author proposes an Inflation Protected Retirement Annuity [IPRA] strategy which occurs in two stages: 1. For the initial stage, purchase of TIPS until a target age is reached (e.g., age 80). The TIPS portfolio is depleted at this time. 2. At the time of the TIPS portfolio purchase, sufficient premium is paid into a deferred annuity that pays out only in the event that the investor survives stage one. For surviving investors, the annuity pays a constant dollar benefit for life. The benefit of an IPRA plan is similar to that of a Defined Benefit Pension. Although purchase of TIPS and annuities represent bets on the interest rate environment, even at a worst case (real return = 0%), a 65 year old male could withdraw 4.59% constant dollar yearly income. Bequests are available only if the retiree dies prior to the exhaustion of the TIPS stage one fund.	MetLife Longevity Income Guarantee-Maximum Income Version Annuity. Single premium at ages 55 through 79 pays out nominal dollar annuity starting at age 85 if buyers survives. Otherwise no benefits payable. Argument is a combination of Scott, Sharpe & Watson [2008]—the 4% solution and Scott's [2008] article on the contingent deferred annuity.
2009	"Making Your Nest Egg Last a Lifetime," Anthony Webb	This article begins with a quick review of three commonly discussed rules of thumb for retirement income: <u>Spend income / conserve principal</u>—this strategy will mean that the	Although many lifecycle models focus on the "optimal" asset allocation, the assumed preference for household consumption is also a critical model

	Center For Retirement Research at Boston College (September 2009), Number 9-20.	investor(s) will forgo spending any of their original capital and, possibly, any of the gain generated by it. The initial asset allocation and not cash requirements dictate household consumption. In fact, the household's consumption needs dictates the asset allocation—a situation that might lead to overweighting of fixed income or tilting towards high-yield stocks. Generation of accounting income takes precedence over establishing a portfolio with a suitable risk/reward profile. <ins>Spend down over life expectancy</ins>—this strategy subjects a household to a 50% chance of outliving their assets. "…a strategy of consuming one's wealth over any fixed period, even if flawlessly executed, has a probability of failure that equals the probability of surviving to the end of that period." <ins>Spend a Fixed Percent Each Year</ins>—the most well-known advice is to fix spending at 4% of original principal. However, this strategy does not account for the withdrawal stress against the portfolio's current value. If a portfolio, starting with a 4% of initial value withdrawal plan in year one, declined by 50% during the year, the new withdrawal percentage amount would equal 8% of portfolio value. Clearly, after one year, a 4% rule and a 8% rule "…cannot both be the right answer to the same decumulation problem." According to the author, the problem of designing an optimal strategy is difficult because the preferred tradeoff between current and future consumption differs over each household. A common economic assumption is that households consumption will decline over the course of retirement "…based on the premise that households prefer consumption in periods when they are more likely to be alive." The extent to which a household is willing to accept reductions in their future consumption has a significant impact in the optimal spending rate. Annuities may have a role to play in the implementation of a decumulation strategy: "Although households may in theory do better by delaying the purchase of an annuity, there is a strong case for annuitizing sufficient wealth immediately on retirement to at least	component. "Technically speaking, economists typically assume constant relative risk aversion with a coefficient of risk aversion lying in the range of two to five. Assuming a risk-free asset, no pre-annuitized wealth (or, equivalently, that pre-annuitized wealth funds basic living expenses that do not enter into the utility function), and a rate of time preference that equals the rate of interest, consumption will decline at a rate that equals the annual mortality risk multiplied by the reciprocal of the coefficient of risk aversion." "In a simple model with a single risk-free asset, the optimal consumption rate is 16 percent higher at a coefficient of risk aversion of two than at a coefficient of risk aversion of five."

		secure the household's required minimum standard of living, given the likelihood and consequences of error."	
2009	"Managing a Retirement Portfolio: Do Annuities Provide More Safety?" John J. Spitzer Journal of Financial Counseling and Planning vol. 20 no. 1 (2009), pp. 58 – 69.	The article draws upon Spitzer's previous research and modeling methodologies. Specifically, Spitzer uses a two-asset class (stocks and bonds) bootstrap approach to determining the delta in shortfall probability as annuities are added to the investor's asset allocation. In this case, a 30-year guaranteed, inflation-adjusted annuity (based on an inflation-adjusted annuity product offered by Vanguard) was integrated into the retirement withdrawal strategy. Spitzer defines the investment problem as follows: "How much to withdraw each year from a retirement portfolio is a complex decision: withdraw too much and the retiree outlives the retirement portfolio; withdraw too little and the good times that might have been are forfeited." Spitzer asserts that "…the recommendation of a 4% - 5% withdrawal amount with 50% - 70% of the portfolio in stocks and the remainder in bonds seems to be generally accepted and widely recommended." The article assumes that retirement wealth is in a Tax Deferred Account [TDA] and, whenever the RMD exceeds the income target (4% of the initial portfolio value adjusted for inflation), the excess is invested in a taxable brokerage account the allocation of which, when combined with the TDA, adheres to the target allocation. Spitzer's base case is six annuity weightings (from 0% to 100%) with the remaining balance, if any, allocated 50% to stocks and 50% to bonds. Given that, at the time of the article's publication, the Vanguard inflation-adjusted annuity was paying 4.3%, and that the income target was 4%, Spitzer concludes that a 100% allocation to an annuity is a risk-free proposition in that the probability of shortfall will fall to zero. He does not consider counterparty risk--the probability of insurance company insolvency. The claim is: "…the annuity strategy provides any level of comfort up to 0% shortfall probability. Providing a high level of	Spitzer justifies use of a bootstrap ("the bootstrap is the superior choice") as follows: "Monte Carlo methods use computers to generate rates of return from certain probability distributions. Unfortunately, the appropriate probability distribution to draw from is unknown. Investigators often base their Monte Carlo simulations on the assumption that rates of return are normally (or log-normally) distributed. For the annual data used here, the assumption of normality cannot be confirmed by statistical testing; hence, Monte Carlo is a dubious choice. Bootstrapping methods sample with replacement from the same dataset, in this instance from the historical data. No assumptions about the underlying distribution are necessary." Note: Spitzer raises the issue of how investors define the term 'safe': "…to some a 19% shortfall rate is acceptable, but to others nothing short of 0% shortfall will suffice. Each retiree must determine what 'safe' means to them." Note: Spitzer cites Milevsky and Young (2007) "Annuitization and Asset Allocation, Journal of Economic Dynamics & Control, pp. 3138-3177 who recommend that "individuals should always hold some annuities." Milevsky and Young conclude that the demand for annuities is increasing with wealth, risk aversion, health assessments, and portfolio volatility. Conversely, a Trust's demand to hold annuities may increase as wealth decreases and the current income beneficiary's economic benefit

		security comes at the cost of not being able to provide a larger estate to heirs." In the Spitzer model, shortfall probability decreases monotonically as the commitment to annuitization increases. Pair-wise differences in the shortfall rates (from a high shortfall probability in the 5.57% to 5.85% range depending on initial value of the TDA) are statistically significant. Using annuities to reduce withdrawal shortfall probability, however, comes at the cost of lower ending terminal wealth (bequests). Spitzer considers a dynamic annuity strategy wherein the retiree annuitizes a portion of TDA wealth upon retirement and, if the equity premium is positive, additional funds can be annuitized in the future: "…the second annuitization locks in the proceeds of the market upswing." Note: Such a strategy would be appropriate for an investor with increasing absolute risk aversion.	becomes more tenuous. Note: Are immediate annuities an asset class or a component of a withdrawal strategy?
2009	"Who Should Buy a Lifetime Income Annuity? And When?" Don Ezra CFA Institute Private Wealth Management(February 2009) available at http://www.cfapubs.org/doi/full/10.2469/pwmn.v2009.n1.10	A brief article for the CFA organization. It develops the concept of annuity-equivalent wealth: "Imagine two societies, A and N (A for available lifetime annuities and N for not available). You and your spouse live in Society A. You've worked out your desired postretirement spending pattern. And you have just enough money to lock in that spending pattern by buying an annuity. Buying that annuity gives you a certain amount of utility…." "Your twin and your twin's spouse live in Society N, and they have exactly the same desired lifestyle and exactly the same amount of money as you. But they don't have annuities available to them. So, they're exposed to longevity risk. What do they do? Being sensible, they set aside some money each year against the chance that they'll live past 90. Thus, they spend less than you and have less utility than you. Depending on what exactly delivers utility—or, as an economist would put it, depending on the shape of the utility function—their utility might be 80 percent or 90 percent of yours; certainly, it will be less than 100 percent because, unlike you, they cannot live their desired lifestyle." "How much more money would they need to raise their utility to your	Provides a justification for a top-down approach to the annuity purchase decision. Annuities are considered as a safety net to protect essential needs. Uses the concept of conditional mortality expectation—i.e., the longer you live, the greater the age of life expectancy—to argue for postponing the option to annuitize as long as possible. [note: a parallel argument is made by Milevsky and Huang [2010—"Spending Retirement on Planet Vulcan"] However, if an investor is on the threshold with respect to the amount required to fund threshold needs, an annuity should be purchased without delay. Only surplus wealth provides an option to delay.

		level? That's what annuity-equivalent wealth measures. It's expressed as a ratio—1.2 or 1.45 or whatever—and it is always bigger than 1.0, of course, because no pooling mechanism exists for longevity risk. That's why you're better off living in Society A than in Society N (Admittedly, I'm ignoring the fact that your twin will be able to leave a bequest, and you won't. But the utility from a bequest is typically much smaller than that from living comfortably yourself)." The author states that as retirees age they must set aside a bigger proportion of remaining wealth every year because the standard deviation of life expectancy is a bigger proportion of life expectancy at older ages. It follows that the benefit of hedging longevity risk through an annuity increases with age. Additionally, the cost of an annuity, all else equal, decreases with age: "therefore, postpone buying an annuity as long as you can." The decision regarding the optimal weighting of annuities in a retirement portfolio is a function of the wealth/consumption ratio: "…if you have, at most, enough money to buy your desired lifetime annuity, buy it, because doing so maximizes your utility. But if you have more than enough for the annuity, you have the luxury of focusing on wealth management first and longevity protection only in certain circumstances." [More Wealth, more Choices]. The weaker the bequest motive, the stronger the propensity to fund cash flow for lifestyle maintenance via an annuity; the stronger the motive, the more a household is inclined to take investment risk.	
2009	"The high cost of a no-fee, no-commission Single Premium Immediate Annuity (SPIA)." http://www.retireearlyhomepage.com/annuity_costs.html	The author asserts: "You can estimate the costs embedded in a single premium immediate annuity by comparing the premium quote you get from the insurance agent to the expected present discounted value (EPDV) of an immediate life annuity. The EPDV is sometimes called an 'actuarially-fair annuity' or 'money's worth annuity.' Economists define the ratio between the EPDV and the premium quote as the Money's Worth Ratio (MWR). For individuals of average mortality. Money's Worth Ratios as low as 0.70 are not uncommon….." The article presents a table of "embedded costs" based on the current [December 2009] real yield on the long-term TIPS [TIPS coupon of	

		2.35%]—for nominal annuities, the yield on AA-rated commercial bonds is a reasonable discount rate. For the subpopulation of contract holders with annuitant mortality adjustments to the Social Security Administration's Mortality table, the embedded cost of an inflation adjusted annuity providing a $12,000 annual benefit to a Male age 65 is Insurance Quote ($218,465) – EPDV of actuarially fair annuity ($192,609) = Embedded Cost ($25,856). The gap between the MWR and the insurance company's premium could be explained by 1. Interest rate reinvestment risk: "To the extent that the longevity of people in the annuity pool exceed [sic] the maturity of readily available fixed income securities, insurers bear some risk that they won't be able to reinvest the proceeds on maturity at an equal or higher interest rate for the portion of the annuity pool that remains." 2. Administrative overhead and Profit. 3. Distribution and Marketing Costs. These include sales commissions, advertising costs, and other incentives.	
2009	"Reality check: The implications of applying sustainable withdrawal rate analysis to real world portfolios," Qianqiu Liu, Rosita P. Chang, Jack C. De Jong, Jr., and John H. Robinson Financial Services Review vol. 18 no. 2 (Summer, 2009), pp. 123 – 139.	The authors note that most early research devoted to determining a maximum sustainable withdrawal [MSR] rate focused on a single two-asset class portfolio under static asset allocations. Their study considers dynamic asset allocations for a range of time horizons: "…dynamic decision-based asset allocation and withdrawal strategies." Specifically, they apply a bootstrapping algorithm over a range of asset allocations and withdrawal rates for 10, 15, 20, 25, and 30-year periods. In general, "…the analysis confirms that equity diversification appears to improve sustainability success rates, especially for investors who require high initial withdrawal rates and have long anticipated retirement periods." A withdrawal strategy which spends the bond portion of the portfolio first is superior to both glide-path strategies or to constant (static) asset allocation. The authors cite the 2007 and 2008 papers of Spitzer and Singh which test portfolio sustainability under various portfolio management rules including: (1) constant asset allocation, (2)	Advocates a 'spend-bonds-first' withdrawal strategy. This is the opposite of the floor + multiplier [CPPI] insured portfolio management approach. It spends 'safety' first and retains risk. Although the numbers may work out given the authors retirement income risk model, the investor's utility function would take an unusual form in order to find this strategy appealing. Compare this article with "Spending Buckets and Financial Placebos," Moshe A. Milevsky Research Magazine (June 1, 2007) and "The Benefits of a Cash Reserve Strategy in Retirement Distribution Planning," Shaun Pfeiffer, John Salter and Harold Evensky, Journal of Financial Planning vol. 26 no. 9 (September, 2013), pp. 49 – 55.

		withdrawing from the previous year's best performing asset class, (3) withdrawing from the previous year's worst performing asset class, (4) withdrawal from equity first, (5) withdrawing from bonds first, and (6) using a glide path strategy. The model developed in this article examines several allocations: (100 stocks-0 bonds; 75-25; 50-50; and 25-75) using the S&P 500 and 10-year treasuries.; and a more diversified allocation where the equity component consists of 45% S&P, 30% Russell 2000, and 25% MSCI EAFE. Data covers the period January 1970 through December 2007. Initial withdrawal rates of 3%, 4%, 5%, 6%, and 7% (adjusted for annual inflation) are tested over planning horizons of 10, 15, 20, 25, and 30-years. Taxes and advisory fees are not considered. The bootstrap algorithm selects an historical month at random. It then records the returns for the various asset classes as well as for the rate of inflation in that month. The authors contend that this preserves the correlation structure for the portfolio. The bootstrap model results generally indicate, for all stock/bond allocations, the diversified portfolio generates higher means and lower standard deviations and, therefore exhibits a higher rate of sustainability when compared to the two-asset class portfolio. This is especially true for longer horizons and higher withdrawal rates. However, the spend-bonds-first strategy produces the best results: "…the bonds first strategy produces a higher median remaining portfolio balance for every time period and every withdrawal rate for each of the three models comprised of both equities and bonds. The absolute advantage increases as both time horizon and equity allocation are increased." These results support the conclusions of Spitzer and Singh. The authors suggest that the bonds first strategy works best because "…it ameliorates serial returns risk better than either the glide-path or constant allocation strategies." They explain: "Under the glide-path and constant allocation methods, investors who retired at the end of 2007	

		would be forced to sell stocks early in retirement when the market is dramatically down--i.e., at exactly the worst time. In contrast, under the bonds first strategy, newly retired investors effectively buy time for the stock market to recover. To the extent that these investors may be more exposed to sharp downturns later in retirement when the bond portion of the portfolio has been depleted, there are two factors that work to their advantage. First, presumably the equity portion of the portfolio has had time grow [sic] since the initial retirement date (or to recover from downturns early in retirement). Second, the time horizon has been shortened by the number of years that income was supported by the bonds. In this light, it seems clear that a benefit of the bonds first strategy is that it may minimize the risk of catastrophic portfolio failure." The argument continues: "…portfolio depletion in year 19 of a 20-year time horizon is not as catastrophic as failure in Year [sic] 9. To the extent that the bonds first approach may help forestall depletion in extreme investment environments until the later retirement years, it may also place retirees in a better position to eventually replace portfolio income through the annuitization of home equity."	
2009	"Structuring Distribution Strategies for Retirees in a Bear Market" Jack Gardner Investments & Wealth Monitor (November / December 2009)	The article, written for the Investment Management Consultants Association, compares retirement income withdrawals during the periods 2000 through 2008 and 1973 through 1981. The sample portfolio has an initial value of $1 million and is invested 60% in the S&P 500 index and 40% in the Barclays Intermediate Term Government Bond Index. The initial year's spending rate is $50,000 increased annually by the rate of inflation. The author defines this withdrawal scheme as a "lifestyle" withdrawal strategy. In both nine-year bear market periods, the portfolio ending values fell to below $500,000. The portfolio deployed in 1973 was fully depleted in 21.5 years. The author maintains that the critical flaw in the "lifestyle" strategy is that the "…withdrawal rates are totally delinked from the performance of the investment portfolio." In contrast to the "lifestyle" withdrawal strategy, the author defines an "endowment" strategy. The endowment withdrawal strategy takes into	A good example of the 'adaptive withdrawal strategy' approach to determining spending in retirement. The author contends that "…withdrawal rates need to be revisited frequently." His decision rule for spending is taken from commonly used "smoothing rules" for endowment portfolios.

		account both the prior year's withdrawal amount and the current year's portfolio value. Although many endowments calculate annual spending targets as a weighted combination of 80% of previous year's spending plus 20% of the target rate multiplied by the current year's portfolio value, the author prefers a 90/10 weighting "…because it moderates the spending volatility a bit more in bear markets." For example, beginning with a $1 million portfolio and a 5% spending rate, spending in year one is $50,000. If the portfolio value at the beginning of year 2 is $800,000, applying a 90/10 rule results in [.90($50,000) + .10(.05)($800,000)] = $45,000 + $4,000 = $49,000. If inflation was 6%, the lifestyle withdrawal strategy in year two results in a distribution equal to $51,940. The 90/10 endowment withdrawal strategy results in the ability to sustain the portfolio launched in 1973 for over 30 years.	
2009	"Immediate Annuity: Fixed vs. Inflation-Protected," Felix Schirripa Working Paper ELM Income Group (April 2009), pp. 1 – 14. Available at the Employee Benefit Research Institute website:http://www.ebri.org/pdf/programs/policyforums/cpi_fixed_annuities_value.pdf	The article suggests that an inflation-adjusted annuity may be preferred to a fixed nominal payout annuity: • "The disadvantage of the lower starting payment from the inflation-protected annuity begins to disappear quickly after adjusting for taxes (non-qualified purchases) and inflation…." • "In the April 2009 interest rate environment, there appears to be little difference between the fixed and the 'real' interest rates insurers may be using to price income annuities." The author measures the cost of the inflation protection in terms of the reduction in the initial pre-tax payment under the inflation-adjusted annuity: "…in the current environment, an inflation-protected life annuity guarantees 'real' payments that start at 25% to 30% below the comparable fixed annuity, before we consider the effects of income taxes." However, given the 2009 interest rates and inflation rates the author estimates that the inflation-protected annuity payouts will start to surpass the fixed annuity payouts in year nine if inflation stays at "an effective annual rate of 4%."	Note: the different method of calculating the exclusion ratio for nominal and inflation-adjusted SPIAs makes an inflation-adjusted annuity attractive to higher bracket taxpayers.

		To determine the tax consequences of receiving annuity payouts, the contract holder must calculate the exclusion ratio. "For fixed annuities, the exclusion ratio is determined by dividing the total investment in the contract by the total expected return. The investment in the contract is generally the gross premium. The expected return is generally the total amount that the annuitant can expect to receive; and in the case of life contingent annuities, expected return is based on IRS tables. In the current environment, the exclusion ratio of a fixed annuity without inflation adjustments is in the ballpark of 65% for a male age 65…." "On the other hand, for inflation-protected annuities purchased with non-qualified funds, the exclusion from gross income is determined based on a fixed dollar amount (not a fixed ratio). The dollar amount of exclusion is held constant, regardless of future inflation rates, until the investment in the contract is recovered. As payments are increased for the effects of inflation a larger and larger portion of each payment is taxed. In effect, all the inflation adjustments are fully taxable. In the current environment, roughly 95% of the first year's payment would be excluded from taxation." The author estimates that the difference in the methodology for calculating the exclusion ratio reduces the initial payout advantage of a fixed nominal annuity from 25 to 30% to a post-tax advantage of approximately 17% for a taxpayer in the 35% bracket. Accumulating the post-tax differences in payouts at an after-tax interest rate of 3% assuming an inflation rate of 4% suggests that the maximum accumulated cost of the inflation-protected annuity is $5,600 for an initial premium expenditure of $100,000 by a male aged 65. The high water cost occurs in year eight; thereafter, the inflation-adjusted payouts decrease the cost with a breakeven point of approximately 16 years. However, if inflation runs at 2% then the fixed annuity has a lower effective cost for more than 30 years.	
2009	"A Dynamic and Adaptive Approach to Distribution Planning and Monitoring" David	The authors recommend that the withdrawal rate decision from a retirement portfolio be revisited annually in order to adjust for changes in investment performance, unforeseen expenditures, and the expected planning horizon—i.e., for client aging. The distribution decision is not one that should be irrevocably determined at the time of retirement:	Explores the exchange of 'sequence risk' for portfolios with a high weighting to stocks for 'longevity risk' for portfolios heavily weighted to bonds. Prudence requires an appropriate balancing of these risks via the asset allocation decision.

	Blanchett & Larry Frank Journal of Financial Planning vol. 22 no. 4 (April 2009), pp. 52 - 66.	"An adaptive approach to distribution planning, where the withdrawal rate is fluid and not constant, can dramatically improve the probability of success of a distribution strategy. Reviewing the withdrawal rate also allows for the withdrawal amount to be increased as situations warrant…." The essay reviews the literature on "decision rules" regarding how and when to alter portfolio distribution policy. The author cites Guyton & Klinger [2006], Bengen [2001] and Stout and Mitchell [2006]. The authors claim that their suggested "revisiting approach" is simpler than the complex decision rules often recommended in academic studies. Returning to first principles, the authors point out that any asset allocation strategy must address two risks: sequence risk and longevity risk. A higher weight on equity exacerbates the deleterious effects of sequence risk while a higher weight on fixed income exacerbates longevity risk. The paper tests four asset allocations [20-80, 40-60, 60-40, and 80-20 stocks to fixed income] where stocks are split two-thirds to U.S. large Caps and one-third to Foreign Large Caps and Fixed Income is split 50-50 between cash and intermediate-term corporate bonds. Initially, a planning horizon is set—e.g., 30 years for a retiree age 65—and an inflation-adjusted withdrawal based on the portfolio's starting value is determined. Depending on the evolution of the portfolio, the withdrawal amount can be either increased or decreased by 3% annually, or can remain the same. These options are in addition to the inflation adjustment. The authors establish the following decision rules: "The withdrawal dollar amount is decreased by 3 percent if: • The probability of failure for the portfolio is greater than 20 percent when the target end date is 20+ years away • The probability of failure is greater than 10 percent when the target end date is 11-19 years away • The probability of failure is greater than 5 percent when the	The set of portfolio spending rules outlined in this essay are based on probable evolutions of a portfolio rather than on strict historical back testing. Rules are now emerging from bootstrapped models rather than from historical realizations. This approach places the essay squarely in the research effort to find a set of rules for retirement spending.

		target end date is 10 years or fewer away • The withdrawal amount is increased by 3 percent if the probability of failure is less than 5 percent • If neither of the above conditions is met, the distribution dollar amount does not change (except for inflation or deflation adjustments)." Portfolio failure rates are calculated based on the results of a bootstrap of the four asset classes from 1927 through 2007. Distributions are taken at the beginning of each year and the test portfolios are rebalanced to their target allocations on a monthly basis. Taxes and fees are not considered. The authors use a static 4 percent of initial portfolio value as their baseline for comparison. Given the (1) range of their historical data, (2) the asset allocations, and (3) the bootstrap methodology, they calculate that "the probability of failure for a static 4 percent withdrawal rate for a 60/40 portfolio over a 30-year distribution period was only 4.07 percent….even for a 50-year distribution period the probability of a failure for a 4 percent initial withdrawal rate for a 60/40 portfolio was only 16.91 percent." Revisiting the portfolio annually to apply the withdrawal rate decision rules reduces the probability of failure significantly. For example, the 4 percent initial withdrawal rate for 60/40 portfolio allocation over a 30-year distribution period has a failure rate of only 2.65%. Furthermore, as the time remaining in the withdrawal period decreases as the investor ages, the withdrawal rate may increase over time.	
2009	"Model Risk in Retirement Simulations" Marlena I. Lee (January 2009)available at http://www.pdfdrive	This is a Dimensional Fund Working Paper. Most Monte Carlo simulations assume a normal distribution of financial asset returns. The models generate independent and identically distributed (i.i.d.) returns which produce means and standard deviations that remain constant through time. Such models cannot incorporate serial correlation, mean reversion, or volatility clustering. Examination of empirical returns, however, suggests that i.i.d. assumptions are unwarranted. This fact introduces	Author argues that expected return, withdrawal and contribution rates have greater impact on outputs of models of portfolio sustainability than does a model's assumption of a normal return distribution.

	.net/model-risk-in-retirement-simulations-dimensional-fund-advisors-e1871898.html	the concept of model risk in the interpretation of simulated outcomes: "Because model risk in Monte Carlo simulations is potentially large, the tool should be used with caution. Retirement simulations can be useful to illustrate the broad impact of retirement choices, but they should not be relied upon to make accurate predictions of future wealth." The author investigates the sensitivity of model outputs to assumptions regarding the statistical characteristics of return distributions. In the main, she concludes that critical evaluative criteria (e.g., portfolio sustainability over long planning horizons) are not particularly sensitive to serial correlation, and higher moments. Models using normal distribution assumptions (e.g., simulations, and bootstraps) do not produce values that are far off from those produced under a normality assumption: "These results indicate that accounting for the non-normality in monthly returns does not lead to dramatically different conclusions about the distribution of wealth and portfolio survival rates compared to simulations with normally distributed returns." Likewise, autocorrelated volatility (ARCH/GARCH modeling) produces only negligible differences in output values when compared to models using i.i.d. assumptions: "These results indicate that while the assumption of constant volatility is not a valid description of historical returns, it is a fairly harmless assumption because it does not substantially bias the distribution of wealth over time." The essay contains a short discussion of return predictability—a tendency for returns to mean revert. Lee notes that there is some evidence for mean reversion in the time series of U.S. stock returns. A commonly used test for mean reversion is to compute variance ratios: "If prices follow a random walk, the variance of k-year log returns should equal k times the variance of annual log returns. If prices are mean-reverting, long-horizon returns are less volatile than would be implied by the random walk hypothesis; and the variance ratio, defined as the k-year variance divided by k times the one-year variance, would be less than 1." Although both the empirical return series and the series	

		outputs of various models exhibit long-term (five to ten year) ratios of less than one, Lee points out that caution must be used in interpreting the outputs: "…variance ratios computed using the monthly market returns do not statistically differ from 1.0 at the 10% level. Additionally, the average variance ratios from the baseline simulation, where returns are truly independent, also fall below 1.0. Academic research suggests that sample variance ratios are downwardly biased. Using the time series of EAFE returns, Lee finds less evidence of decreasing variance ratios over the long term. However, model outputs are highly sensitive to inputs regarding expected returns, savings/contributions, and withdrawals/distributions. For example, "…using the historical average to estimate expected returns would lead one to conclude the portfolio has a 76% probability of survival to age 90. However, if the equity premium is actually closer to 3% to 4%, survival rates at age 90 are only around 17% – 24%. Additionally, small changes in savings or withdrawal assumptions are shown to have noticeable impact on simulation results."	
2009	"A Framework for Portfolio Decumulation," Richard K. Fullmer The Journal of Investment Consulting Vol. 10, No. 1 (Summer, 2009), pp. 63 – 71.	The article begins by considering two "fallacies." The first fallacy is a form of "the fallacy of time-diversification risk." Although a high allocation to stocks may improve the likelihood that a portfolio can sustain high periodic withdrawal rates, it also makes a retiree vulnerable to early portfolio depletion because bear markets create asset declines of large magnitude. Such declines, conditional on their occurrence, increase the likelihood that a portfolio will run out of money early in retirement leaving the retiree with a substantial shortfall in resources to fund consumption in later years. Loading for equity increases sustainability at the cost of creating potentially catastrophic shortfalls if the investor fails to realize the expected equity risk premium. A second fallacy, in the author's opinion, is the underlying assumption in most simulation methodologies that the initial investment strategy will be maintain throughout the planning horizon. A variation on this assumption is found in the equity glide-path strategies employed by target-date funds. The author assets: "The only way such an assumption	In certain respects, the two fallacies serve as "strawmen" arguments—that is as rhetorical devices designed to place his later observations in a more favorable light. His first point is that simulators are used to justify high equity allocations (and larger distributions?). Although such an allocation strategy may reduce the long-term rate of portfolio failure, it may also lead to a more catastrophic failure if risk assets decline in value. However, this seems more like an argument against bad simulators. Simulators have been adopted specifically because they acknowledge tail risk and allow a practitioner to make an approximation of it. Using a simulator should, if its assumptions are correct, allow the practitioner to determine in conjunction with the client a good

makes sense, however, is if the investor's risk exposure is static over time." Given the nature of retirement income planning, however, such an assumption is never true: "Holding more-aggressive portfolios early in retirement and less-aggressive all0ocations later in retirement leads to the retiree taking the most investment risk precisely when it is most risky to do so—when the risk of outliving the portfolio (longevity risk) is greatest."

Pivoting from his discussion of "fallacy" two, the author further contends that longevity risk subsumes other forms of commonly identified risks including: (1) spending risk, (2) investment risk, and (3) mortality risk. This train of thought leads to the conclusion that "The beauty of life annuities is that they fully hedge longevity risk." [subject, as the author points out, to the claims paying ability of the insurer]. Therefore, investors are justified in establishing an "'annuitization hurdle' above which the portfolio balance needs to remain." The fair market value of assets should exceed the current cost of an annuity designed to produce the investor's targeted lifetime cash flow.

The study next explores characteristics of the 'annuitization hurdle.' The annuity cost curve declines with age assuming that the yield curve, insurance carrier profit objectives, and population mortality remain fixed. The annuity cost curve level moves inversely with the direction and magnitude of interest rate change. The greatest impact on duration sensitivity occurs at younger ages. At advanced ages, the insurance carrier expects shorter investment horizons during which to realize interest from its underlying investment portfolio.

A portfolio "should be structured so as to retain the investor's option to annuitize (guarantee) the required cash flow stream for life…." In this framework, the risk to the investor "…is simply the risk of falling short of the annuitization hurdle." Changes in spending or asset allocation are levers the investor can use to reduce the risk of assets falling below the annuity's cost. Therefore, the investor should embrace the prospect of dynamic portfolio management vis-à-vis the annuity cost benchmark. This, in turn, requires asset allocation models to "…be capable of

long term strategy which embraces the higher likely returns from equities and which allows the user to recognize, as time passes, whether that failure rate is increasing and therefore allow them to adjust.

The second fallacy is that the simulators assume a static strategy which increases risk asset allocation at the beginning - exactly when the longevity risk is greatest. This is a misunderstanding of longevity risk. Longevity risk is greatest as age increases. If you are 80 and planning for 4 years of additional life, living a few extra years can create big problems. On the other hand, if you are 60 and planning for 24 years, an extra 2-5 years doesn't make much difference. Of greater importance are the returns on the portfolio over the next 10+ years; and that is exactly what a higher equity allocation is designed to improve.

Moving on, his discussion of accumulation and decumulation raises a series of questions all of which are best answered with well-designed simulators. However, he states that a simulator isn't great because it leads to static decisions - again this suggests a deficiency in certain simulators rather than a deficiency in the general approach.

And so he reaches a conclusion - buy an annuity when necessary! The idea of purchasing an annuity is, as presented in this article, also a bit of a fallacy. The downside of an annuity purchase is hardly discussed—i.e., no matter when you buy the annuity, doing so will always constrain the set of future consumption opportunities. This is one aspect of the risk/reward tradeoff that the investor should consider. For example, it may be more advantageous to purchase an annuity later in life

		modeling dynamic strategies." A simple case study illustrates the consequences of the author's ideas. For example: "If markets are poor, the portfolio may decline in value. In this case, the portfolio value in excess of the annuitization hurdle may decrease, resulting in an increase in longevity risk in the couple's financial plan. Longevity risk also can increase if interest rates fall, causing the annuitization hurdle to rise….in response to this increase in longevity risk the couple could spend less, invest more conservatively, or annuitize a portion of the portfolio…." The case example continues: "If the surplus shrinks enough, annuitization may become the only option that satisfies the couple's risk tolerance. If the surplus shrinks so far that a shortfall results, the couple will need to either reduce spending or gamble that favorable market conditions will get them back over the hurdle."	when the investor is in a position to reduce annuity costs through mortality credits derived from risk pooling. However, because longevity risk increases with age, it is a coin toss whether someone lives 2, 4 or 6 more years. A different aspect of longevity risk created by an annuity solution is "brevity risk." The main body of the article then looks at using the annuity cost as a feasibility measure. This is a more useful approach which leads to better conclusions—i.e. make some initial decisions; track the results; make some reasonable alterations in strategy; and, if things go bad, plan to buy an annuity late in the game. Buying the annuity late may make sense as risk reduction, but in the presence of excess wealth (i.e., when a bequest is almost certain) it represents a sub optimal solution because portfolio failure becomes less likely. This is the annuity as safety-net option rather than the annuity as the ideal first strategy out the door option.
2009	"The Case for Flexible Retirement Planning" R. Gene Stout and John B. Mitchell Journal of Personal Finance vol. 8 (2009), pp. 63 – 78.	The authors define flexible retirement planning as "retirement withdrawal management as a means of reducing the probability of running out of money in retirement…." They draw an analogy to piloting an aircraft: "periodic withdrawal changes serve as a correction to the portfolio amortization path: much as a pilot corrects a flight path to ensure a soft landing." Using data for the period 1926 through 2006, they simulate yearly returns for a portfolio allocated 65% to U.S. large cap stocks and 35% to U.S. government bonds. The historical (arithmetic average) real return was 6.77%. If this number is plugged into a spreadsheet and if portfolio variance is ignored, the portfolio amortizes to zero over a 30-year span at an annual withdrawal rate of 7.37%. By simulating return sequences, however, over a 30-year horizon, withdrawal rates of 3.5% to 6.5% of the	Note—annuity calculations use several "averages"—expected remaining life span, average historical portfolio ROR, etc. Results are "unconditional" and are not reflective of the current circumstances that may characterize the investment regime. Article highlights the advantages of flexible withdrawal policy over autopilot formulae. The authors value the retirement cash flow liability as the present value of an annuity due. However, the discount rate is the portfolio's average historical rate of return. The planning horizon is based on life expectancy.

		initial portfolio value result in failure rates ranging from 3.37% to 47.69%. Thus, the authors conclude that a 7.37% withdrawal rate is unrealistic for most retired investors. Additionally, there is considerable variance over 30-year intervals. A retiree in 1955 would have earned at annual real return of 3.62% in a 65-35 allocation; a 1932 retiree would have earned 7.52%. Incorporating mortality into the analysis significantly reduces the rate at which portfolios run out of money over a simulated 30-year horizon. For an age 65 investor, a constant real withdrawal rate of 4% over the earlier of death or a 30-year planning horizon results in a 2.33% probability of ruin. Increasing the withdrawal rate to 5% generates a ruin probability of 7.63%. Given the historical data and simulated results the authors wish to test the effects of a flexible withdrawal strategy on ruin probabilities. A set of decision rules substitutes for the initial irrevocable decision to withdraw a fixed percentage of the beginning portfolio value. The decision rules represent a process that incorporates "…year-by-year tracking of the plan's performance, changes in the client's circumstances, and taking limited correction actions based upon the client's revealed risk tolerance when the portfolio is threatened by financial ruin." Given the current withdrawal rate, the rules call for calculating the expected life span at the investor's current age. This number will decrease as age increases. The target income stream for the expected remaining life span is annuitized [present value of an annuity due] where the annuity factor is determined by a discount rate equal to the average historical return over a comparable period of the 65-35 portfolio. The calculated value is then compared to a pre-determined upper limit and lower limit. If the annuity-adjusted value is greater than the upper limit, an increase in the withdrawal rate is forthcoming; if below the limit, the investor implements a decrease in the withdrawal rate. Values within the upper/lower bounds do not call for any withdrawal rate change. A "higher" lower limit means that the investor is more risk averse and is more willing to implement a decrease in withdrawals in the face of poor	

		performance.	
2009	Travis L. Jones and Jack Brown, "Integrating Asset-Liability Risk Management with Portfolio Optimization for Individual Investors II," [An updated version of the article that appeared in the Journal of Wealth Management, Vol 12, No. 3 (Winter 2009), pp. 51 -60.	The authors assert that "…a straightforward application of multi-period risk management (via ALM), combined with common forms of optimization, can yield an optimal portfolio that can be easily adjusted to specifically fund liabilities or take on higher levels of risk." They review briefly some previous studies on Asset/Liability Management [ALM]; and they contrast this approach to one in which the advisor uses mean-variance optimization [MVO] to optimize the portfolio subject to the pre-calculated liability constraint. "The goal of multi-period risk management is to provide a map of liabilities (future portfolio outflows for lifestyle needs) that is clear enough to determine the sorts and amounts of investments needed to fund them." The authors indicate two solutions to creating such a map: Cash flow matching: "…a bond could be bought for each liability, with maturation values matching the size and timing of each liability….The problem is that this is an expensive proposition…and few pension sponsors or individuals have the means or willingness to implement such a plan." Bucketing: "…short-to-intermediate liabilities are matched with fixed-income investments commensurate with the present values, and the duration of such liabilities and equity investments are designated to fund future long-term liabilities, with an individual's risk tolerance impacting the length of the matching periods designated for short-to-intermediate liabilities." The authors recommend a nine-step process for incorporating liabilities into an MVO software application. Key steps include: • Mapping the magnitude and timing of all future liabilities • Discounting the liabilities to a single net present value number [Note: this seems to lose a wealth of information regarding the term-structure of the liabilities. It also seems to side-step the stochastic nature of deterministic liabilities because of changes in applicable discount rates]	Although the article is a general description of how the authors use mean-variance optimization in an Asset/Liability Management approach, the primary case example concerns the asset allocation decision for an investor about to retire. Note: A third iteration of the 2009 article appears in Jack Brown and Travis L. Jones, "An Application of Asset-Liability Management for Financial Planners," Journal of Financial Planning, Vol. 24, No.5 (May 2011), pp. 62 – 69. This bibliography includes a brief discussion of the 2011 version. Note: The literature review of ALM is somewhat perfunctory. The authors use a discrete-time approach; and, although they claim that the horizon is multi-period, most, if not all, of the discussion and case example deals with optimization at a single period—presumably, the liabilities are re-calculated from time-to-time, and the portfolio is periodically re-optimized based on new inputs. The question at hand is: given the investor's liabilities, which is the best allocation today? The "multi-period" aspect of the analysis occurs because liabilities extending over several periods are discounted to a single net-present value. In the Merton continuous time approach, the optimal portfolio consists of a three-fund solution: (1) the "Merton Optimum" investment portfolio; (2) the "state-hedge" portfolio (e.g., a currency overlay portfolio; or, a portfolio to hedge against deteriorating investment opportunities within the macro economy; and, (3) a risk-free asset. Incorporation of liabilities into this context results in

		• Determining the length of time during which the investor wishes to 'cash-match' the liabilities—cash matching the entire time-line of liabilities may be too expensive; the investor may wish to use a shorter period. [Note: this is a form of feasibility test: "Discounting these liabilities to their present value helps determine if the residual portfolio assets are enough to meet long-term liabilities, through a cash-flow matching approach for the entire lifespan. If a high percentage of overall assets is required to be allocated towards immunizing assets in the matching period, then it may be determined that the residual percentage of assets may not be enough to meet long-term liabilities."] • The NPV of "immunizing assets" enters the MVO application as a minimum constraint on the fixed-income position; the remainder of assets are optimized accordingly (i.e., short positions may be disallowed, tax and trading costs may be included, limits may be placed on maximum exposures to an asset class, etc.) • Securities are selected once the MVO application determines the optimal asset allocation; and, finally, • "Rebalancing is simply a matter of updating the above steps." The case example focuses on an investor who is about to retire. It outlines both short- and long-term liabilities which must be funded from a $2 million financial asset portfolio. At the start of retirement, the discounted present value of liabilities over the next ten years amounts to $806,753. For liabilities occurring after year 10, the authors contend that the investor can rely on equity investments because stocks have high expected long-term returns. Consequently, the MVO allocates a minimum of 40.34% of the portfolio to fixed income. The optimizer determines the optimal mix of fixed income (approximately 75% of fixed income in the aggregate bond market and 25% in high-yield bonds) with remaining assets allocated to U.S. and foreign stock asset classes. "This leaves the investor with an overall expected return and standard deviation of 9% and 12% respectively." At this point, the case example becomes somewhat muddy. The authors	the addition of a "liability-hedge" portfolio; or, a four-fund solution. This discussion is lacking in the Jones-Brown article—in part, because they contend that both investors and advisors are unable to understand and work with complex financial concepts and models. Readers will have to decide if the above observations constitute a flaw—too great of an oversimplification; or, a contribution to the profession—an effort to bring the dimension of investor liabilities into practices relying primarily on MVO software.

		write: "For this same investor, we apply a traditional mean-variance approach. To do this, we will optimize around the same constraints used for the ALM method, but without an immunized constraint. We will instead add an aggregate portfolio risk constraint….We will assign a portfolio standard deviation risk constraint of 10%...." The differing standard deviation values destroy the validity of the comparison. The 10% SD-constrained MVO portfolio bumps the fixed income asset allocation from 40% to 45% and results in a slightly lower expected return: 8.0% vs. 8.5%. The table of results presents a MVO portfolio ("scenario 6") with a 12% constraint on standard deviation (30% allocation to fixed income with an expected return of 9%); but this scenario is not discussed. The apparent advantage of incorporating the point-estimate of future liabilities in terms of a constraint in the MVO approach is that "This method provides investment practitioners with a means to better customize the portfolio for a given investor's situation, and gives individual investors the confidence that their future liabilities can be funded out of their portfolio without a major change to the asset allocation."	
2009	Noel Amenc, Lionel Martellini, Vincent Milhau and Volker Ziemann, <u>Asset-Liability Management in Private Wealth Management</u> EDHEC-RISK Asset Management Research (September, 2009).	The article asserts that investment advisors employ analytic tools that, in the main, are inadequate to the task of providing investment advice and portfolio management to private wealth clients: "…most existing financial software packages used by private bankers to generate asset allocation recommendations rely on single-period-mean-variance asset-portfolio optimization, which cannot yield a proper strategic allocation for at least two reasons. For one, optimization parameters [expected returns, volatilities and correlations] are defined as constant across time, a practice which is contradicted by empirical observation and does not make it possible to take into account the length of the investment horizon. For another, and most importantly perhaps, liability constraints and risk factors affecting them, such as inflation-risk on targeted spending, are neither modelled not explicitly taken into account in the portfolio construction process." Finally, the authors note that most commonly-used optimization tools cannot directly incorporate the	A large section of the study deals with asset management for retirees seeking to secure a lifetime inflation-adjusted income. The monograph is written from a "Eurocentric" perspective in that its descriptions and criticisms of private wealth management assume that investment advisors are employed at bank-owned private wealth management departments. Many observations and conclusions, however, are appropriate for both U.S. investors and advisors. The intention is to outline "…new forms of welfare-improving financial innovation inspired by the use of asset-liability management techniques…." A small criticism is that the tone often suggests, albeit

		specific objectives and risk constraints of individual clients. The article endorses an ALM investment management process for private client wealth management: "Asset-liability management (ALM) refers to the adaptation of the portfolio management process to the presence of constraints relating to the commitments represented by the investor's liabilities….we use a broad definition of 'liabilities,' which encompasses any commitment or spending objective, usually self-imposed…." The article presents a stochastic model that (a) includes risk factors influencing both asset and liability values, and (b) estimates dependency relationships between them. Assets are proxied by commonly used stock, real estate, bond and commodities indexes tracking publicly traded and liquid securities. Three month US Treasury Bills and a 20-year constant-maturity bond return are additional fixed income asset classes used in the model. Predictive variables incorporated into a vector-autoregressive [VAR] modeling approach are (1) dividend yield, (2) credit spread, and (3) term spread (10 year minus 3-month T-Bill rate). The proxy for a retirement income liability is the return on TIPS (adjusted for periods of missing data) over the relevant planning horizon—the inflation risk premium is constant. The proxy for a real estate acquisition objective is the return on the FTSE NAREIT index. Historical data are used to construct a VAR system with estimated variance/covariance matrices and residual correlations. A matrix is estimated for the vector of expected asset class returns, for covariance between assets and liabilities, for covariance of residuals, and for a matrix "…that selects the vector of excess returns on the assets and the liability from the state vector **z**." The outputs of the VAR model imply that stocks are not particularly risky for the long-term investor: "This effect is explained by the presence of implied mean-reversion in stock returns." By contrast, the volatility of T-Bill investments increases with the planning horizon: "…due to the uncertainty involved in rolling over short-term debt in the presence of stochastic interest rates." Volatility is also increasing with time for real	subtly, that such 'observations' are actually 'discoveries' freshly made by the authors. To be fair, the paper explicitly states that most of the strategies and techniques recommended for private client asset management were previously developed for institutional client portfolio management. Despite the observations concerning the inadequacy of commonly used portfolio optimization software employed in an asset-only approach to investment management, the work does not veer off into behavioralist finance. The ALM discussion clearly differs from behavioralist 'goals-based' management: "Satisfaction of the investor's long-term objectives is fundamentally dependent on an ALM exercise whose aim is to determine the proper strategic inter-class allocation as a function of the investor's specific objectives, constraints, and time horizon. In other words, what will prove decisive is the ability to design an asset allocation programme that depends on the particular risks to which the investor is exposed. Similarly, the very concept of a risk-free asset depends on the investor's time-horizon and on his objectives. Hence, a five-year zero-coupon Treasury bond will not prove a perfectly safe investment for a private investor interested in a real estate acquisition in five years….In other words, the first benefit of the ALM approach is perhaps its impact on the menu of asset classes, with a focus on including an asset class that exhibits the highest possible correlation with the liability portfolio." The authors take care to position their study in the MPT tradition: "An initial attempt to introduce liability

		estate and commodity investments. Whereas changes in liability values are often sensitive to the inflation rate, it is important to note the correlations of various asset classes with the CPI as a function of time. In particular: • "T-Bills have negative correlation with realized inflation over short horizons, but this correlation becomes positive and relatively high over horizons exceed ten years." • "…stocks…have good inflation-hedging properties over horizons that exceed twenty years….thus equity investments should offer significant inflation protection over longer horizons…." An important implication of these findings is that the use of a "…standard one-period optimization model, as is customary in private wealth management, is a great oversimplification that does not allow investors to benefit from the life-cycle effects induced by time-varying opportunity sets." The article introduces a dynamic asset allocation model in which the funding ratio [assets ÷ liabilities] is an additional state variable. The model assumes Constant Relative Risk Aversion and log-normal distribution of asset returns. This collapses the model to a mean-variance problem where optimization is defined in terms of the funding ratio [surplus optimization]. The efficient frontier (the set of feasible mean-variance ALM portfolios) is defined in terms of the value of the funding ratio. The mathematics of the efficient frontier calculation leads to "…a fund separation theorem, dictating allocation of a fraction α of the wealth to the performance-seeking portfolio (PSP) and another fraction 1-α to the liability-hedging portfolio (LHP). The efficient frontier can than [sic] be drawn by letting α vary between 0 (which generates 100% allocation to the LHP) and 1 (which generates 100% allocation to the PSP)…." This procedure contrasts with the typical Asset-Only allocation that ignores investor liabilities. The hedging portfolio arises because of the presence of liabilities: "…indeed, the funding ratio is the ratio of assets to liabilities…." The optimal portfolio asset allocation in the face of stochastic state	constraints in optimal portfolio selection theory was made by Merton (1993), who studies the allocation decision of a university that manages an endowment. In this particular strand of the finance literature, are papers by Rudolf and Ziemba (2004) who have formulated a continuous-time dynamic programming model of pension fund management in the presence of a time-varying opportunity set, and Sundaresan and Zapatero (1997) whose work also involves an endogenous retirement decision." We note that this monograph precedes the Das, Markowitz, et al. paper published in 2010 [Sanjiv Das, Harry Markowitz, Jonathan Scheid and Meir Statman, "Portfolio Optimization with Mental Accounts," Journal of Financial and Quantitative Analysis, Vol. 45, No. 2 (April 2010), pp. 311 – 334] which attempts a synthesis of modern portfolio theory and behavioral approaches. The authors speculate that "…it would be interesting to try to cast the ALM approach to private wealth management in a context in which the investor has a behavioural objective. One challenge here is that recent advances in behavioural finance, while they provide very useful insights into investors' behaviour, do not provide much guidance to the design of a formal normative analysis of optimal asset allocation decisions. A possible approach would involve capturing some of this complexity by adding a set of suitably specified investor-dependent goals and constraints to the standard expected utility maximization paradigm." The article also contains some interesting

		variables leads to a three fund solution: (1) the performance seeking portfolio, (2) the liability-hedging portfolio, and (3) the risk-free asset. The authors argue that "…failing to take an ALM approach to long-term investment decisions and sticking to the sub-optimal asset-only perspective will generate very substantial opportunity costs for the private investor." Of particular interest is the example of a retired 65-year-old investor with a funding ratio at retirement date equal to 100%. The investor wishes to provide a lifetime inflation-adjusted income with a horizon to age 100. The article considers four model variations: (1) an asset allocation scheme, based on a short-term planning horizon [myopic], which fails to include a liability-hedging portfolio (LHP); (2) an asset management scheme, based on a long-term planning horizon, which fails to include a LHP; (3) an asset management scheme based on an ALM model without a LHP; and (4) an asset management scheme, based on an ALM model, which incorporates the perfect LHP portfolio over the correct planning horizon. The perfect LHP portfolio is a "menu of assets…sufficiently rich to allow for a perfect hedge of liability risk." The first scheme reflects a commonly used Asset Only {AO] single period optimization approach; the second scheme is a multiperiod AO approach that fails to include liabilities; the third scheme is an ALM approach that fails to determine the specific properties of a liability hedging portfolio; and, the fourth scheme corresponds to the authors' recommended three-fund ALM asset management approach. The retirement planning efficient frontier in the AO management space—model variations 1&2—generates the curve of annualized expected return versus annualized return volatility over planning horizons of 1, 5, 10, and 25 years. For all planning horizons, variation 2 dominates variation 1: "…when the horizon is very long (T = 25 years), the opportunity cost of using a static optimization model with a short-term objective is substantial." As the analysis moves to the ALM model--variations (3&4)--the efficient frontier is generated by tracing the curve of the annualized expected funding ratio value against the annualized funding ratio volatility—i.e., the surplus efficient frontier. The myopic	observations regarding difficulties in implementing a TIPS-oriented strategy.

		AO values are also graphically presented for comparison purposes. The model demonstrates that variation 4 provides the most favorable risk/return tradeoffs. The TIPS asset class proxy for the composition of the liability-hedging portfolio manifests several difficulties: "…it is important to emphasise [sic] that investing in inflation-linked instruments is neither the only nor necessarily the most cost-efficient means of obtaining protection from inflation uncertainty. For one, the capacity of the inflation-linked securities market is not sufficient to meet the collective demand of institutional and private investors….In addition, real returns on inflation-protected securities, negatively impacted by the presence of a significant inflation risk premium, are usually very low, which implies that investing in inflation-linked securities, when feasible, is costly." The article concludes: "…taking an ALM approach to private wealth management generates two main benefits. First, it has a direct impact on the selection of asset classes. In particular, it leads to a focus on the liability-hedging properties of various asset classes, a focus that would, by definition, be absent from an asset-only perspective. Second, it leads to defining risk and return in relative rather than absolute terms, with the liability portfolio used as a benchmark or numeraire."	
2009	"The 4% Rule—At What Price?" Jason Scott, William Sharpe, and John Watson Journal of Investment Management vol. 7 no. 3 (2009), pp. 31 – 48.	The authors posit a market model similar to that used for ALM pension management. They argue: "A 2% risk-free real rate is broadly consistent with the historic record for U.S. Treasury STRIPS and TIPS investment returns." Using a 30-year retirement horizon, the cost of generating a real dollar in each of the 30 years is $22.40. [Purchase of 30 discount bonds with sequential maturities purchased at a 2% compounded discount rate—this formula reduces to the PV of a 30-year annuity without loads or expenses] Thus, the guaranteed withdrawal rate is $1 divided by the price of the annuity/replicating bond portfolio [$22.40], or 4.46%. In essence, the retiree buys 30 separate pieces of retirement consumption. Upon the maturity of each piece, the retiree spends exactly the inflation-adjusted maturity value of that piece. This is a risk free retirement consumption strategy that never has a	An important contribution to research on the 'cost-of-retirement consumption' topic. The purchase of a 30-year bond sequence, however, may overstate the cost of consumption for many retirees. The assertion stands in contrast to Milevsky's argument that only an annuity can measure retirement costs. These articles are important precursors to the issue of how best to measure and monitor retirement cost. The 30-year TIPS ladder is a static cost measure. Analysis assumes complete markets and log-normal parameters.

		surplus or a deficit. Any strategy that matches constant spending with volatile (uncertain) return outcomes is suboptimal in several respects: (1) there is a positive probability of ruin; (2) there is a cost to generate an "unneeded" / low-utility surplus and, (3) the cost of the spending distribution is higher than the 30 year-bond sequence cost. The article has a table of failure rates that is in line with previous studies about the interaction of portfolio volatility, expected returns and distributional stress. With respect to surplus, volatility shifts money away from consumption and towards "surpluses that waste money." Increased portfolio value also means that the cost of obtaining funds sufficient for the spending distribution is also higher than necessary.	Argument assumes that investment surpluses are "costly." A utility penalty is imposed for strategies that generate consumption at a level either below or above target. Note: the points made in this essay are further developed in "What Makes a Better Annuity?" Jason Scott, Wei-Yin Hu and John G. Watson Journal of Risk and Insurance vol. 78 no. 1 (2011), pp. 213 – 244.
2009	"Comparing Strategies for Retirement Wealth Management: Mutual Funds and Annuities" Gaobo Pang and Mark J. Warshawsky Journal of Financial Planning vol. 22 no. 8 (August 2009), pp. 36 – 47.	This study, written by two Towers Watson Ph.D.s, compares six strategies in terms of (1) their ability to provide income throughout retirement and (2) their end-of-life account values. It does not include a consideration of taxes but incorporates investment product fees. The benchmark annual income is $45,000 inflation adjusted for investors retiring at age 65 with a $1 million portfolio. The portfolio is allocated 50% to "high-risk" assets (stocks) and 50% to "lower-risk" assets (bonds & annuities). Fees for mutual funds equal 1.2% per annum. Total cost of variable annuities is 2.4% per year with an additional charge of 60 basis points for Guaranteed Minimum Withdrawal Benefit Rider [GMWB] per year. The six strategies under consideration are: 1. Systematic withdrawal from mutual funds. This is a unitrust withdrawal formula: "Investors in this strategy are assumed to take a systematic withdrawal as a constant percentage of mutual fund balance in each period. This strategy, by design, will not exhaust the wealth entirely, although it may come close to low or zero dollars in highly adverse situations…." 2. Fixed payout immediate life annuity. "Retirees in this strategy are assumed to make a one-time purchase of a fixed nominal payout straight life annuity, converting all wealth accumulated." 3. Immediate variable annuity for life. "…retirees are assumed to	The risk model is a vector autoregressive process. The simulated average value for inflation is 4% (with a standard deviation of 2.8 percent). The corresponding statistics for stocks are 8.8% with an SD of 17.1 and for bonds are 6.4 percent with an SD of 6.7%. The authors note that modeling a variable annuity with a guaranteed minimum withdrawal benefit is difficult because "The majority (approximately 70 percent) of the VA+GMWB providers….state in their prospectuses that, upon the automatic step-up or the investor-elected step-up of GIB, the contracts will increase, may increase, or reserve the right to increase the annual rider percentage charges, subject to the contract maximum rates. Changes in market conditions may also trigger such fee hikes."

		purchase an immediate variable straight life annuity that delivers variable income for life, with no residual. At the time of purchase the investor selects an assumed interest rate (AIR)." 4. Variable annuity plus guaranteed minimum withdrawal benefit (VA+GMWB). "…the minimum is guaranteed by the rider to be a certain percentage of the nominal guaranteed income base (GIB). The GIB is non-decreasing and can step up on the rider anniversary date if the market performs well….Note that the extra GMWB rider fee does not have a direct effect on the GIB or the resulting income payments. This fee, however, reduces the account value, depresses the likelihood of the GIB step-ups, and therefore, has a potential negative effect on the future income stream." 5. Mix of withdrawals from mutual funds and fixed payout immediate life annuity, one-time wealth split at retirement. "Investors adopting such a strategy get a certain percentage of the mutual fund balance in addition to the annuity payout." In this strategy, the investor converts 30% of retirement wealth to an annuity. The allocation in the mutual fund is adjusted to maintain the 50-50 risk allocation. 6. Mix of mutual fund withdrawals and fixed payout life annuity, gradual annuitization at certain ages. "To make income levels less skewed by one-time conditions in the annuity market, investors in this strategy allocate a larger fraction of wealth to mutual funds in the early years of retirement, escalate the shift to a fixed life annuity with increasing age, and eventually convert all mutual funds into a fixed annuity by a certain age." By the end of age 75 the investor is fully annuitized. Equities are proxied by the S&P 500; bonds are proxied by the US Government Bond index. The paper reports all results in constant dollars. "The dynamics of asset returns and inflations [sic] are modeled as a vector autoregressive (VAR) process. The VAR coefficients and variance-covariance matrix, estimated on the 1962-2008 quarterly data, are embedded in the simulations to generate a large number of 36-year	

		series of rates and returns. This approach captures the serial correlations among variables and the contemporaneous correlations of market shocks. Moreover, the VAR-based simulations reproduce the persistent structural shifts or long-run mean reversions of variables, the differing short- and long-term correlations between them, and the changing risk-return trade-off of bonds and stocks across investment horizons (a 'term structure'). These characteristics are observed prominently in the historical data." The strategies are first compared on an ability-to-preserve-real-wealth at the earlier of death or age 100 metric. Strategies one and five exhibit the most favorable distribution of outcomes. They are then compared to the ability to sustain a real annual constant-dollar income of $45,000. Both the mutual fund only strategy and the variable annuity strategies exhibit substantial shortfall risk. The fixed annuity delivers slightly better performance than the variable, especially in the early years of retirement. "…a mix of fixed annuities and mutual funds in Strategy 5 delivers similar, or even higher, income flows than the VA-GMWB does…." "This result reveals that income stability offered by VA-GMWB only rests at the nominal level….The security of the insurance company guarantee in extreme financial conditions, however, is unknown." Finally, "Emphasizing real income stability, the 10-year gradual annuitization in Strategy 6 offers another alternative….Investors can reasonably expect to receive significantly improved annuity payouts. The median real incomes are greater than those generated by the one-time annuitization strategies. The overall inward shift (reduction) of income shortfall risk is substantial."	
2009	"Optimal Annuity Risk Management," Ralph S.J. Koijen, Theo E. Nijman and Bas J.M. Werker Working Paper August 2009	This article defines the problem of retirement income within a life-cycle model context. It incorporates a constant relative risk aversion function to calculate the utility of various strategies. The model makes several simplifying assumptions: • Annuities are fairly priced; • Annuity markets are complete (e.g., investors are not concerned about the risk of unexpected health costs);	An important discussion of 'annuity risk.' For example, annuitization in a low-interest rate /low inflation environment may prove to be suboptimal depending on changes in the future economy. The concept of 'annuity risk' is a variation on the notion of 'annuity timing risk.' The study generates 10,000 trajectories of state variables over the

		• Annuitization of all financial wealth occurs at retirement age 65; and, • The investor lacks a bequest motive. Given the above assumptions, most life-cycle models indicate that immediate annuitization of wealth upon retirement is optimal—e.g., the Yaari lifecycle model. However, the authors point out that such a strategy exposes the investor to 'annuity risk': "…the utility derived from the annuity payoffs may disappoint if financial market conditions turn out to be unfavorable at retirement." Annuity risk is exacerbated due to the irreversibility of the transaction. One cannot rebalance or dynamically change the annuity portfolio once the annuity portfolio is set at retirement —the model assumes that a variable annuity will be invested only in equities. The loss of liquidity/flexibility adds to 'annuity risk.' The investor can manage this risk in two ways: 1. Incorporating current information regarding the state of financial markets at the time of annuitization—i.e., setting up the annuity portfolio conditioned on economic state variables; and, 2. Hedging the pre-retirement portfolio so that the investor can achieve satisfactory annuity payoffs at the time when the investor converts the financial asset portfolio to annuity instruments. The study seeks to demonstrate that both strategies are welfare improving and it calculates both the optimal annuity allocation strategies post-retirement and the optimal hedging strategies pre-retirement. The model incorporates time-varying elements including inflation, interest rates, and equity risk premia. By contrast, earlier life cycle models usually assume that one or all of these elements are constants. The mathematical expressions reflecting the stochastic nature of these	relevant planning horizon and then calculates the corresponding annuity payouts. The optimization procedure considers the set of 10,000 'conditional' economies that the investor faces at the moment of the annuitization decision. The study creates a distribution of annuity payouts contingent upon the state of the economy at the time of annuitization. This distribution of future annuity payoffs can be compared to Feng Li's calculations of the distribution of future annuity contract costs. Both studies provide valuable information and insight.

		variables are, however, complex. The authors present a financial market that assumes an Ornstein-Uhlenbeck process for the real interest rate and the expected inflation rate. The drift component of this process reflects single risk factors while the innovation term reflects a five-dimensional vector of independent Brownian motions. Likewise the stock price process incorporates a drift term plus a Brownian motion innovation. However, the drift term is further decomposed into an element reflective of the level of the nominal short-term interest rate, the current stock price level, and a vector of prediction variables that includes both term structure and dividend yield information. This allows equity risk premia to change in response to the evolution of the vector of relevant underlying stochastic variables. Bond risk premia are also modeled in a way that allows for time varying risk premia. The authors present an annuity market consisting of nominal, inflation-adjusted, and variable annuities. The three types of annuity payouts allow, given the simplifying assumptions of the model, for a complete post-retirement annuity market. The three risks that are of concern to a retiree—interest rate risk, inflation risk and equity performance risk—are spanned by the annuity market. The article offers mathematical expressions for the pricing of each contract as time unfolds and as the planning horizon changes. The variable contract assumes a 4% AIR. As a life-cycle model, it incorporates labor income and assumes that prior to retirement the investor can create a portfolio consisting of a cash account, a three and ten-year maturity nominal bond, a ten-year maturity inflation-linked bond, and stocks. Utility is derived only from consumption; and, the value function for the retirement optimization problem maximizes consumption with risky asset allocation as the pre-retirement control variable and annuity contract allocation as the post-retirement control variable. The value function assumes a 4% subjective time discount factor. Model parameters are estimated from stock, bond	

		and inflation monthly data over the period January 1952 through May 2002. The unconditional equity risk premium is 4.3%, the nominal bond 10-year risk premium equals 1.8%, and the 10-year inflation-lined bond risk premium (difference between nominal and inflation-linked bond of same maturity) equals 1.2%. Setting the vector of state variables equal to its unconditional expectation, the authors graph, through time, the mean and volatility of annuity payouts assuming that the investor annuitizes $100,000 at retirement. The inflation-adjusted annuity provides a constant real riskless income. This payout appears as a straight horizontal line on the graphs. The expected real income level is lower than a nominal annuity's payout until age 75 and is lower than the expected payout of a variable annuity at every age. However, the real payouts for both the nominal and variable contracts are risky. "When initial expected inflation is low, the initial payoff of nominal annuities is low and the expected payoff stream is more stable in real terms. When initial expected inflation is high, the initial payoffs of nominal annuities are high, but decline rapidly in expectation. The opposite occurs for variable annuities, since the equity risk premium is negatively related to the level of expected inflation. The level of expected inflation is therefore likely to impact the investor's annuity choice and utility derived from inflation-sensitive annuity products….Different values of the dividend yield have substantial impact on the expected annuity payoffs of variable annuities…." Unfortunately, the optimal annuity allocation at retirement is a non-linear function of the state variables: expected inflation, the real interest rate, and the dividend yield on stocks. However, the authors consider a first-order approximation of the optimal solution by running cross-sectional regressions for the optimal annuity weights on the state variables where each variable is standardized. The optimal annuity allocation expression is:	

$$\alpha_T^i(Y) \approx \alpha_0^i + \sum_{j=1}^{3} \alpha_j^i Y_j$$

In this expression, the constant term is the unconditional allocation to a particular annuity product; the Beta or slope terms are the percentage change in the annuity allocation given a one standard deviation in the state variable under consideration.

For example, for an investor with a coefficient of relative risk aversion equal to 5, the optimal unconditional annuity allocation at retirement—assuming no pre-existing pension income—is:

- Nominal Annuity: 8% of financial wealth
- Inflation-Linked Annuity: 50% of financial wealth
- Variable Annuity with 4% AIR: 42% of financial wealth.

The optimal allocation percentages, however, vary when conditioned on the state of the economy at the time of the annuitization decision. If the state variables of interest are one standard deviation above their unconditional (historical) values, the marginal allocation to the three annuity contracts changes. For example, a one standard deviation increase in the real rate of interest results in a marginal change in allocation percentages as follows:

- Nominal Annuity: 0% (no change)
- Inflation-Linked Annuity: +2%
- Variable Annuity: -2%.

Likewise, a one standard deviation increase in expected inflation results in a marginal change in allocation percentages as follows:

- Nominal Annuity: +8%
- Inflation-Linked Annuity: +3%
- Variable Annuity: -11%.

Finally, a one standard deviation increase in the equity dividend yield results in a marginal change in allocation percentages as follows:

- Nominal Annuity: -1%
- Inflation-Linked Annuity: -27%
- Variable Annuity: +28%.

		The authors note that optimal pre-retirement investment strategies designed to hedge annuity risk often require taking substantial short positions in financial assets. The estimated welfare costs of not implementing a pre-retirement hedge strategy range from 1 to 10% depending on the value of the investor's risk aversion coefficient.	
2009	"The safety first expected utility model: Experimental evidence and economic implications," Haim Levy & Moshe Levy, Journal of Banking & Finance Vol. 33 (2009), pp. 1494 - 1506	Although a strict application of Roy's Safety First preferencing rule is not generally found in portfolio choice literature because it can result in paradoxical decisions, the authors nevertheless believe that it should not be ignored. Empirical evidence suggests that investors make decisions that are combinations of a safety first criterion and an expected utility maximization criterion. According to Roy's rule, investors avoid investment strategies that place them below a level considered to be economically disastrous. Roy recommends that investors select portfolio allocations that minimize the probability of a downside catastrophe. Avoiding an economic calamity is of greater importance than maximizing expected utility. Formally, given a disaster level 'd' and two investment distributions F and G, F is preferred to G if, and only if: $Pr_F(x<d) < Pr_G(x<d)$. When the nature of the distributions is unknown, the decision maker may use Chebyshev's inequality to obtain an upper bound on the disaster level probability. The fact, as Roy asserts, that most investors are primarily concerned with avoiding a disaster, means that investors tend to ignore the region of the distribution above 'd'. Different individuals will, of course, define 'd' differently. The authors incorporate a disaster level 'd' into their model by defining it as a no gain/no loss point—the point at which 'd' = 0. Given this rule, the investor prefers distribution F [-800, $200, $1500] to	The article provides an interesting alternative to life-cycle models that incorporate utility functions that are additive and separable with preference going to the distribution or strategy that maximizes expected utility. The EU-SF model can adapt to a variety of utility functions including linear, CRRA and other hyperbolic risk aversion functions. Although the EU-SF criterion may look similar to the models used in behavioral finance, there are significant differences—particularly with regard to loss aversion. The article provides a good summary of Prospect Theory's basic decision-making elements: 1. Cash flows are viewed relative to current wealth. 2. Individuals are risk-averse regarding gains and risk-seeking regarding losses. 3. Individuals are loss-averse: losses are weighed more heavily than gains. 4. Probabilities are subjectively weighed.

		distribution G [-$400, -$100, $1,400] where each outcome has a 1/3 probability because distribution F has less chance of producing a below $0 outcome. The authors point out that such a rule can generate extreme outcomes "…such as a zero investment in stocks when d < r. Nevertheless, incorporation of a downside avoidance point is intuitively pleasing when considering how investors make their portfolio selection decisions. The authors' solution is to introduce an expected utility- safety first model of investment choice. Normalizing the traditional expected utility function so that U(a) = 0 and U(b) =1, expected utility is the integral: $$E_F U(x) = \int_a^b f(x)U(x)dx$$ and $$E_G U(x) = \int_a^b g(x)U(x)dx$$ Utility under the safety first decision criteria is the preference set [$U_{SF}(x)$] equals: $U(x) - k$ for $x < d$ $U(x)$ for $x \geq d$. with 'k' acting as a penalty weight for failure to attain the minimum acceptable investment outcome. A preference function combining the standard utility criterion with the safety first criterion yields the following expression: $$\int_a^b [f(x) - g(x)]U(x)dx - k\int_a^d [f(x) - g(x)]dx$$ Where the expected utility term on the left evaluates the integral's entire area, and the safety first term on the right evaluates only the area under the function that lies below the threshold 'd.' The disutility	

		produced by failing to achieve the objective is weighted by ‘k,’ with larger penalties associated with a larger value of ‘k.’ The above preference function is a weighted average of the expected utility and the safety first criteria—hence the EU-SF nomenclature. Whenever a distribution evidences first degree stochastic dominance it will rank portfolio choices exactly as the EU-SF criterion. A similar ranking, however, is not assured under second degree stochastic dominance because of the penalty term ‘k.’ Depending on the nature of the utility function ‘U,’ higher order moments can influence the investor’s choice. Using experimental evidence, the authors estimate that the weighting of the safety first element is approximately 0.1 and the weighting of the expected utility element is approximately 0.9. The 0.1 weighting given to the SF term, implies that investors will generally lower their preferred allocation to equity by approximately 30 percent.	
2010	“Post Retirement Financial Strategies from the Perspective of an Individual Who is Approaching Retirement Age,” Arnold F. Shapiro Society of Actuaries’ Pension Section	This monograph, sponsored by the Society of Actuaries’ Pension Section, provides a literature review on financial strategies appropriate for an investor approaching retirement. The literature is both academic and practitioner oriented. Published research is categorized according to topic. Important positions and insights are briefly summarized, and an extensive bibliography provides guidance for further reading.	
2010	“Evaluating the Advanced Life Deferred Annuity—An annuity people might actually buy,” Guan Gong and Anthony Webb Insurance: Mathematics and Economics vol. 4	In 2005 Milevsky wrote about a new annuity concept: the inflation-adjusted advanced life deferred annuity [ALDA]. The basic idea is that annuity payments would begin at an advanced age if, and only if, the annuitant was alive. No insurance firm as, to date, offered an inflation-adjusted ALDA. A primary purpose of this study is to determine the merits of offering ALDAs to 401(k) plan participants. On a preliminary basis, the authors cite three potential advantages of an ALDA: 1. An ALDA enables households to preserve liquidity at retirement	Authors reference a 2001 article by Estelle James and Xue Song [“Annuities markets around the world: Money’s worth and risk intermediation”] stating that insurance companies may be able to offer money’s worths greater than 1.00 because they “…invest at least part of the premiums in risky assets.” The calculations embedded in the tables reflect the assumption of CRRA. The authors point out that this

	(2010), pp. 210 – 221.	date because the cost of an ALDA is far lower than the cost of an immediate annuity. 2. Although a full (immediate) annuity provides complete longevity insurance, even at actuarially unfair prices, households prefer ALDAs because they gain almost as much longevity insurance at a lower level of actuarially "unfairness." 3. An ALDA "…dominates an optimal decumulation of unannuitized wealth." It improves and simplifies the process of retirement wealth decumulation. The study estimates the money's worth of both nominal and inflation-adjusted ALDAs for a joint and two-thirds survivor annuity. They estimate the cost of an inflation-adjusted ALDA by (1) comparing nominal to inflation-adjusted immediate annuities in insurance carriers selling both products; (2) comparing the money's worth of the nominal and inflation-adjusted immediate annuity products, and (3) inferring likely relationships between nominal and inflation adjusted ALDA products. In calculating an annuity's money's worth, the choice of an interest rate is important. This is especially true for ALDAs because the payment deferral provision makes these long duration instruments. The article discusses the merits of various interest rates including the risk-free nominal treasury rate [Treasury STRIPS], the risk-free TIPS rate, the authors' preferred rate [AA grade corporate bonds] and BAA corporate bond rates. They report on the money's worth of a joint/survivor annuity and ALDA purchased either at household age 60 or age 65 for both immediate annuities—nominal and inflation-adjusted, and for ALDAs—nominal and inflation-adjusted—beginning payments at ages 70, 75, 80, and 85. The tables report results both for households with annuitant population mortality and for households with general population mortality. The tables are based on an examination of products offered by four insurance carriers selling both immediate annuities and ALDAs. Nominal and Inflation-adjusted immediate annuities, when evaluated at the Treasury or TIPS interest rates generally offer the annuitant mortality	assumption may not be realistic: "The above calculations are contingent on a utility function that does not appear to be very predictive of current behavior….care needs to be taken when estimating the distribution of welfare gains with an expected utility framework that has substantive predictions so at odds with observed behavior." Compare the high welfare gains of annuitization estimates in this paper to the low estimates in the 2011 paper "Optimal Portfolio Choice over the Life-Cycle with Flexible Work, Endogenous Retirement, and Lifetime Payouts," Jingjing Chai, Wolfram Horneff, Raimond Maurer, and Olivia S. Mitchell.

		population a money's worth greater than 1.00. Furthermore, "…the provision of inflation protection has little effect on annuitant money's worths." At an AA corporate rate, nominal immediate annuities have a money's worth value slightly above 1.00 [cannot evaluate an inflation-adjusted annuity with a nominal interest rate benchmark]. At a BAA corporate rate, the annuitant mortality population generally experiences money's worth value less than 1.00—at some ages, the value is substantially less. Again, all annuities are joint and 2/3 survivor. For nominal ALDAs purchased at age 65 with payouts commencing at age 85, the money's worth is generally above 1.00 when discounting at the Treasury rate, and below 1.00 when discounting at the AA and BAA corporate rates within the annuitant mortality population. The next section of the paper calculates "annuity equivalent wealth" [AEW] for households with varying coefficients of relative risk aversion. AEW is defined as: "…the factor by which unannuitized wealth must be multiplied so that the household can enjoy the same expected utility through an optimal decumulation of its unannuitized wealth as it would enjoy were it to purchase an actuarially fair annuity with that wealth." Stated otherwise, it is the factor value at which a household is indifferent between an optimal decumulation of financial assets and the purchase of an actuarially fair immediate annuity or an ALDA. Calculating AEW for an ALDA is complicated because several elements must be jointly determined: (1) optimal amount of initial wealth to spend on an ALDA, (2) optimal decumulation plan for non-annuitized wealth, and (3) optimal time to commence ALDA payments. The authors employ a "relative risk aversion utility function" incorporating separate male and female consumption needs (lambda of one means that all consumption is joint—i.e., no distinct male/female consumption pattern). Utility of each period is added over the applicable horizon with time-preference utility discounting equal to the interest rate. Here is the procedure: "We first calculated the household's expected utility if it buys an	

		actuarially fair annuity at retirement. We then close the annuity market. We use numerical optimization techniques to calculate an optimal decumulation of the household's wealth and the expected utility of that decumulation plan. We then calculated the amount by which the household's wealth must be increased so that its expected utility equals that obtainable when it annuitizes. This increased amount is divided by the household's original wealth to obtain the household's annuity equivalent wealth. We assume that the household and the insurance company are both able to invest in a single risk-free asset yielding 2.35%, the average yield on long dated TIPS in February 2007, the month we started running the programs, and that this also equals the household's rate of time preference." For example, the AEW for an age 65 household purchasing an actuarially fair joint 2/3 survivor immediate annuity is as follows: RRA coefficient 2: 1.264 RRA coefficient 3: 1.307 RRA coefficient 4: 1.336 RRA coefficient 5: 1.356 The AEW for an age 65 household purchasing an actuarially fair joint 2/3 survivor ALDA with payments commencing at age 85 is as follows: RRA coefficient 2: 1.174 RRA coefficient 3: 1.209 RRA coefficient 4: 1.233 RRA coefficient 5: 1.246 Given an actuarially fair ALDA commencing at age 85, the optimal percentage of age 65 wealth to commit to the ALDA is RRA coefficient 2: 13.5% RRA coefficient 3: 13.7% RRA coefficient 4: 13.8% RRA coefficient 5: 13.8% The authors remark that "…even at age 85, the ALDA provides more than half the longevity insurance provided by the annuity, at a fraction of the cost in terms of foregone liquidity."	

		The next section of their paper recalculates AEW for various levels of actuarial unfairness. Specifically, they calculated AEW for ALDAs assumed to have a money's worth of 1.00 for annuitant mortality population and, alternately, for ALDAs assumed to have a money's worth of 0.90. Assuming age 65 retirement, the election to annuitize all wealth immediately creates AEW values ranging from 11.5% to 19.6% better as the coefficient of risk aversion ranges from 2 to 5. The improvement is measured relative to results achieved under the optimal decumulation of non-annuitized wealth plan. By contrast, the corresponding values for an ALDA commencing at age 80 are 11.7% to 18.6%. When an annuity's money's worth value is 90%, assuming age 65 retirement, the election to annuitize all wealth immediately creates AEW values ranging from 0.3% to 7.7% better as the coefficient of risk aversion ranges from 2 to 5. The improvement is measured relative to results achieved under the optimal decumulation of non-annuitized wealth plan. By contrast, the corresponding values for an ALDA commencing at age 80 are 10.6% to 17.1%. Note: the optimal decumulation plan trades off the benefits of higher consumption early in retirement against the risk of lower consumption in later years given a lower probability of surviving to those years. The authors conclude: "At 90% annuitant money's worth, the household is better off delaying….The optimal period of delay is greater at lower coefficients of risk aversion. We calculate that for a household aged 65, the optimal delay ranges from thirteen years when CRRA equals two, to five years when CRRA equals five." Furthermore, "…ALDAs with optimal deferral periods dominated both immediate and deferred annuitization. At higher money's worths than those assumed in our table, immediate annuitization comes to dominate the ALDA. At lower than our assumed money's worths, unannuitized decumulation comes to dominate immediate annuitization. At lower assumed levels of money's worth, the optimal ALDA deferral period increases. But at any money's worth other than zero, an ALDA with an optimal deferral period will dominate an	

		optimal decumulation of unannuitized wealth."	
2010	"A more dynamic approach to spending for investors in retirement," Colleen M. Jaconetti and Francis M. Kinniry Vanguard Research (November 2010).	This Vanguard research paper explores the financial consequences of implementing and adhering to three retirement spending strategies throughout future simulated economic conditions: 1. Dollar amount grown by inflation 2. Percentage of Portfolio 3. Percentage of Portfolio with Ceiling and Floor. The authors use Monte Carlo simulation analysis with the parameters for means and variance calibrated to reflect initial market conditions as specified by the Vanguard Capital Markets Model. The base case example compares each pre-tax withdrawal strategy—first year spending equals 4.75% of the portfolio's initial value of $1 million allocated 35% to US stocks, 15% to international stocks, 50% to US bonds and rebalanced annually. Not surprisingly, the dollar amount grown by inflation [constant dollar annuity] shows the lowest portfolio survival rate over a 35-year planning horizon (71%) while providing the greatest budgetary. However, when the portfolio failed to meet the spending target, spending dropped to zero. The Percentage of portfolio withdrawal strategy [unitrust] shows both a zero percent portfolio depletion rate over the planning horizon, and the highest withdrawal amount variability. In the worst case scenario, the annual inflation-adjusted withdrawal drops to $2,850. Finally, the authors characterize the Percentage of Portfolio with Ceiling and Floor [unitrust with a collar] strategy as a dynamic method which is "…a hybrid of the two others." The distribution of simulated results depends on the width of the collar. Results generally lie between the constant dollar and percentage of corpus distribution ranges. The authors stress that flexibility in spending is a key driver of long-term financial success. They conclude "…no strategy should be followed blindly; indeed, it is essential for investors to periodically evaluate their income strategies, assess their portfolios, and consider whether alterations are needed."	The three spending strategies examined in this paper are extensively analyzed in articles on the topic of total return unitrusts. This literature appeared in legal journals for the previous decade. The fact that the Vanguard bibliography omits entirely this analysis is a good indication of the 'disconnect' among the legal, actuarial and investment professions. See, for example, "Financial Consequences of Distribution Elections From Total Return Trusts," Patrick J. Collins, Sam L. Savage and Josh Stampfli, Real Property, Probate and Trust Journal vol. 35 no. 2 (Summer 2000), pp. 243 – 304; "Promises and Pitfalls of Total Return Trusts," Patrick J. Collins and Josh Stampfli, ACTEC Journal Vol. 27 (Winter 2001), pp. 205 – 219; Patrick J. Collins, "A Risk Primer for Investment Fiduciaries," California Trusts and Estates Quarterly Vol. 8 No. 3 (Fall, 2002), pp. 4-24; and Patrick J. Collins, and Mark C. Griffin Esq., "The Lawyer as Trustee: Duty to Monitor and Review Investments." Maryland Bar Journal (March/April, 2003), pp. 54 - 57. These articles explore the subjects raised in the Vanguard Research paper and provide an extensive bibliography to comparable research written for the estate and trust legal community.

		Note: they give no indication of how to systematically go about this process. Monitoring and managing a portfolio is a topic that is beyond the scope of their analysis. The article provides a strong rebuttal to the "trust the market" approach to retirement portfolio management. They caution investors that: "rigid spending rules cannot eliminate investment volatility; they simply push its consequences into the future. Spending strategies insensitive to returns are risky, inasmuch as they rely on the assumption that the portfolio will recover before a crisis point is reached…."	
2010	"A liability-relative drawdown approach to pension asset liability management," Arjan Berkelaar & Roy Kouwenberg, Journal of Asset Management Vol. 11, Nos. 2-/3, (2010), pp.194-217.	The article compares and contrasts a variety of "optimized" portfolios within the context of asset management for Defined Benefit [DB] pension plans. The authors summarize their objectives and define the various strategic approaches as follows: "…we consider both surplus optimal portfolios and drawdown optimal portfolios and compare the results with standard mean-variance optimal portfolios." Surplus Optimal Portfolios: "…maximize expected utility over the funded ratio at the end of the investment horizon for a given level of surplus variance (that is, the variance of the terminal log funded ratio)." Maximum Drawdown Optimal Portfolios: "…maximize expected utility over the funded ratio at the end of the investment horizon for a given acceptable worst-case drop in the funded ratio over the investment horizon." Optimal 90% Conditional Drawdown at Risk [CDaR] Portfolios: are similar to the Maximum Drawdown Optimal Portfolios "…but under a restriction on the expected decline in funded ratio in the 10 per cent worst-case scenarios for the fund." CDaR is the conditional value at risk of the distribution of drawdowns as determined by the authors' 500-scenario risk model. Minimum Risk Portfolio: "…in the case of standard mean-variance analysis is the portfolio with the lowest variance in cumulative asset returns at the end of the investment horizon."	The authors' model yields a variety of interesting econometric insights and a succinct discussion of conditions which must be met for model tractability (e.g., eigenvalues of the variance/covariance matrix must be less than one in absolute value is a stationarity condition). The model leads to a discussion regarding the term structure of volatility for various assets. For instance, the volatility of stocks decreases over the investment horizon due to mean reversion. REITs exhibit a hump-shaped volatility; the volatility of commodities, however, increases with the horizon. Volatility is measured by return in excess of the risk-free rate.

		Minimum Surplus-Risk Portfolio: "…is the portfolio with the lowest surplus variance (variance of asset returns over liability returns) at the end of the investment horizon." Minimum Drawdown Portfolio: "is the portfolio with the lowest liability-relative maximum drawdown ($LRDD_{100\%}$) in the worst-case scenario." The $LRDD_{90\%}$ CDaR portfolio is a variation on this approach. Traditional Mean-Variance Portfolio: the efficient frontier as calculated in an asset-only context. Among the risk metrics of interest are: LASR or Liability-Adjusted Sharpe Ratio: ratio of (expected surplus return) / (standard deviation of plan surplus) $LACR_{100\%}$ or Liability-Adjusted Calmar Ratio: ratio of (expected cumulative surplus return) / maximum liability relative drawdown) $LACR_{90\%}$ or Liability-Adjusted Calmar Ratio: ratio of (expected cumulative surplus return) / 90 per cent CDaR measure). A major thesis of the article is that plan sponsors should prefer drawdown optimal portfolios to both traditional (asset-only) optimal portfolios and surplus optimal portfolios because the drawdown optimal portfolios provide better protection, have lower weightings to equity and provide higher expected returns. "Liabilities should be at the center of designing investment policies and serve as the ultimate reference point for evaluating and allocating risks and measuring performance. The goal of the investment policy should be to maximize expected excess returns over liabilities subject to an acceptable level of risk relative to liabilities." The authors develop a vector autoregressive [VAR] model that incorporates state variables, yield curve projections and the evolution of returns for a variety of assets including stocks, bonds, hedge funds, securitized real estate, and commodities. The model assumes constant weighting of each asset throughout planning horizons of 5, 10, and 20 years. The article distinguishes between solving portfolio optimization problems through (1) dynamic programming, and (2) stochastic	

		programming. Dynamic programming, in the authors' view, requires a pre-set structure within which the researcher applies a somewhat inflexible return-generating process to solve the portfolio choice problem. Stochastic programming is more flexible in that it models investment results from a more flexible return generating process—in this case with time-invariant asset weights—to arrive at a distribution of scenarios. The risk model considers two liability-risk measures: (1) the variance of the natural log of the funded ratio (plan assets – plan liabilities) or, Surplus Variance measured at the end of the planning horizon; and, (2) Liability Relative drawdown risk which is cumulative excess asset returns over liabilities. The risk measures require a covariance matrix of cumulative excess returns over liabilities. The goal is to maximize expected log utility subject to a constraint on the maximum allowable drawdown. With the introduction of suitable auxiliary variables, the authors convert the problem into a linear optimization problem with nine assets, 40 periods, and 500 scenarios. The model allows the investor to short cash in order to fund positions in long-term treasuries with durations similar to those of the DB plan's liabilities. The state variables in the VAR model are the level, slope and curvature of the yield curve (i.e., the Nelson-Siegel yield curve model is used to generate future yield curve evolutions), the dividend yield on the S&P 500, the default spread between Moody's Baa and Aaa corporate bonds, and the (aggregate) consumption-wealth ratio (where consumption, wealth—assets—and labor income share a common trend or cointegrating factor). The optimization model utilizes forward-looking return expectations from the World Bank pension fund. Sensitivity tests on parameter values are not reported. DB plan liability calculation assumes no new employees. Liabilities are non-indexed, and are "…the present value of current expected benefit payments discounted at the term structure of nominal Treasury rates." The liabilities increase due to accrual of additional benefits by current employees. Given these assumptions, the duration of liabilities suggests that a long-term treasury portfolio should provide a reasonable hedge	

		against interest rate risk. The authors acknowledge: "Pension liabilities are exposed to other risks such as longevity risk, changes in the demographics of a pension fund and labor income risk. Currently, these risks cannot be hedged with financial instruments….The additional uncertainty from these risk exposures may, however, provide an additional rationale to seek excess returns." The model's portfolio optimization results section offers a detailed comparison of various types of optimal portfolios. The data indicates that the maximum drawdown optimal portfolios are superior to other portfolios both in terms of expected returns and maximum drawdown risk. The authors emphasize that the asset-only mean/variance portfolios expose a DB plan to potentially extreme drawdown risk. The following is an excerpt from Table 5 which provides the optimal asset allocations for various portfolios at different planning horizons. Minimum Variance Portfolio—20 year horizon US Treasuries: 75.5 Stocks: 10.7 Commodities: 10.9 Real Estate: 2.9 Hedge Funds: 0 Minimum Surplus Variance Portfolio—20 year horizon US Treasuries: 91.7 Stocks: 4.4 Commodities: 3.6 Real Estate: 0.4 Hedge Funds: 0 Minimum $LRDD_{100\%}$ Portfolio—20 year horizon US Treasuries: 69.2 Stocks: 13.5 Commodities: 8.8 Real Estate: 1.3	

		Hedge Funds: 7.3 Minimum $LRDD_{90\%}$ Portfolio—20 year horizon US Treasuries: 70.7 Stocks: 14.6 Commodities: 7.5 Real Estate: 0.5 Hedge Funds: 6.8 The maximum drawdowns (log of worst-case funded ratio minus log of previous funded ratio) of each allocation are: 106.2, 45.1, 34.0, and 43.3 respectively. Although the Minimum $LRDD_{100\%}$ portfolio exhibits a more favorable drawdown risk metric than the Minimum $LRDD_{90\%}$ portfolio's metric, the authors opine that the 90 per cent CDaR portfolio "…is less sensitive to extreme values in the scenarios and leads to more stable optimal portfolios."	
2010	"Optimizing the Equity-Bond-Annuity Portfolio in Retirement: The Impact of Uncertain Health Expenses," Gaobo Pang and Mark Warshawsky Insurance: Mathematics and Economics vol. 46 no. 1 (2010), pp. 198 – 209.	The shock of unexpected health expenses has often been cited as an explanation for the "annuity puzzle." Although the 2005 article by Davidoff, Brown & Diamond suggests that late-in-life heath expenses increases the demand for annuitizing, other researchers argue that health shocks reduce longevity expectations and hence reduce the demand for annuities. This paper allows for partial or complete annuitization at any age and considers the annuity to be a separate asset class: "The annuitization decision is modeled as a portfolio allocation choice because a life annuity basically represents a class of financial assets with its own unique risk and return features." The authors argue that it is rational to shift retirement assets away from equities into bonds as a precautionary savings measure against the economic consequences of uninsured health costs. Specifically, "It is optimal for households to hold precautionary savings in the equity-bond bundle prior to annuitization when the annuity return (considering some load) has not yet exceeded the reference returns on the conventional assets. The shift to annuities also provides greater leverage than do bonds for higher-risk-and-return equity investment in the remaining asset portfolios."	Note: a low value for elasticity of intertemporal substitution [EIS] indicates that the investor is more interested in smoothing consumption throughout retirement rather than linking consumption to reflect changes in portfolio value. Adjustments to spending should reflect investor utility—not a set of pre-determined risk metrics. A low coefficient of risk aversion means greater risk tolerance. Investors with low risk aversion will tend to find annuities unattractive given an expectation of an adequate equity risk premium. Annuities are viewed as a distinct asset class. The article argues that an increase in wealth increases the demand to hold annuities.

		The authors observe that “Households in higher income groups, compared with the lower-income, need to save more because they have higher probability of living to advanced ages (differential mortality) and tend to face larger health expenses (differential health expenses).” However, the timing of health expenses is critical. If expenses occur early in retirement there is a high demand for liquid assets, if they occur later in retirement an annuity income stream might better fund ongoing health costs: “If, however, major illnesses are associated with both a shorter life expectancy and a preference for consumption in early life, the demand for annuities may be reduced by health shocks.” To ascertain the effects of uninsured health expenses on households of various income levels the authors set up a life-cycle model in which households are assumed to have Epstein-Zin-Weil preferences [CCRA with elasticity of intertemporal substitution that is possibly not isoelastic]. The model incorporates survival probabilities, discount rates reflective of time preferences for consumption, risk aversion, intertemporal elasticity, and possibly bequest preferences. Household expenditures for strict necessities do not generate utility. However, they supplement the model by assuming utility for health care spending and by specifying the fraction of total health expenditures that generate positive utility. The real bond return in the model is a constant while the equity return follows a log-normal distribution. The annuity “asset class” is a Joint and 100% Survivor annuity. The pricing factor for the annuity is the sum of the geometrically-linked discounted returns adjusted for the probability of survival: $\alpha = (1+\upsilon)\ddot{\sum}_{k=1}^{T} \sum \prod_{j=t}^{t+k} \phi_J)R_{\alpha}^{-k}$ As the survival rate shrinks as people age, the annuity return increases nonlinearly. Non-annuitized wealth, which is constrained to be non-negative, thus evolves according to the following dynamic: Return from Existing Annuities + Return on Stock/Bond Asset Portfolio – Health Care Costs – Consumption – Wealth spent to acquire additional annuities). Investors	

		optimize investment and consumption decisions over their lifetime to maximize utility. The utility maximization is found by backward calculations from the end of life to the initial period through a process of numerical integration done through the Gaussian quadrature method. Once the optimal decision path is identified, the authors conduct a Monte Carlo simulation to identify optimal asset allocation, consumption, and annuity purchase amounts and timing. The model sets the time preference discount factor to 0.96; the coefficient of relative risk aversion to 5 ("reflecting a moderate to low risk tolerance for retired households."); and the elasticity of intertemporal substitution to 0.5 [1% change in rate of earnings on financial portfolio results in a 0.5% change in consumption]. The annuity expense load is set to 15%. When investors do not have access to annuities, they decrease consumption as they age. "This is because households prefer to consume sooner than later as the marginal utility of consumption shrinks with the increase in the effective discount rate (higher mortality rate and a positive time preference)." Introducing health shocks makes expenses more volatile. The added uncertainty increases the amount of assets devoted to precautionary savings and investors shift assets from stocks to bonds to reflect their risk tolerance. The authors note that under these conditions the optimal portfolios bear a strong resemblance to the asset allocation paths followed by life cycle funds. When investors have access to annuities, they "…start annuitizing their wealth around mid 70s and fully annuitize in their 80s." If. However, they have a bequest motive, they continue to keep a substantial portion of wealth in stocks and bonds. Interestingly, the greater the health spending risk, the greater the demand for voluntary annuitization either with or without a bequest motive. Beyond certain ages, the presence of additional health costs does not alter the dominance of annuities over other financial assets. Additionally, "…both annuity payouts and health expenses are life contingent and both annuity returns and health costs are increasing with age….For health expense in later years, the optimal hedging strategy is to annuitize so as to capture the increasing-with-age annuity returns (mortality credit)." The study concludes: "…household	

		would be better off by eventually converting these precautionary savings, at least partially, to annuities. Life annuities support late-life consumption and significantly improve household welfare." The study concludes: "It is optimal for households to annuitize their wealth, fully or partially depending on the bequest motive, when the expected annuity return is greater than the reference returns on bonds and equities….The addition of the background health spending risk acts to enhance the demand for annuities over bonds." Were annuities not available, households would substantially reduce their exposure to equity. Thus the acquisition of annuities makes a risk-averse household more comfortable with equity risk exposure: "A life annuity is not just a passive replacement of bonds as an insurance against risk; it is also an integral part of the asset allocation strategy for wealth creation because of its effective accommodation of higher-risk-and-return portfolios." Irrespective of bequest motivation, the model demonstrates that all households benefit—I.e., increase utility—from the availability of annuities. "The welfare gain is greater for households in higher income deciles because they have a lower beginning degree of annuitization. Finally, the uncertainty in healthcare spending enhances the welfare gain of life annuities, more so for higher-income households because such annuities are more likely to hedge their longer life and higher health spending." Note: the definition of a high income household is one that spends $12,000 or more on annual necessities.]	
2010	"Asset Allocation in a Crisis," Brian Jacobsen CFA Institute Magazine vol. 21 no. 2 (March/April 2010), pp. 11 – 13.	Short article arguing for dynamic asset allocation to reflect changes in investor circumstance and market conditions. However: "The danger is that a supposedly *temporary* deviation might actually be a *structural* change in the markets. This is why it is important to understand not only the mechanics of the markets but also the politics and economics of the time. If there is a regime change such that empirical tendencies are no longer relevant, being early to recognize the change will create an opportunity to add value."	It may be "different this time." You cannot simply rely on a tendency for recessionary economies to rebound because you don't know if the economy is on the verge of a depression. Changes in equity prices contain information regarding the future desirability of holding the investment. Price change is more than a supply/demand equilibration. Dynamic asset allocation must consider risk perception as well as risk tolerance.
2010	"Primer on	"In the event an insurance company becomes insolvent, the California	Good checklist of issues faced by trusts owning

	Commercial Annuities for Trust and Estates Attorneys," S. Andrew Pharies Estate Planning 2012 The School of Law University of California, Los Angeles, and California Continuing Education of the Bar Chapter 8 (2010), pp. 351 – 383.	Life and Health Insurance Guarantee Association provides a limited amount of protection to the annuity owner. It protects 80 percent (up to a maximum of $100,000) of the present value of annuity benefits including net cash surrender value or net cash withdrawal value, up to a maximum of $250,000 for all life insurance and annuity policies." "Many commercial annuities generate high commissions for those who sell them. For this reason, many estate planning attorneys shy away from recommending commercial annuities as an income security vehicle." "…if the annuitant dies before all the basis in the contract is recovered, the owner may be entitled to an income tax deduction for the unrecovered basis." "…each payment received is part taxable gain and part tax-free return of basis. The amount of each payment that is considered tax-free return of basis is determined by an 'exclusion ratio'….The exclusion ratio is calculated by dividing the 'investment in the contract' by the 'expected return' of the contract….the expected return is simply the amount of each periodic payment multiplied by the expected number of period payments based on the annuitant's life expectancy using the tables set forth in IRC §72…..The exclusion ratio is applied only until the tax-free basis is fully recovered. IRC §72(b)(2)…..If the annuitant dies before the basis of the contract is fully recovered, the owner receives a deduction for the unrecovered basis §72(b)(3)." "The Internal Revenue Code provides virtually no guidance on the taxation of annuities owned by trusts." "In some cases, it may make sense for an estate plan to direct a trustee to purchase a commercial annuity in lieu of creating and administering an irrevocable trust after the settlor's death. This technique could be used when income security is desired for a beneficiary but no individual is available to serve as trustee and the principal amount available is too small to justify a professional trustee."	annuity contracts. Max protection = 80% to $100K for each contract to a max of $250K for all contracts. Article explores the issue of high commissions for annuities v. trustee duty of cost consciousness. Should small-sized trusts be terminated in favor of purchasing an annuity for the current beneficiary and making a distribution of the excess to the remaindermen?
2010	"Revisiting Retirement Withdrawal Plans	The paper considers the sustainability of the "4R" and "5R" plans where the 4R plan follows the 4% withdrawal rule: Initial withdrawal equals 4% of portfolio value; subsequent withdrawals increase the initial dollar	A good discussion of the inflation variable and its impact on the sustainability of withdrawal strategies. Both the rate of inflation and the

	and their Historical Rates of Return," Chris O'Flinn & Felix Schirripa available at: http://ssrn.com/abstract=1641382	withdrawal amount by the rate of inflation. The study does not consider taxes. Assuming 420 monthly payments (35 years), the *real* IRR needed to support withdrawals is 2.06% per year; for a 5R withdrawal rate, the required *real* 35-year IRR is 3.62%--ending wealth is zero. For all 35-year periods from 1926 through June, 2009, the inflation-adjusted return on the S&P 500 averages 6.8% per year; the minimum real return for any 35-year period is 4.8%. Historically, however, a withdrawal plan failed in 8% of all 4R withdrawal plans and 30% in all 5R plans. The success of retirement portfolios depends on the 35 year period under consideration. For example, a 1930 retiree would have succeeded in a 4R plan despite the fact that the portfolio earned only a nominal 3% during this period. By contrast, a 1965 retiree invested 100% in the S&P 500 falls short despite earning a nominal return of approximately 7%. The key variable is the impact of inflation. When inflation is factored into the calculation, a nominal IRR of 8% (net of expenses) is required to reach the 85th success percentile based on historical results. The starting dates also result in significant differences in the IRRs earned by savers vs. the IRRs earned by withdrawers: "For example, near the peak of the market, in December 1999, the 35-year 'saver' had enjoyed an internal return of 14.4%, yet the retiree following the 5R plan ending on that date would have experienced an internal annual return of only 5.9%." "The nominal IRR required for a successful 35-year 4R plan in the 1926-2009 period ranged from just below 4% to just above 8%....Both the rate and the volatility of inflation affect the monthly amounts required by the withdrawal plan." The following table summarizes results of the authors' historical rolling-period analysis in which the model makes monthly withdrawals equal to 1/12th of the yearly 4% or 5% dollar amount	volatility of inflation are key factors in determining success. Retirement income strategies that track inflation are most likely to succeed. The authors recommend inflation-adjusted annuities and/or TIPS. The paper contains a series of charts for allocations of 100%, 75% and 50% stock. The authors solve for the amount of the initial portfolio that an investor would have to trade for an inflation-adjusted annuity that, in combination with the earnings from the remaining financial asset portfolio, would have historically resulted in a 100% success rate for income sustainability. In some cases, shorter time periods did not require purchase of an annuity; in other cases (longer time periods), if the annuity did not pay an income at a sufficiently high rate, there is no solution to the income sustainability goal. The authors prefer a methodology rooted in historical IRR calculations to a simulation-based approach to the income sustainability issue: "Gauging sustainability of withdrawal rates using Monte Carlo analysis has prompted some controversy due to the wide disparity in results…."

adjusted for inflation/deflation. Bonds are represented by U.S. long term corporate bonds with data coming from Ibbotson Associates:

		4R Withdrawal Plan		5R Withdrawal Plan	
Allocation	**# Years**	**Average Annual Nominal IRR**	**Times Plan Failed**	**Average Annual Nominal IRR**	**Times Plan Failed**
100% Stock	20	10.8%	1%	10.5%	8%
0% Bond	25	10.6%	2%	10.1%	17%
	30	10.3%	4%	9.6%	24%
	35	10.0%	8%	9.2%	30%
75% Stock	20	9.8%	0%	9.6%	7%
25% Bond	25	9.6%	0%	9.2%	16%
	30	9.3%	4%	8.8%	25%
	35	9.0%	8%	8.4%	33%
50% Stock	20	8.5%	0%	8.3%	9%
50% Bond	25	8.3%	2%	8.0%	20%
	30	7.9%	8%	7.5%	36%
	35	7.6%	15%	7.1%	51%

The authors test the rolling period withdrawal plans to determine the percentage of initial portfolio value—subsequently adjusted for inflation—that is sustainable at a 100% historical probability rate. The following table summarizes these results:

Allocation	# Years	Withdrawal Plan generating 100% success rate
100% Stock	20	3.4R
0% Bond	25	3.1R
	30	2.9R
	35	2.9R
75% Stock	20	4.4R
25% Bond	25	4.0R
	30	3.7R
	35	3.6R
50% Stock	20	4.3R
50% Bond	25	3.8R
	30	3.5R
	35	3.4R

Of the three allocations under consideration, the highest level of sustainable income for a 100% success rate is the 75% stock and 25% bond portfolio.

Given the variability in success rates, the authors examine the role of adding an inflation-adjusted annuity to the retirement portfolio. The authors solve for the amount of the investment portfolio that must be exchanged for an inflation-adjusted annuity in order to have guaranteed a 100% historical success rate for sustainable income over the applicable planning horizon. "For example, a 75/25 stock/bond portfolio is selected to fund a 30-year 5R Plan. Table 2 above indicates that 3.7R would come from the portfolio of stocks and bonds. Chart 21 below indicates what portion of the portfolio would be used to buy an inflation-adjusted income annuity to make up the difference between 3.7R and the 5R desired. If the inflation-adjusted annuity is paying an annual benefit equal to, say 7.5% of the premium, we find that 35% of the portfolio would need to be spent on the annuity."

		The authors opine that "…a portfolio with returns that closely track inflation is a key to a successful withdrawal plan." They indicate that the two most readily available options are inflation-adjusted annuities and TIPS.	
2010	"Retirement investor risk tolerance when risk is range: experimental survey evidence from tranquil and crisis periods," Hazel Bateman, Towhid Islam, Jordan Louviere, Stephen Satchell, and Susan Thorp. Australian School of Business Research Paper No. 2010ACTL10.	The paper explores the preferences and risk attitudes of investors when offered the opportunity to invest in a spectrum of portfolios exhibiting a wide range of risk and rewards. Not only does the paper compare the empirical evidence determined by investor choice, it (1) compares the evidence to predictions of behavioral and basic finance theory; and (2) tests to see if investment choices and risk attitudes change significantly from a period of relative investment calm in March 2007 to a period of turbulence in October 2008. The null hypothesis is that "…investors use a mean-variance approximation for utility…." This hypothesis is tested by offering a discrete choice of five portfolios ranging from 0% equity / 100% cash to 100% cash / 0% equities. Additionally, an Australian "retirement savings account" is on the choice menu—this is an investment selection with no variance in returns (Cash evidences variance over time in their model). Various levels of fees are also incorporated into the choice set. Utility preferences are determined by asking each participant to identify their most and least preferred portfolio choice. Additional information concerning each investor's wealth level, demographic profile, and other circumstances is gathered. Finally, each participant completes a risk profile questionnaire "…of the kind typically used by financial advisors." The results of the experiment are surprising: "…the correlation between inferred risk preference and the risk profile scores from the AMP questionnaire was significant and negative for both best and worst choices, indicating that respondents tend to make choices that go against their risk profile." Older and higher income participants tend to choose higher risk options while younger and lower income investors select high cash options. The observed results between 2007 and 2008 do not differ significantly: "…the full-blown financial crisis of October 2008 suggests a mild moderating of risk tolerance, with a slight decrease	One reason, as the authors speculate, for these somewhat unexpected results may lie in the presentation format used to present the portfolio selections to participants. Each participant is presented with both expected 10-year investment values and a range of values at various percentiles of the distribution. Absolute best and worst results derived from a bootstrapped distribution based on Australian stocks and bonds are also presented. Presenting risk as "range" differs from presenting risk through labels ("safe," "moderate," aggressive," etc.), or by presenting risk as frequency—number of positive or negative returns over a time horizon. The design of the experiment asks participants to invest a hypothetical $1,000 retirement savings contribution.

		in the preference for higher share weightings." "In both samples preference classes populated by older and high income retirement savers tend to choose the riskier options, with low income and younger retirement savers opting for the high cash alternatives. The marginal effects are also similar with older-high income retirement savers making risk-loving choices and younger respondents choosing as expected under mean-variance theory."	
2010	"Freedom at 55 or drudgery till 70?" Nabil Tahani & Chris Robinson Financial Services Review vol. 19 no. 4 (2010), pp. 275 – 284.	The authors develop a stochastic model in which the rate of return and the rate of savings are both random variables. The model calculates the probability of reaching any wealth accumulation goal given a specified number of years, a return distribution reflective of the investor's asset allocation, and a specified initial endowment. They emphasize that "Telling a client the standard deviation of returns utterly fails to portray the risk of falling short of a goal." Shortfall risk is often illustrated through the use of simulations. However, the authors develop an analytical solution for the probability of a pre-retirement wealth accumulation shortfall. Their model incorporates both stochastic returns and variability in the investor's savings amount. The stochastic future value [SFV] of the portfolio at a future date "…is a random variable that can only be known through its probability distribution." The model employs standard methodology of stochastic calculus and assumes that both the savings process and return generating process follow two correlated Geometric Brownian motions. Because the SPV is similar to a continuous sum of lognormal variables, a closed-form solution is not available. Therefore the model approximates the SFV distribution by the lognormal distribution by matching the first two moments.	The authors' model is programmed into an Excel spreadsheet and is available at http://www.yorku.ca/ntahani/Research/TargetRetirement08.xls. The model jointly solves for the optimal asset allocation and the probability of successfully reaching the savings goal.
2010	"Issues in the Issuance of Enhanced Annuities," Robert	Two actuaries discuss underwriting approaches and public policy issues for standard, enhanced, and impaired annuities. They compare the UK, US and Canadian marketplaces. Among their observations are: "Annuity prices have also increased markedly over the past decade as	Annuities represent a product-oriented solution to longevity risk. This article provides a good list of annuity pros and cons.

	L. Brown and Patricia L. Scahill Social and Economic Dimensions of an Aging Population SEDAP Research paper No. 265 (May 2010) available at http://66.216.104.121/library/journals/actuarial-practice-forum/2007/october/apf-2007-10-brown-scahill.pdf	interest rates have dropped and life expectancy has improved, thus making life annuities a relatively 'expensive' product to the consumer." "The life annuity insurer underwrites the life expectancy and has an actuarial gain in the event of an early death—the opposite of the result from a life insurance policy. As a result, the interests of the company (making a profit through early death) and the interests of the annuitant (the desire for a long life and the financial benefit from receiving more annuity payments) are not aligned." "…wealth relative to living expenses is an important factor in the individual's ability to self-insure the longevity risk." "Planning one's post-retirement income strategy without the benefit of pooling the 'longevity' risk is very difficult. Either you accept less than the optimum income through your retirement lifetime or you run a significant risk of running out of money altogether."	
2010	"Spending Rates, Asset Allocation, and Probability of Failure," Dimensional Fund Research Paper James L. Davis (May 2010)	The author notes that the asset allocation and spending rule decisions must be made jointly because both affect retirement consumption opportunities. After noting some of the difficulties with a Monte Carlo simulation based on the assumption that returns are normally distributed, the author recommends a two asset class bootstrap as a way to avoid pre-selecting a probability distribution. A second advantage of a bootstrap is that it captures cross-sectional correlations. Under both Monte Carlo and bootstrap approaches, however, the model must use historical returns to compute expected future returns. The standard error of the mean is equal to the standard deviation of the sample divided by the square root of the number of observations. Assuming that the standard error is normally distributed, there is an approximately 5% probability that the true mean of the distribution is more than 1.645 standard errors below the historical mean of the sample: "therefore, one conservative estimate of the expected return of the index is the sample average	The study is a Dimensional Fund Working Paper. The author considers either a fixed level of spending or a fixed spending rate as "extreme solutions." Note: this approach assumes parameter stability. What is the objective—either produce a conservative estimate of the expected mean return or provide a realistic indication of the risk/reward tradeoffs faced by the investor? The article emphasizes the weakness in the argument that investors can use any autopilot distribution formula. Concluding section of article provides rationale for the necessity of close portfolio monitoring.

		return, minus 1.645 standard errors. Since the likelihood of the expected return being below this estimate is small, it provides useful perspective for stress-testing the ability of a portfolio to support a given spending rule." The sample is the monthly returns of the CRSP cap-weighted index and one-month T-Bill returns from 1926 through 2009. All returns are discounted for CPI. Spending rules are imposed on bootstrapped forty-year samples where the two-asset class portfolio is annually rebalanced to the asset allocation target. The spending rule consists of a weighted average of two components: (1) prior spending over the previous 36 months (this is a smoothing function that mitigates volatility of spending), and (2) a percent of current value rate (over the range of 3% to 7%). The weightings determine the relative importance of prior spending versus current percent of value spending. Investors preferring spending stability will give greater weight to the prior spending component; investors preferring to avoid portfolio insolvency will give greater weight to the current percent of value spending component. The base case gives a 50-50 weighting to each component with a 4% current spending rate and a 50% stock / 50% T-Bill asset allocation. At the end of 40 years, real monthly spending ranged from $1,000 to $9,000 at a 90% confidence interval. Changing the spending rates suggests that longer retirements will benefit (achieve more income stability) by selecting lower spending rates and keeping the smoothing parameter from approaching a value of 1 (weight = 100%). Not surprisingly, higher allocations to stocks result in higher median retirement spending. For a 50-50 stock to bond allocation, "insisting on a very high degree of smoothing is a recipe for disaster. When the smoothing parameter is set at 0.9, all the failure frequencies are above 15%. The combination of 0.8 smoothing and 90% in stocks produces a failure frequency of 5.3%. For smoothing parameters below 0.8, all the frequencies are either equal to or very close to zero."	

		High allocations to stocks increase the volatility of consumption. Bootstrapped results for portfolios with low stock allocation, however, exhibit greater spending declines because the variation in spending takes place around a lower mean. The lower mean is a drag on consumption growth: "When the allocation to stocks is low, consumption growth tends to decline at a smooth, reliable rate. This is not what most people have in mind when they talk about smoothing consumption." A downward trending median suggests that the investor will not be able to maintain his standard of living as retirement unfolds. Given a smoothing parameter of less than 0.8, model results indicate: "…when the spending rate is 3%, stock allocations of 40% or more result in an upward-sloping median. When the spending rate is 4% it takes a stock allocation of at least 60%, and a 5% spending rate requires a stock allocation of at least 80%. These results say that we can have a reasonable expectation of maintaining living standards, if we are willing to live with a low spending rate and/or a high allocation to stocks." The lower adjusted mean, however does not exhibit an upwardly slopping median for *any combination* of spending rules or asset allocations. The author acknowledges the limitations of his model: (1) it fails to capture serial correlations in returns, (2) it considers only two asset classes, and (3) it does not take age and investor circumstances into account. He cautions against trying to develop a set of autopilot rules: "Rather than trying to develop such a mechanical rule for time-varying parameters, it may make more sense for investors (and their advisors) to periodically review the appropriateness of current parameter values. This would allow the individual's current circumstances to be factored into the decision."	
2010	"Sequence Risk: Managing Retiree Exposure to	The paper begins with a critique of past research which, according to the authors, focuses largely on using static simulation models to determine a safe initial retirement income withdrawal rate which is sustainable	The authors state: "A measure of success, ruin or failure should not be misinterpreted by advisers as a measure of risk. Risk is a measure of the

	Sequence Risk Through Probability of Failure Based Decision Rules," Larry R. Frank Sr., John B. Mitchell, David M. Blanchett ssrn.com/abstract=1849868	throughout the applicable horizon. Such an approach fail to recognize "…that all the variables are dynamic over time." The approach of the authors is to revisit the initial outputs of a retirement income risk model by updating all variables to their current values. The values are (1) current value of financial asset portfolio, (2) time remaining in the planning horizon, (3) the portfolio's asset allocation, and (4) the current withdrawal rate defined as dollars needed divided by current portfolio value. Given that markets produce unpredictable sequences of above and below expected returns, each retiree is exposed to sequence risk. An unexpectedly large withdrawal need, or a significant market decline are events that effect portfolio sustainability. Adjustments to either asset allocation or withdrawal amounts may be required in order to keep the probability of portfolio depletion—probability of failure, or POF—within reason. Although the authors do not implement a dynamic programming model, in essence, they seek a set of complex decision rules to maintain an upper bound on POF with allocation and spending as the control variables. The trigger for invoking a decision rule is a change in portfolio value greater than .5 standard deviations above or below expected return. Given such an outcome, either the weighting to equity changes or the portfolio's withdrawal amount adjusts by a fixed 3% relative to the amount currently being withdrawn. They model a wide array of outcomes based on a five-asset class portfolio parameterized by historical values from 1926 through 2009. Testing a spectrum of withdrawal rates, they explore the sensitivity of POF to periodic change in asset allocation or in spending. For investors with high spending rates a change in asset allocation generally has only a *de minimus* impact on POF.	consequences of events while probability is a measure of likelihood of those events." This distinction, of course, is merely the difference between the probability of a shortfall and the magnitude of the shortfall should it occur. See Dus, et al. [2005] for a discussion of the expected present value of a shortfall. Although the paper presents an extremely granular analysis based on a somewhat simplistic retirement income risk model, it provides a useful and credible rationale for avoiding auto-pilot retirement spending policies. It is one of a many recent studies that place increased importance on ongoing periodic portfolio monitoring.
2011	"Portfolio Success Rates: Where to Draw the Line," Philip L. Cooley, Carl	The article begins: "The idea of a portfolio success rate has served as a useful metric since we first wrote about it in 1998. It measures 'the percentage of all past payout periods supported by a portfolio despite annual withdrawals.' The "sustainable withdrawal rate" literature	"In our opinion, most practitioners and clients should require at least a 75% portfolio success rate in the selection of a withdrawal rate."

	M. Hubbard, and Daniel T. Walz Journal of Financial Planning vol. 24 no. 4 (April 2011), pp. 48 – 60.	suggests that a sustainable rate is a relatively low rate that "…will deprive a client of funds in the early years of retirement but will provide greater support in the later years, or simply a larger estate." The "sustainable rate" rules have been criticized because they impose too high a cost in terms of consumption during early retirement. The authors distinguish between two sets of research. The first is the literature on portfolio sustainability: "…authors calculated portfolio success or failure rates based on constant or inflation-adjusted periodic withdrawals from continually rebalanced portfolios of stocks and bonds for specific payout periods, or until the portfolios were exhausted." The second is a literature on changes in retirement income planning to reflect unexpected changes in the value of the retirement portfolio. This literature considers "…whether adaptive or variable payouts over statistical life expectancies and various portfolio rebalancing plans improve the sustainability of withdrawal rates." "The principal objective of our analysis is to calculate retirement portfolio success rates for various monthly withdrawal rate assumptions and various portfolio asset allocations from 1926 to 2009, and show how an adviser can use the findings to manage portfolio withdrawal rates adaptively." The article uses the methodology of rolling periods and calculates monthly portfolio returns for the S&P 500 and the Salomon Brothers Long-Term High-Grade Corporate Bond Index and the S&P monthly high-grade corporate composite yield data for overlapping 15, 20, 25, and 30 year periods from January 1926 through December 2009. Portfolio allocations range from 100% bonds to 100% stocks at increments of 25%. The rolling periods methodology is preferred to Monte Carlo simulations and bootstrap studies: "…the rolling periods approach has the unique advantage of retaining the effects of the actual sequence of security returns and variances in bull markets and bear markets over the 84 years of data…." When withdrawals are limited to constant nominal percentages of the	The paper examines various "adaptive" withdrawal strategies under a shortfall metric preferencing criterion. It examines historical returns, via a rolling period methodology, to discover the rules that exhibit high rates of portfolio sustainability. In a severe bear market, the implication is that investors should limit portfolio withdrawals to accounting income (dividends and interest) only. At the extreme, the adaptation seems merely to advocate slashing spending drastically and hoping for a quick market recovery. It mirrors the advice of Bengen in 1994 ["Determining Withdrawal Rates Using Historical Data," William P. Bengen Journal of Financial Planning vol.7 no.4. (October, 1994), pp. 171 – 180].

		portfolio's initial value, withdrawal rates of up to 7% exhibit success probabilities in excess of 90% over 30-year periods for portfolios allocated 75% to stocks and 25% to bonds. An allocation of 100% stocks drops the rate to 87%, 50-50 to 85%, 25-75 to 38% and 100% bonds to 24% for the nominal 7% withdrawal rate. In the 55 30-year periods from 1926 through 2009 eight simulated portfolios failed. Four failed because they were "…initiated in the late 1920s and early to mid-1930s." However, for an inflation-adjusted withdrawal rate of 7%, a 75% stock portfolio succeeded at only a 45% rate. No portfolio reached a 90% success rate for a 30-year period unless the withdrawal rate was lowered to 4%. The results, according to the authors, can assist advisors in formulating adaptive policies. "In a sustained bear market such as 2008 to early 2009, clients preserve the opportunity for portfolio recovery by reducing withdrawal amounts to no more than the dividend and interest income from the portfolio, thus avoiding the liquidation or shares or bonds at low values. Planning budgets in retirement and allocation to liquid assets should include the possibility of such market conditions."	
2011	"A Safer Safe Withdrawal Rate Using Various Return Distributions," Manoj Athavale and Joseph M. Goebel Journal of Financial Planning vol. 24 no 7 (July 2011), pp. 36 – 43.	The four percent withdrawal rate rule is usually tested by researchers using one of two methods: (1) historical back testing using overlapping periods or, (2) simulation analysis. The article defines the rule as follows: "In its most general form, the rule suggests that a retiree with a reasonably diversified retirement portfolio may make inflation-adjusted annual withdrawals equal to 4 percent of the initial portfolio balance….[it] tries to balance the two sides of the withdrawal rate dilemma: withdraw too much and face the negative consequences of outliving the retirement portfolio or withdraw too little and under-live the retirement potential." The article cites some examples of past research studies using one or both of the above-listed methods. Most of these studies suggest that a reasonably diversified portfolio could sustain a 4 to 5 percent withdrawal rate over a 30 to 35 year planning horizon at a failure rate of 10 percent	A discussion of the 4%withdrawal rate rule. The authors point out that the methods used to test the rule—i.e., historical back testing and simulations of parameterized normal distributions each have serious flaws. The article contrasts the results of simulated returns drawn from a normal probability distribution with returns drawn from other non-normal distributions. They also provide a helpful discussion of why average or expected return is not determinative of ultimate success in any historical period. Rather, it is the timing and sequence of returns that is of

		or less. However, a failure rate analysis based either on the single path of historically realized returns or on a simulation model assuming a normal or lognormal return distribution "…may lead to misleading inferences." There is no guarantee that the future returns will correspond either to historical results or to any specific statistical distribution. They note: "…the distribution of stock returns is known to be non-normal and heteroskedastic (Nelson and Kim 1993). The empirical distribution of returns has been observed to have more distributions around the mean and fatter tails than the normal distribution, and skewed distributions offer more flexibility in modeling returns by removing the constraint of symmetry in returns. Harvey and Siddique (2000) and Dittmar (2002) found that higher-order moments are relevant in explaining equity returns, Levy and Duchin (2004) therefore conducted a study to determine which theoretical distribution best fits the observed distribution of returns for various asset classes and holding periods, and found that the logistic distribution was generally the best fit, though in a few instances other distributions could also describe the observed returns." In order to maintain comparability with past studies, the authors consider a portfolio providing an annual real return of 5.1 percent with a standard deviation of 12 percent. They draw random annual returns for 35 periods. In addition to making the draws from a normal distribution, they make draws for the following probability distributions: Beta, Extreme, Gamma, Laplace, Logistic, Lognormal, Pert, Rayleigh, Wakeby and Weibull: "…most of which are characterized by multiple parameters to represent the location, scale, and shape of the distribution." There are 10 scenarios for each of the ten distributions for a total of 100 retirement scenarios. The authors observe: "Although most scenarios resulted in portfolio success (the portfolio was able to sustain a 4-percent withdrawal rate over the 35-year period), we were surprised by the proportion of scenarios that resulting in portfolio failure—18 of the 100 scenarios. In	primary importance.

		order to be consistent with some of the other studies mentioned previously, we refined portfolio success by shortening the retirement period to 30 years. The portfolio failure rate dropped to 14 percent….” They further note: "…scenarios in which average returns exceed the withdrawal rate do not necessarily lead to portfolio success….scenarios in which average returns are lower than the withdrawal rate do not necessarily result in portfolio failure….although larger returns and smaller standard deviations contribute to portfolio success, these are not sufficient conditions to ensure success, and other factors including the timing of returns and the occurrence of negative or positive runs may also be important." The study concludes: Our results indicate that a 2.52 percent withdrawal rate could be sustained in each of our randomly generated scenarios, but a 4 percent withdrawal rate is associated with an 18 percent probability of portfolio failure."	
2011	"Products and Strategies for Lifelong Retirement Distributions," Mark J. Warshawsky available at http://www.dol.gov/ebsa/pdf/TowersWatson091510.pdf	[Text of testimony given September 15, 2010 at the Joint Hearing of the Depts. Of Treasury and Labor on Certain Issues Relating to Lifetime Income Options for Participants and Beneficiaries in Retirement Plans] Warshawsky's testimony begins with the observation: "…the household faces a distressing choice—either it should reduce consumption in order to maintain assets to finance a possibly long retirement, or maintain its standard of living and risk having to reduce spending suddenly and significantly if it outlives the average and runs out of money." Although Warshawsky explores a variety of strategies including purchase of annuity contracts, he begins by pointing out two elements of these contracts (1) "…insurer insolvency, although a rare event, could be catastrophic if the relevant state-guaranteed funds do not cover the losses;" and, (2) "…annuity purchasers are likely to come from higher socioeconomic groups than non-purchasers, and the wealthy generally have lower mortality. Our research found that the impact of the 'adverse selections' adds about 10% of the annuity price, compared to what	Warshawsky contends that combining annuitization with ongoing management of an investment portfolio is too complicated a task for the average retirement plan participant: "…these combination strategies have not yet been 'automated' in the market-place, they unfortunately appear to be too complex for a household to pursue."

		would have been charged if the mortality of the general population were to apply.” The testimony briefly covers the use of nominal immediate annuities, inflation-adjusted immediate annuities, variable annuities, and mutual fund withdrawals (5% from a 50/50 equity/bond asset allocation decremented by 120 bp in fees). Although the mutual fund withdrawal strategy avoids many of the shortcomings and risks associated with annuity contracts, it produces “…the highest probability of not meeting minimum real income targets and averages the lowest real income flow.” Warshawsky suggests that the preferred strategy to generate retirement income is a blend of annuitization and mutual fund withdrawals: “…annuitization should begin around the mid-70s for an individual and continue until about the mid-80s for significant annuitization, but still keeping aside about a fifth to a quarter of the original account balance for other liquidity needs.”	
2011	M.A.H. Dempster and E.A. Medova, “Asset liability management for individual households,” British Actuarial Journal, Vol. 16, Part 2, (2011), pp. 405 – 439. Accompanying group discussion in British Actuarial Journal, pp. 1 – 27.	The essay presents an interactive “decision support tool” to a meeting of the Institute of Actuaries in England on February 22, 2010. The financial planning application was developed by a team of actuaries including the two authors of the articles. The authors characterize the application as an “…individual asset liability management (iALM) meta-model.” The authors present a brief review of the state of financial planning advice: “current best practice of leading financial advisors and private wealth managers is to employ static Markowitz mean-variance portfolio allocations based on current market views, while projecting future portfolio returns from the optimal allocation using Monte Carlo analysis to calculate the probabilities of achieving various goals.” By contrast, the iALM application solves “… a dynamic multistage stochastic *optimization problem* in discrete time.” The model brings together “…the principal ideas from behavioural finance, classical finance and stochastic optimization theory to help individuals with *long term financial planning decisions*.”	While this presentation to the Institute of Actuaries does not focus specifically on retirement income planning, it is nevertheless of interest because it presents a comprehensive financial planning model that “optimizes” for sustainable consumption at several levels--minimum, acceptable and desirable—in terms of an investor’s prioritized goals and liabilities. In addition to education, housing, and other expenses, a key goal is often to fund a lifetime income during retirement. Any decline in investment value is less “frightening” if the investor sees that achievement of the minimum goals has a very high probability—only the desirable funding level may have to be trimmed and such a setback may only be temporary. Much of the accompanying discussion focuses on

		Much of the article is devoted to an explanation of the “black box” elements of the program—including several advances in dynamic stochastic programming [DSP] modelling language. Depending on the type of financial time series under consideration—e.g., stock, bond, real property, cash, inflation series, etc., the article discusses various processes that are used for simulations. These include geometric Brownian motion (stocks), Ornstein-Uhlenbeck process (bonds), and geometric Ornstein-Uhlenbeck process (cash and inflation series). The authors claim that statistical testing [Kolmogorov-Smirnov and Jarque-Berra] of domestic equity, international equity, corporate bonds, government bonds, commodities, alternatives, and real property indicate “…empirical return distributions for normality which showed monthly returns acceptably normal at below the 20% significance level.” The simulations of future returns are based on monthly historical time series for ten-year periods from June 1997 through May 2009. A quasi-maximum likelihood approach was used to calculate the variance/covariance matrix. The model works within a decision tree structure where scenarios branch off from nodes which represent important random events (e.g., death, illness, etc.) in the lifetime of an investor. An extremely complex structure of liabilities (indexed, nominal, contractual, and goal-oriented) operates within the decision tree structure. Cash flows are tracked over a large range of sources and uses. Of particular interest is the prioritization of goals and liabilities in terms of time and subjective importance. Prioritization incorporates elements of the classic behaviorist value function in which the client’s current financial position is the assumed reference point where the slope of losses is steeper than the slope of gains—i.e., a kinked value curve. Each goal can have its own priority, inflation/discount rate, cash flow constraints, etc. Behavioral finance concepts are also employed in framing the output into “narrow framing” which focuses on the likelihood of achieving a particular goal; and “broad framing” which looks at real sustainable lifetime spending: “The primary goal of iALM is thus to increase the real spending that a portfolio can sustain.” The objective function is “…to	the model’s input assumptions and structure. It is an optimizer and, if constraints are not imposed on allocation weightings, an extreme “corner solution” emerges. For example, the recommended allocation in 2007—just prior to the collapse of the real estate market—was 46.5% for the sample British investor. [The difficulty with optimizers finds a nice expression in Terry Marsh and Paul Pfleiderer, “Alpha Signals, Smart Betas, and Factor Model Alignment,” The Journal of Portfolio Management (Special Issue, 2016): “…any portfolio optimization approach, unless it is fairly tightly constrained, has a well-known tendency to take relatively extreme positions to exploit what appear to be ‘near-arbitrage’ positions if perceived alpha spreads are believed to be associated with little or no systematic risk.”] A number of critical observations—as well as favorable comments—was forthcoming from the audience: “Having a control mechanism which gets more pessimistic when everyone else does, and vice versa, is a prescription for getting poorer in the long term….” “I notice that the assumptions that the authors are adopting appear to be based merely on historic returns over the last ten years….the authors have not taken into account models that are fat-tailed, even though such models appear to describe the world better than ones that do not exhibit such features.” “Kahneman and Tversky have written quite a lot, and a fair amount of what they have written shows

		maximize the expected utility of lifetime consumption….” Risk is managed by inputting the investor's risk tolerance in terms of a drawdown limit—i.e., the optimized portfolio should not result in a wealth drawdown greater than x% over a designated time period. This constraint can also be discretized with respect to funding for individual goals and liabilities. Much space is devoted to a sample case. The process begins with a "reality check." This is a simple projection of inflation and wealth accumulation under constant returns to evaluate the likelihood that lifetime goals are feasible given current resources and expected income and outgo. The example illustrates the application's recommendations pre-and post-crisis. Finally, the appendix provides a brief introduction to the principles of Dynamic Stochastic Programming. This is used to explain the generation of scenarios from "…a discrete time, continuous state, multi-dimensional stochastic data process….The evolution of the discrete state simulated data process across time is given by a *scenario tree*….All decisions, at intermediate nodes of the tree, take into account the possible evolution of the stochastic data process from that point forward."	that the preferences that people express when faced with market researchers with clipboards are not consistent. The danger is that you are coding-in irrational behavior and then just extrapolating that or extending that irrational set of preferences into the future. So it is not obvious to me exactly what service is being provided by calibrating this utility function to certain questions and then coming up with an optimization." "The objective function is described as maximizing future consumption. I do not think that is how individuals think: what individuals want to do is to minimize future pain. The key thing that a complex computer model should try to do is help an individual understand his own attitude to risk and to learn about his own individual response to things that may happen to his investments." "A male at 65 today has about a one-in-1000 chance of living to twice their life expectancy and life expectancy is, perhaps, 21 years. So at 65 are you going to live to 107? Probably not. But when you reach age 85, you have a one–in-ten chance of living to [twice] your life expectancy. In the same way that…we need to take account of the variability of returns on investment assets so, I believe, we may need to look at the variability of longevity to determine the optimal time to buy an annuity."
2011	"An Aged-Based, Three Dimensional, Universal Distribution Model Incorporating Sequence Risk," Larry R. Frank Sr., John B. Mitchell,	The paper seeks to develop insights for distribution strategies where the distribution period under consideration in based on the retiree's current age. A sustainable withdrawal rate (WR%) is a time dependent variable that is based, in part, on age. Thus, the authors envision a model that includes "…the three dimensions of distributions (allocation, withdrawal rate, time) with a focus on Probability of failure (POF) which is a time independent variable…." The definition of the Distribution Period (DP)is	An important distinction to keep in mind for planning retirement income portfolios is: Average Life Span v. Actual Life Span. One half of the population will live longer than the average life span; and, in some cases, the individual's life span may be many years above the average [mortality distributions, or "the force of mortality" approach

	David M. Blanchett Journal of Financial Planning available at: http://www.onefpa.org/journal/Pages/An%20Age-Based%20Three-Dimensional%20Distribution%20Model%20Incorporating%20Sequence%20and%20Longevity%20Risks.aspx	"…the <u>current</u> life expectancy for the *current* retiree's age." In order to evaluate how longevity impacts the model, the authors vary the life-expectancy variable by establishing certain thresholds under which either 75% of the population are expected to outlive the life expectancy, 50% outlive, or 25% outlive. The allocation model incorporates five asset classes: 1. 30-day T-bill 2. Long-term Corporate Bond Index 3. S&P 500 4. Ibbotson Associates US Small Stock Index 5. International Large Equity Returns are constant dollar returns from an assumed log-normal distribution. Withdrawal rates are established for each distribution period and the distribution period is adjusted each year the retiree ages. Thus, as the retiree ages, he or she rolls through various DPs and WR%s. The WR%s are managed so that the retiree maintains a constant target exposure to the POF rates—e.g., 5% probability of failure at age 'x' at a y% withdrawal rate. This is a dynamic adjustment or the withdrawal rate to maintain a specific targeted failure rate probability bound. The authors conclude "…a set withdrawal rate, e.g., 4%, is not optimal for all retirees because not all retirees are the same *age*." Note: the "differences between Outliving Expected Longevity *increases* [sic] as the retiree ages." The authors then turn to the question: "What happens when the retiree continues to survive? What happens to retiree withdrawal values when the retiree continues to live beyond expected longevity?" The answer to these questions leads to an Aged-Based model. The impact of aging is to shorten the DPs and to thus increase the WR%. "De-cumulation should be viewed as a dynamic, rather than set-and-forget, exercise." Sequence risk, according to the authors, does not exist merely at the	exponential distributions exhibiting long tails]. Longevity risk (the likelihood of outliving resources) is a stochastic variable not simply an "average." E.G., see, <u>Investment Management for Taxable Private Investors</u> by Jarrod Wilcox, Jeffrey Horvitz, and Dan diBarolomeo (Research Foundation of CFA Institute, 2006): "The *actuarial* life expectancy is not fixed at birth but is conditional upon having reached a greater age, so the life expectancy of a person at age 10 is less than that of a person who has reached age 60." "…the uncertainty of remaining lifespan increases as the retiree ages even though there is a reduction in the number of expected remaining years." Sequence risk exists throughout retirement. This article provides a good justification for an active portfolio monitoring process.

		start of retirement. Rather, "…a retiree may experience 'good' markets to be above the 50th percentile during one period of time (e.g., age 60 simulation) and experience 'bad' markets to be below the 50th percentile during another period of time (e.g., age 62, or any subsequent age simulation) because sequence risk is ever present during de-cumulation years." Thus, the paper argues: "…that WR% is a dynamic function of other variables, including portfolio allocation, each of which should be evaluated three dimensionally in relation to each other to access the current transitory state of a retiree during de-cumulation years."	
2011	Michael Ashton, "Maximizing Personal Surplus: Liability-Driven Investment for Individuals," <u>Retirement Security in the New Economy: Paradigm Shifts, New Approaches and Holistic Strategies</u> SOA 2011 Conference: https//www.soa.org/news-and-publications/publications	The author begins with a commentary on Cooley, Hubbard and Walz's 1998 "Trinity Study." He notes: "Using a lengthy data series covering a wide range of market environments greatly improves upon an approach that uses historical mean returns as the benchmark for what can be withdrawn. But it also limits the possible outcomes to those observed in the historical data set." Given the realized historical results, the Trinity Study suggests that a relatively high weight to equity improves portfolio sustainability when faced with periodic spending demands. For Ashton, however, this is a questionable conclusion for individual investors: "If this were a pension fund, we would say the liabilities are roughly fixed in real space and the assets are very volatile." The purpose of his study is to explore the success of portfolios that "…jointly consider both the asset mix and the spending requirement. That process is called Liability-Driven Investing, or LDI." Things are a bit trickier when applying an LDI approach to an individual. Many personal "liabilities" are not, in fact, contractual. A pension fund enjoys the actuarial "smoothing" that comes with operating over a large population of participants and beneficiaries. An individual, by contract, "…is exposed to the randomness of a single spin of the wheel of fortune when it comes to his own longevity or the possibility of large medical bills due to his own poor health." Nevertheless, the investor still benefits from allocations that more closely match asset and liability valuation changes because (1) such allocations "…reduce the importance of the	The author uses the Cooley, Hubbard and Walz [CHW] 1998 "Trinity Study" as a comparative benchmark for the results presented in his study. He concludes that CHW equity-weighted portfolios evidence high success rates not because of an inherent advantage of stocks over bonds; but, rather, because of "…the peculiar configuration of our actual history that produces the high portfolio success rates." The historical period under consideration "…can promote over-optimism…." Rather than hoping that the future range of economic/investing environments mirrors the past, "we would like to reduce as much as is practical this reliance on future conditions being quite like past conditions." The study occupies an interesting position in the spectrum of "safety-first" (immediately lock in the required cash flow at retirement lest the portfolio suffer an unacceptable decline in value) to "safety-net" (invest for growth by giving yourself the opportunity to capture higher expected returns from equity and consider annuitization only if the portfolio suffers large reversals or when the

		assumption that the experienced returns, variances, and correlations will be like those of the historical data set" and (2) "managing the asset/liability match" decreases the likelihood for large declines in the surplus portfolio, and reduces "…the behavior-management challenge." The study considers a simple decomposition of individual liabilities into nominal (e.g., home mortgage and car payments) and inflation-adjusted (e.g., health care) expenses. Investigating the investor's liability structure, allows for an asset allocation more closely matched to the goal of preserving portfolio surplus. The investment goal shifts from maximizing return on the financial asset portfolio, to providing a low-risk strategy to provide an acceptable level of lifetime spending. Ashton makes the valuable observation that a simulation model's long-term success coefficient for a given asset allocation and periodic spending policy is potentially misleading: "…because they do not distinguish between portfolios that fail toward the end of the planning period from ones that fail spectacularly, closer to the beginning of the planning period. Each is recorded as a single portfolio failure, but the former circumstance is clearly preferable." To illustrate his point, Ashton tilts the asset allocation process away from substantial equity weighting towards combinations of nominal and inflation-linked bonds depending on the investor's liability profile. Specifically, he compares the output of a simplified simulation model for a 75% stock / 25% nominal bond portfolio to a portfolio allocated 80% TIPS / 15% bonds / 5% stocks. Both portfolios exhibit roughly the same long-term success probability (c. 58%). However, the investor owning the 75% stock / 25% nominal bond portfolio stands a 10% chance of portfolio depletion by year 15; the investor owning the 80% TIPS / 15% bonds / 5% stock portfolio does not incur a 10% chance of depletion until year 22. After considering the results of several other simulations, the article asserts: "…choosing an asset portfolio to more-closely match liability is not *less* important if the withdrawal rate is more conservative, but *even more important*….When our expected outcome is well above the "safe" level, failure is more remote. Portfolio failure now requires the **joint** occurrence of (a) poorly-matched assets and liabilities combined with (b)	mortality premium grows large). He provides a thumbnail sketch of thinking on the topic of retirement planning: • Initially the goal is to maximize assets—more money is always better than less; • More recently, the goal is to maximize constant-dollar or nominal income either through "self annuitization" and the use of adaptive rules for withdrawal (subject to shortfall probability constraints), or through purchase of annuity contracts; • Currently, an LDI approach is beginning to take hold. This is an exercise in surplus management focused on controlling the risk to surplus rather than pushing the asset portfolio towards earning higher and higher returns.

		bad luck. As noted earlier, the primary reason for matching assets and liabilities closely is to minimize the impact of bad draws from the return distribution." Finally, the author cautions that asset allocation design must consider both the character (constant dollar or nominal) and the timing of liability payments. Additional risk control is possible by insuring against potential end-of-life liabilities like long-term care needs. Annuities can insure against longevity risk.	
2011	"De-Risking Retirement Income," Stephen J. Huxley and Brent Burns CFA Institute (2011), pp. 1 – 4.	The article advances the proposition that "Behavioral risks denote flaws in individual decision making caused by emotional responses to changing financial conditions." In order to counteract such flaws, the authors assert that investors should (1) immunize their near-term income needs—e.g., one to ten years—with an income-matching portfolio, and (2) establish a long-term growth-oriented portfolio of risk assets (stocks). The income matching portion of the portfolio consists of bonds that are maturity matched to the investor's cash flow needs. The bonds are held to maturity and thus rendering rising interest rates harmless—the duration of liabilities matches the duration of assets as long as the maturities/coupons exactly match the expense payment schedule. Reinvestment risk is also eliminated because the smart bond ladder generates periodic income through coupon interest payments and redemptions—no income is reinvested. Timing risk is eliminated because the bond maturity dates are exactly calibrated to the need to generate income. The authors continue by pointing out that the investor can eliminate default risk by funding the ladder with insured CDs or government / municipal bonds. A TIPS latter can reduce inflation risk, and longevity risk is mitigated "…because the time-targeted strategy of income matching should increase capacity for a more aggressive equity (growth) portfolio." They turn their attention towards identifiable "behavioral risks."	A short article that is squarely in the annuitize-ASAP-school of thought. The authors make the interesting argument that pre-funding the initial period of retirement with a 'smart bond ladder' is a way to leverage the behavioral decision making tendencies of investors in such a way that they (1) remain comfortable with their long-range plan and resist the tendency to bail out during rough economic periods, and (2) enhance the probability of achieving lifetime retirement income goals. Usually decision making based on behavioral tendencies produces suboptimal results. However, the authors assert that their strategy—partial liability-driven investing— leverages such tendencies to produce a salutary financial outcome. Depending on investor circumstances the bond portion of the portfolio can take the shape of a rolling or non-rolling ladder. Note that the strategy espoused in this article [fixed income portfolio to fund the initial retirement years / risky asset long-term portfolio to fund later years] reverses the more commonly found strategy of (1) maintaining a risky asset portfolio for a limited initial period followed by (2) income from a deferred annuity (Advanced Life Deferred Annuity =

		1. Impatience risk is reduced because, with the pre-funding of near-term income needs, the investor can shift focus to longer-term planning objectives. 2. Income matching reduces ignorance risk because the financial plan is easy to understand and the purpose of each section of plan is easy to understand and the purpose of each section of the portfolio can be clearly articulated. 3. Regret risk is greatly reduced because holding the individual bonds to maturity makes the worst-case results known in advance. The authors assert that their preferred approach solves many of the difficulties investors face when attempting to stay the course with a plan based on a total return approach.	ALDA) at the end—contingent upon the investor's survival. The authors provide no analysis regarding comparative costs, utility, risk, etc. [Perhaps due to space limitations]. Of course, there is a benefit to overcoming adverse psychological tendencies.
2011	"Annuities in the Context of Defined Contribution Plans" A study for the U.S. Department of Labor, Employee Benefits Security Administration, Michael J. Brien and Constantijn W.S. Panis (November, 2011), pp. 1 – 19.	The study provides an overview of the U.S. annuity market by examining a sample of annuity prices from 1986 to 2010. The sample is taken from data available at Annuityshopper.com which is a website that markets annuity products. Price quote data for qualified annuities is available from February 1986 through January 2001; and for non-qualified annuities from January 1992 through July 2010. The study examines both qualified and non-qualified single-premium immediate annuity quotes for a 65 year old male for a premium of $100,000. The study identifies high, low and average monthly payments from the insurance carriers marketing products through this website. For non-qualified annuities, the average monthly payout drops from $842 in January 1992 to $613 in July 2010. This decline is approximately 27 percent. In January 1992 the payout ranged from a low of $751 to a high of $916. The corresponding range in July 2010 is $570 to $648. The decline in annuity payouts reflects both the downward trend in bond yields and a decline in mortality rates over a comparable period. The study documents a negative correlation between the monthly annuity payout and the financial rating of the issuing insurance company: "Generally speaking, insurance companies with good credit ratings	A good contribution to the time-series of annuity data. The study also provides: (1) data for joint and last survivor annuity contracts, and (2) a survey of corporate sponsored plans to determine the percentage of plans offering annuity options to participants.

		command higher prices than those with lower credit ratings."	
2011	"In Search of the 'Best' Retirement Strategy," William Klinger Journal of Financial Service Professionals vol. 65 no. 1 (January 2011), pp. 62 – 73.	This article identifies eight strategies for generating retirement income and, after some modifications to assure comparability among the strategies, creates a simulation model assuming log-normal asset returns in order to illustrate the patterns of income and terminal wealth (legacy). The eight strategies are: 1. The 4% Rule adjusted for inflation 2. The Floor & Ceiling Strategy where the 4% rule is subject to a yearly upper and lower bound 3. The Modified 4% Rule which is a 4% Unitrust distribution formula with a further modification that portfolio losses in a particular year will also reduce the next year's distribution to 95% of the subsequent year's withdrawal amount 4. The Decision rules Strategy of Guyton and Klinger 5. The Safe Reset Strategy where the withdrawal rate is a function of the retiree's age 6. The Aggressive Strategy which begins with a high rate of withdrawals and decreases them each year either by a set amount or by certain formulae 7. The Half-Annuity Strategy where 50% of the portfolio is used to buy an annuity immediately upon retirement 8. The Delayed-Annuity Strategy where an initial 15% of the portfolio is set aside and invested to fund an annuity purchase at age 85. The article notes that the patterns of retirement income produced under the various strategies can differ significantly. Whereas each investor has a unique set of preferences, there may be no such thing as a single best strategy across the entire population of retirees.	A succinct review of withdrawal strategies that have been "recommended" in previous articles. Uses a simple simulation model to demonstrate how different withdrawal patterns emerge from each rule. Retiree's job is to select the pattern that best fits with their preferences rather than forcing consumption preferences to conform to a pre-set rule.
2011	Chuck Yanikoski, "Creating a Reality-Based Financial Decision-Making Model for Older Americans,"	Yanikoski begins his presentation with the assertion that most financial advisors offer flawed recommendations to older clients. In large part, this is due to the fact that advisors are trained primarily to assist investors in the wealth accumulation stage rather than in the retirement stage. The orientation towards techniques for wealth accumulation, in Yanikoski's opinion, "…has created the mistaken impression that, when	Yanikoski, coming from a background with New England Mutual Insurance Company, has a good perspective on the investment advisor / practitioner community. However, some assertions may reflect a lack of familiarity with academic investigation. For example, the assertion that the "…first

	<u>Retirement Security in the New Economy: Paradigm Shifts, New Approaches and Holistic Strategies</u> SOA 2011 Conference: https//www.soa.org/ news-and-publications/publications	people actually do retire, accumulation simply becomes 'decumulation,' that we can just put a minus sign in front of the savings rate, and posit (or assume) that *withdrawal* rate and investment performance are now the two critical elements." By contrast, Yanikoski asserts, "…investment return is no longer an overridingly critical variable, while withdrawal rate is a highly inappropriate concept that should simply be abandoned." Yanikoski also points to other aspects of flawed, misleading, or inappropriate investment advice: • Increased allocation to equity in the hope of increasing return often translates into increasing either the probability that the retiree will either die wealthy or go broke. Neither objective, however, is high on the priority list. He argues: "…investment risk is adversely priced for retirees, so they would be irrational to 'buy' it. It is adversely priced, because the risk-return trade-off is not the same for retirees as it is for institutional, wealthy, or younger investors. The risks are greater and the rewards are smaller. The risks are greater, because if retirees lose the gamble, they usually have no good options for recovery; time is working against them, not for them. The rewards are smaller, because in a scenario of withdrawals rather than deposits, the effect of compounding is reduced, so even when markets rise, retirees benefit less from it." • "…in a serious retirement plan," such a thing as withdrawal rate should rarely be discussed. His rationale comes from observing that retirees almost never follow a level or a smoothed percentage pattern of withdrawals throughout the relevant planning horizon. Among the factors that can change the budget are: inheritances, insurance benefits, medical costs, gifts to or support of family members, residence change, tax-rate change, etc. Bottom line: Yanikoski recommends that planning for retirees encompass a wide range of variables that affect cash flow. He advocates incorporating both assets and liabilities (contractual and goal-oriented) in the retirement planning model. He argues that a credible	published recognition of the problem of making regular withdrawals from volatile investment funds…" occurred in a New England Mutual publication: Guide to the Personal Retirement Market, 1991-1992. Given the academic work of Fishburn, Yaari and others throughout the 1960s and 70s this statement should probably be amended to something like: "the first publication aimed at financial planners and insurance agents…." The article contains a brief but interesting discussion on the topic of annuitization. Some retirement risk models include annuities in the asset allocation as a source of guaranteed income and/or as a vehicle to reduce overall portfolio risk. Other models assume that an annuity contract is primarily a cash-management instrument where the guaranteed cash flow can cover necessary expenses while the return from the investment portfolio is available to fund discretionary expenses. Other models consider annuities as tax shelters or as a longevity risk hedge. However, "…annuitization carries an imposing downside: loss of control over one's assets—leaving them unavailable for other needs, including potentially severe ones. Granted, newer annuity products offer access to funds, but only at a pretty steep cost. So the annuitization election is nontrivial and ought to be made only when it produces a clear improvement in the odds of not running too low on money."

		model should incorporate the following factors that may impact (positively or negatively) a household's risk of running out of money [not all factors impact every household]: • Choice of standard of living (expense management), • Probability / necessity of moving to another residence, • Date of retirement / part time work election / election to return to work on a full-time basis, • Order in which assets are liquidated, • Plans for future mental incapacity, • Plans for future long-term care expenses, • Annuitization, • Asset Allocation / expected rate of return, • Medical insurance coverage options, • Life Insurance portfolio, • Debt management, • Social Security benefit elections, • Pension withdrawal options, • Support for parents, children, grandchildren, siblings, • IRA planning elections, • Trust and estate planning elections. A comprehensive model needs to be granular. Expenses can be decomposed by time remaining until debt liquidation, by nominal or constant dollar costs, by probability of occurrence, etc. Yanikoski states: "The question that these models are really trying to explore is: How little could you reasonably tolerate living on, if you had to? But dividing expenses into 'necessary' and 'discretionary' columns doesn't answer that question." The above-listed types of risks faced by retired investors suggest that generating a precise probability of financial success for a retiree is an exercise in futility. Some risks cannot be measured: "So the overall risk of someone running out of funds before death cannot be measured or even reliably approximated, no matter how comprehensive the model." In such an environment, Monte Carlo models "…do a disservice…because	

		the client cannot see how and why specific scenarios succeed or fail…." Rather than stochastic modelling, the author recommends using a "static model with adverse assumptions." In a comprehensive model, the user can "plug in an extra-long lifespan, an underperforming rate of investment return, a high inflation rate, higher income tax rates, future reductions in pension or Social Security benefits, higher than normal medical and/or long-term care expenses and other adverse circumstances." He asserts: "If the output from the model includes projections of income, expenses and assets by category, then the client can see specifically where he or she is vulnerable. Unlike stochastic modeling, this provides real insight into the financial dynamics of a particular household." The type of model building promulgated by Yanikoski results in integrated planning because a financial decision in one area often affects the consequences of decisions in other areas. A model should incorporate the ability to integrate multiple decisions not only in the financial arena, but also with respect to non-financial family circumstances—e.g., desire for a second career, hidden tensions in a marriage, geographic relocation for personal or family reasons, and so forth.	
2011	"Retirement withdrawals: Preventive reductions and risk management" John B. Mitchell Financial Services Review vol. 20 no. 1 (January 2011), pp. 45 – 59.	Mitchell reviews recent literature on optimizing (1) controls on withdrawal rates and (2) asset allocation to enhance the probability of portfolio sustainability throughout retirement. He notes that many previous studies utilize a bootstrapping methodology to modeling future portfolio evolutions. This methodology is sub optimal because it inputs only historically realized results. By contrast, Monte Carlo simulation allows for extreme results despite the fact that they have not yet been experienced. Mitchell tests a set of distribution rules that are customized to the investor's risk/reward preferences and cash flow objectives. The application of the rules is a function of dynamically changing portfolio values as well as expected remaining lifetime of the investor.	Mitchell notes: "…anticipating adverse events and taking corrective action increases our chances of avoiding larger more catastrophic problems." In this case, the 'event' is the probability of financial ruin. The issue is one of monitoring a portfolio based on a shortfall risk metric (likelihood of success) v. monitoring on a solvency metric (portfolio feasibility or sufficiency). Mitchell notes: "Existing research does not address the question of acceptable probabilities of failure (running out of money before the end of the planning horizon) although the need to do so is

		The controls flow from algebraic calculations. For example, If the Portfolio's current value is greater than the upward threshold [UT] times the Present Value of an Annuity Due [$PVIFA_{DUE}$] sufficient to pay the target income over the expected remaining lifetime of the investor times the initial Portfolio Value times the previous period's withdrawal rate, then algebraic manipulation shows that the current withdrawal rate should equal the current value of the portfolio $PORT_t$ ÷ (The UT * $PVIFA_{DUE}$ * Initial Portfolio Value]. This formula is comparable to that used in the 2006 article by Stout and Mitchell. The discount rate for the annuity is the historical portfolio—not the current or historical risk free rate—earnings rate given the client's asset allocation. The UT term in the equation determines how much "excess" value must exist before a change in withdrawal should be considered by the investor. In a nutshell, unless the ratio of a portfolio's current mortality-adjusted annuity value times the original dollar value exceeds the portfolio's current dollar value by the threshold amount, no change occurs in the withdrawal rate. The innovation in the formula outlined in this article is in the Downside Threshold [DT]. A DT of 1.5, for example, requires the portfolio to have a 50% excess over the discounted, mortality-adjusted value of the expected lifetime withdrawal income stream. The higher the DT, the more conservative in the sense that there must be a greater amount of "excess" value" before withdrawals can be increased. There is a maximum [MAX] and minimum [MIN] allowable withdrawal rate under all circumstances. The final controls are on the percentage of calculated increase or decrease that can be actually taken in any one period: "For example, if a retiree amortizes their portfolio over their expected remaining life span at historic rates of return and finds the portfolio could sustain a 10% withdrawal rate as compared to a current 6% withdrawal rate, a 40% UR would allow them to only increase the withdrawal rate to 7.6% that is (6% + [.4 X {10% -6%}]." [Note: UR = Upward Adjustment Rate]. Mitchell uses the 2006 Stout & Mitchell article as a base case for	noted both by Terry (2003) and Bengen (2006)." Note: Mitchell uses an annuity measure for the liability. The annuity however is a virtual annuity calculated over expected remaining lifetime. Mitchell's withdrawal strategy is a 'start-low-and-increase-when-feasible' strategy. The shortfall probability risk metric trumps a utility of consumption metric in this approach—which is acknowledged and discussed by Mitchell. Note: when Mitchell updates data through 2008, the probability of ruin doubles.

		comparison purposes: "Stout and Mitchell (2006) report, for example, for a 4.5% Initial Withdrawal Rate (INIT), 2.4 UT and 1.0 DT…., 0.2 UR and 1.0 DR, 40% MAX and 3% MIN. These control yield a 6.63% average withdrawal rate, 4.33% probability of ruin to age 100, and averaging ending portfolio 1.07 times the beginning amount based on 1926 – 2004 data. The same control, updated to 2008, yield a 6.47% average withdrawal rate, 8.68% probability of ruin to age 100…, and average ending portfolio 0.88 times the beginning amount. Thus, merely incorporating two years more results into the underlying data set means that there is an approximate doubling of the portfolio failure rate….."What is optimal today may not be tomorrow; presumably because of heteroskedasticity." Additionally, updated returns also cause a revision in the SAFEMAX results reported by Bengen: "The problem of needing to satisfy the most extreme outlier is dealt with here by defining the SAFEMAX rate as having a probability of ruin of 0.1%, that is the lowest 50 of 50,000 results are ignored. The SAFEMAX fixed rate based on 1926 – 2008 data and 50,000 iterations is 1.96% for age 108 and 1.97% for age 100. The essay explores the impact of a DT greater than one. Generally, increasing the value of this control is effective in preventing portfolio depletion. Additionally, raising the value of DT does not have a marked impact on achievable withdrawal rates. However, a higher DT may not meet the preference of all retires: "Retirees may prefer greater consumption at younger ages when they are more active, and therefore prefer the lower DT, at the expense of reduced consumption if they superannuate." This is an important observation regarding the wisdom of set-in-stone decision rules for retirement income portfolios. Utility may be maximized at the cost of risking low probability events at advanced ages.	
2011	Jack Brown and Travis L. Jones, "An Application of Asset-	The authors present a brief recap of attempts to offer private investors an ALM approach to retirement income planning. They contend that such attempts were unsuccessful because of two factors: (1) perceived	The essay falls into a 'two-fund' approach to retirement income planning. The first fund is a risk-minimizing bond portfolio matched to the duration

	Liability Management for Financial Planners," <u>Journal of Financial Planning</u>, Vol. 24, No.5 (May 2011), pp. 62 – 69.	complexity ("Fully understanding, and hence immunizing, the liabilities of an investor requires a deep probing of client circumstances….actual liabilities tend to be stochastic in nature based on changing costs, tax considerations, one-time expenses, etc.); and (2) because of the complexity of implementation including the need for suitable software. Investors are risk sensitive following stock losses in the 2008-2009 recession. Given the decline in financial asset portfolios coupled with falling interest rates, investors would need to move all or the preponderance of their investment portfolio into fixed income assets leaving little or nothing remaining for growth. This course of action leaves the investor at high risk of eventual portfolio depletion in the event of unexpected future expenses: "…consider an investor with discounted liabilities roughly in line with his investment portfolio. If this investor chooses to fully immunize his liabilities, then a high priority is placed on spending needs and portfolio exhaustion is a greater risk over the long-term." The authors' solution? "A shorter immunization period, while taking on more portfolio volatility risk, reduces the risk of portfolio exhaustion." [Note: Geometrically, this is a akin to a 'risk barbell.' The remainder of the article uses a simple risk model (static inflation, fixed planning horizon, fixed spending rate, etc.) to test various asset allocation schemes. In each case, the fixed income weighting required to duration match the liabilities is first calculated. This weight is set as a constraint in a mean-variance portfolio optimization. The optimization algorithm then calculates the optimal asset mix for the remainder of the risky asset portfolio given that the investor needs to have a growth rate sufficient to fund future inflation-adjusted spending. After considering several iterations of the risk model, the authors suggest that allocation of 67.5% to fixed income and 32.5% to equity achieves a reasonable balance between the goals of income safety and future wealth growth.	of investor liabilities; the second fund is an alpha-generation fund tasked with providing the opportunity for future real growth of wealth. Given the low interest rate environment at the time of publication (2011), a strict cash-flow matching of the portfolio is ruled out for most investors. Likewise, the authors recognize that immunization (duration matching) may also be impractical given that the low discount rate for liabilities produces a net present value approaching the value of the corresponding financial asset portfolio. [It is interesting to note that the article presents a case study in which the liability NPV is higher than the current market value of the investment portfolio. Such a condition precludes, as the authors acknowledge, immunization]. They present two workarounds: partial horizon immunization—a half-a-loaf-is-better-than-no-loaf approach—and a fallback to the actuarial practice of discounting Defined Benefit Plan liabilities with the AA-rated corporate bond yield curve. But DB plans are not limited by the life-span of any individual and are exposed to population risk factors as opposed to idiosyncratic single-life risk factors.
2011	"When to Commence Income Annuities," Jeffrey K. Dellinger	This SOA monograph addresses the situation of a retired investor seeking to maximize income while simultaneously minimizing the probability of outliving the income. Dellinger sets up a comparison between an income	Provides a good example of how the Annuity Mortality Credit increases periodic payouts. The base comparison is between an investment

	The Society of Actuaries (2011)	withdrawal program financed from an investment portfolio and a variable annuity which has the same underlying investment configuration. Assuming no fees, expenses or loads on the annuity, it is demonstrable that it will produce—at any level of return—a higher income stream due to the annuity's mortality credits. The annuity's mathematical advantage, assuming that the money is invested in identical underlying investment portfolios, is due to the incorporation of the probability of survival throughout the planning horizon: "Income annuities are really a long series of pure endowments; that is, each future payment is discounted for both interest and survivorship and the sum of all those present values equals the net premium." Furthermore, if income is needed currently, a decision to delay annuitization means that the extra income available to annuitants because of the mortality pooling will be sacrificed. Again, assuming identical underlying portfolios and assuming equal withdrawal amounts from the annuity and non-annuity portfolios, there will be less wealth to purchase an annuity at a more advanced age than is currently available. Each year that an annuity purchase is postponed, the investor deprives themselves of the 'extra' income that could have been generated through mortality credits. Given that annuities are priced by discounting all future years for the probability of survival, any decision to delay means that the investor will receive only the more substantial price discounting for later years and will forfeit the discounting during the years of delay. All else equal annuities make any level of income stream less expensive. During the time of delay, Dellinger assumes that the investment account is depleted at the same level of income that an annuity would have provided. "…delaying income annuity commencement under the notion that delay is beneficial because a single premium will translate into her periodic income benefits at a more advanced age can be suboptimal if one requires additional income now. In essence, spending down non-life-contingent assets in the interim—before the delayed income annuity	portfolio and a no-fee variable annuity holding the exact same financial asset portfolio. The return from the annuity will always exceed the return from the investment portfolio given the annuity risk-sharing principle—i.e., mortality credits. Also provides interesting data on the relationship between the level of annuity payouts and the ratio of actual mortality experience to expected mortality experience. Annuitants receiving high benefit levels exhibit significantly lower ratio values than annuitants receiving low benefit levels—i.e., there appears to be a positive correlation between income and longevity. See, also "The Composition and Drawdown of Wealth In Retirement," James Poterba, Steven Venti and David Wise [2-11] who make a similar observation.

		commences—can significantly reduce retirement income." Finally, if an investor does not need income than it makes sense to postpone annuitization of wealth. Dellinger extends his analysis to include product loads and fees. If a mutual fund and a variable income annuity [VIA] have comparable front end loads and annual costs, a decision to delay annuitization becomes worse. An investor electing to invest now and annuitize later pays two sets of front-end loads rather than one. "If the underlying fund were identical for both the VIA and the withdrawal program alternative, then the gross investment return I would be the same. The decision to delay annuitization thus hinges on whether the benefit of survivorship (present in the VIA but absent in the withdrawal program) has the stronger positive effect or whether lower fees in the withdrawal program product, giving it a higher net investment return, has the stronger positive effect." "The one-year benefit of survivorship, $1/p_x$, is simply illustrated. Suppose 10 individuals aged x each contribute an identical sum of money at the beginning of the year to a common investment pool. At year-end, the pool with be shared equally among survivors. Suppose q_x = 0.1, and suppose this mortality assumption bears out in real-world-experience, with one individual dying during the year and nine surviving the year. At year-end, even with no investment gain, each individual's share of the pool has risen by 11.1 percent, because the decedent's stake….is distributed equally among survivors (e.g., those annuitants still alive and still requiring income). Here $1/p_x$ = 1/.09 = 1.111….Suppose the investment pool returns 6 percent. The collective $1,000 grows to $1,060 and is shared equally at year-end by the 9 survivors, each receiving $1,060 ÷ 9 = $117.78." Dellinger makes several observations concerning annuity pricing: • "…fixed immediate annuities contain a 'spread' between the earned rate on underlying assets (e.g. bonds and other fixed-	

		income securities in the insurer's general account portfolio segment backing fixed immediate annuity obligations) and the credited rate on the liability (e.g. fixed immediate annuity reserve). This spread, in essence parallels its expense charge counterpart on registered products." • "When pricing fixed immediate annuities in actual practice, a vector of interest rates is used. In a traditional (i.e., positive sloping) yield curve environment, the interest rate used to discount any annuity benefit payment is time-dependent, with progressively higher interest rates associated with annuity payments due further into the future." • The load, when it exists, is typically a 'percentage of premium' charge used to cover acquisition expenses such as wholesaler compensation, financial advisor sales compensation, policy issuance, record set-up and other policy acquisition expenses as well as state premium tax, if applicable" • "Mortality improvement—recognized in immediate annuity pricing by projecting progressively lower mortality rates (i.e., q_x values) for the same age for each year into the future—means that p_x values are increasing over time for any given static age x. Hence, $1/p_x$ benefit-of-survivorship values are anticipated to decrease over time, raising the optimal income annuity commencement age….mortality is modeled as the product of the mortality rate for a base year and a reduction factor, which follows an exponential decay. Of course, the mortality improvement factor for a given age and sex combination needn't be one uniform value in all years. Rather a vector of mortality improvement factors could be employed…." • "Front-end sales commissions on income annuities are often in the 3- to 4-percent range. There are income annuities for which no sales commissions exist." • Immediate annuitants electing a "life only" payout option tend to exhibit lower mortality rates than those electing a refund feature. These distinctions can meaningfully affect mortality	

		rates. For example, in annuitant mortality experience study, nontax-qualified, non-refund immediate annuities with annual income amounts of $7,500 and over exhibited an actual to expected (A/E) mortality ratio of 0.595, whereas those with annual income amounts of less than $2,500 exhibited a 1.446 A/E mortality ratio."	
2011	"Optimal Portfolio Choice over the Life-Cycle with Flexible Work, Endogenous Retirement, and Lifetime Payouts," Jingjing Chai, Wolfram Horneff, Raimond Maurer, and Olivia S. Mitchell copyright by authors in 2011. Published in Review of Finance vol. 18 no. 1 (2014), pp. 147 – 188.	The study extends develops a life-cycle model that incorporates flexibility in the number of hours worked, the age of retirement, and the age at which the retiree claims social security benefits. The capital market includes stocks, bonds, and annuities. When workers have flexibility in these variables—i.e., they are endogenous—the model suggests optimal solutions that differ from those suggested by other life cycle models but which, nevertheless, exhibit patterns corresponding to empirical evidence regarding U.S. retirees. The consumer's life cycle objective is to maximize the value of a utility function incorporating both consumption and leisure. The life-cycle model assumes CRRA utility and can reflect various preferences for consumption, leisure, and bequests. The investment process is modeled using real rates of returns on stocks and bonds under the assumption of lognormality of the return distribution. Annuities can be either fixed or variable (variable = "investment linked"). Wealth evolves according to labor innovations (temporary and permanent shocks to income), annuity payout income, tax code decrements, investment evolutions, and Social Security entitlements. Consumption is subtracted from wealth in each period. Depending on the nature of labor income shocks (positive or negative) and returns on retirement savings investments, individuals will maximize utility by deciding on the number of hours worked, the age of retirement, and the age of Social Security benefit claiming. Both the model and empirical literature reflect the fact that there is a large consumption drop off at retirement age: "…older households in retirement are more willing to substitute purchased goods for leisure time once this is feasible….After	The authors point out that setting the AIR in a variable annuity fully allocated to treasury bonds to the risk-free rate of interest changes the variable annuity into a fixed annuity. The significant drop in post-retirement consumption distinguishes this life-cycle model from others. It has important consequences for setting optimal post-retirement distribution policy: "All retirement groups suffer a negative change in consumption, which is much higher for households electing to retire late than for those who retire early (-35% for retirees at age 66 versus -5% for retirees at age 62)….The expected change is -19% and the median decline is -20.3% with a high standard deviation of 18.9%....individuals in the lowest wealth quartiles show a much greater consumption drop of 35% compared to the 7.5% drop for those in the highest wealth quartile." It is interesting to note that Mitchell is a co-author. This paper suggests that fixed annuities do not generate substantial increases in investor utility. Mitchell's earlier work on Annuity Equivalent Wealth [AEW] suggests the opposite.

		the age of 70, when individuals enjoy full leisure, their effective rates of time preference become higher (because of increasing mortality)." This sharp decline in consumption is not found in most other studies using life-cycle models lacking labor income, retirement and SS benefit flexibility. "This discrepancy between standard life-cycle model predictions and empirically observed drops in consumption at retirement has been referred to as the 'retirement consumption puzzle.' By contrast in our model, agents endogenously reduce their expenditure levels around the time of retirement at a fast pace, suggesting that the so-called 'consumption puzzle' described by many analysts may, in fact, not be a puzzle at all." Under the authors' model, investors also generally allocate a substantial amount of wealth to stocks: "…until the mid-30s, the individual invests 100% in stocks. This is because future labor income can be thought of as a high implicit bond position, so investors will seek to diversify their overall portfolios consisting of both human and financial wealth….Around the age of 57, the fraction invested in stocks falls to its lowest level, approximately 65%. Thereafter, the individual is exposed to declining labor income risk as the number of work years remaining falls and the individual becomes eligible to claim Social Security benefits. The fraction invested in stocks then increases again, until age 65." They continue to report the findings of their model: "At age 66, the allocation to bonds is close to zero, whereas the fraction of total wealth invested in annuities is 25%. Thereafter, annuities also start to crowd out stocks to take advantage of the further increasing survival credit. Around age 80, almost the entire portfolio of financial wealth is invested in fixed annuities….Yet the resulting payments from these annuity holdings are not large, amounting to only about 7% of total income at age 80." Finally, the authors observe that the utility loss from failing to following optimal asset allocation is relatively low relative to a fixed 60% stock / 40% bond allocation: "…this could be a reason why many households do not devote much attention to managing their investment portfolios,	

		although they do optimize their work hours and claiming date." Having fixed annuities is only marginally advantageous "…the additional utility gains of 0.1% are very low in a world with fixed annuities, as compared one [sic] without annuities. Investment-linked annuities, however, can raise lifetime utility by a more substantial amount.	
2011	"Creating Portfolios That Confront Retirement's Risks," James X. Xiong and Thomas Idzorek, Morningstar Advisor (December/January 2011), pp. 40 – 43.	A short essay in which the authors define the retirement income challenge as follows: "Some retirees unnecessarily adopt frugal lifestyles out of fear of facing financial ruin, while others spend too much too quickly, depleting their savings to unsustainable levels." When building a retirement income portfolio, there are five factors that "…primarily drive the product-type allocation decision": • Age • Risk tolerance • Wealth vs. Retirement expenses • Subjective life expectancy • Bequest goal. After running simulations to determine optimal asset allocations, results indicate that allocation to immediate fixed annuities increases slightly with age, allocation to VA+GMWB contracts decreases with age—the contract acts as a put option or as portfolio insurance. The longer the planning horizon, the more valuable the option, all else equal. Investors are more likely to allocate funds to fixed immediate annuities as risk aversion increases. The ratio of wealth to retirement expenses has a large impact in the asset allocation decision. "If we take the person's wealth (total net worth) and divide it by his or her annual funding gap (annual income need minus annual amount of guaranteed income), we get the wealth-gap ratio. The lower the ratio, the more the investor needs other sources of guaranteed income to cover his or her income needs for life. (Or the person needs to lower those income needs to realistic levels)."	Introduces the concept of the wealth-gap ratio. The ratio suggests the desirability of securing annuity income for investors with a low ratio value. Investors with higher ratio values will tend to avoid allocating funds to annuity contracts because their portfolios are unlikely to suffer catastrophic declines in market downturns.

		A higher subjective life expectancy will motivate the investor to purchase "longevity insurance." Investors lacking a bequest motive will increase allocation to annuities: clients who want to spend their retirement income in their lifetime (the 'spend-it-now' approach) should have a sizable allocation to longevity insurance, with the allocation tilted towards immediate fixed annuities."	
2011	"What Does Retirement Really cost," Moshe Milevsky Research Magazine (September 2011) available at http://www.thinkadvisor.com/2011/09/01/what-does-retirement-really-cost	A short essay stressing that the cost of funding adequate retirement income is not magically reduced simply by loading a portfolio with higher expected return assets: "Enter the retirement planning software used by confused—or unscrupulous—financial advisors and they seem to offer a better and more soothing answer. If you invest more aggressively then you don't have to use the small, pathetic and depressing 1.5% real return…." However, this is a mirage—"You can't tweak expected return (a.k.a. asset allocations) assumptions until you get the numbers that you like." Stock returns are uncertain and "pricing" the cost of retirement based on expected stock returns is the equivalent of making a bet. "…assuming a more aggressive rate of return…and then claiming that retirement has suddenly become 'cheaper' is a dangerous fallacy that will end up costing many retirees quite dearly. " The best price indicator of the cost of retirement in the marketplace is the cost of a lifetime annuity from an insurance company. "…the annuity price is actually a market signal of what retirement really costs."	Argues that an annuity is the true measure of the cost of providing retirement income.
2011	"Sustainable Retirement Income for the Socialite, the Gardener and the Uninsured," Chris Robinson and Nabil Tahani Financial Services Review vol. 19 no. 3 (Fall 2010),	The paper notes: "retirees can and often do adjust their spending to some extent to respond to changes in their endowment due to higher or lower than expected investment returns." Spending adjustments can reduce the risk of shortfall. Additionally, rather than a fixed consumption policy, spending often follows a declining pattern over time. The sustainability of the portfolio depends on both the initial endowment and the pattern of spending that unfolds over the retiree's lifespan. Most research on portfolio sustainability either presents a series of ad hoc rules for spending change or assumes a constant amount of real	The authors opine about prudent risk: "What do we consider to be an acceptable risk of shortfall? That is a decision for every retiree or planner to think about, but our choice is 10%. We think that many people would choose 5%; but we know of no formal evidence on this question." Monitoring and surveillance are prerequisites to prudent asset management. What risks are your

	pp. 187 – 202.	consumption. This paper considers consumption as a stochastic variable with a drift component of –α and a volatility of β (a geometric Brownian motion process). When α is a positive number, the –α drift represents an exponential decline in consumption. Further, consumption may be correlated to the portfolio's real return. When the stochastic present value [SPV] of consumption is greater than portfolio value [wealth], there is a positive probability of ruin. Given that lifespan can also be represented as an exponential random variable, the mathematical expression for the probability of ruin is: $P(SPV > w) \approx \text{GammaDist}\left(\overline{\alpha, \overline{\beta, \frac{1}{w}}}\right)$ Using this analytical expression to calculate ruin probabilities for a portfolio earning a real 4% per year with a standard deviation of 14%, the study finds that • "The most significant effect on probability of shortfall of the innovations in this paper is the different patterns of consumption." • "A person who retires in the normal age range of 60 – 65 cannot generally expect to sustain an initial consumption rate in retirement greater than 4% of initial wealth." • "A person who plans to continue a constant rate of consumption, 'the socialite,' cannot sustain a rate of more than 3%" • "In the most wildly optimistic case, a 'gardener' whose initial consumption declines a significant rate during retirement, who invests very aggressively and does very well, and who also adjusts consumption partially as real investment return varies, may be able to sustain an initial rate of consumption of 6% of initial wealth starting at age 65." • "Making consumption stochastic has a relatively small impact on the probability of shortfall."	monitoring— shortfall probability, spending sustainability, terminal wealth, etc.? The choice of risk metrics is important because it suggests which asset management elections will be useful.
2011	"The Composition and Drawdown of	This study draws on data from the Health and Retirement Study. It seeks to determine how much additional annuity income would be available to	Discusses the amount, composition and trajectory of wealth at and during retirement.

	Wealth In Retirement," James Poterba, Steven Venti and David Wise Journal of Economic Perspectives vol. 25 no. 4 (Fall 2011), pp. 95 – 118.	retirees if they elected to annuitize financial assets held both personally and within retirement accounts. The study finds that retirees hold financial assets and home equity as precautionary savings in the event of the death of a spouse or a substantial medical bill. The data show that "…half of the households between the ages of 65 and 69 in 2008 have net financial assets of less than $15,000; roughly one-third have almost no financial assets. Seventy percent have less than $70,000 in net financial assets. The same pattern emerges for assets in personal retirement accounts….the low median value in these accounts--$5,000—is not a surprise." The study makes some interesting observations including: • "Those who hold housing wealth until very late in life may be less concerned than others, without such wealth, about the need to insure against longevity risk." • "…households in the top quintile of the wealth distribution report rising net worth until about age 85, and those in the middle three quintiles report relatively stable net worth." "There is relatively little evidence that households in the upper half of the wealth distribution spend down financial assets in the early decades of retirement." • "Age-specific mortality rates are negatively correlated with socioeconomic status, which means that as we track a given age cohort over time, the survivors will be disproportionately those who had higher wealth levels at earlier ages."	
2011	"Stock Market Returns and Annuitization: a Case of Myopic Extrapolation," Alessandro Previtero IFID Conference, Fields Institute November 24, 2011.	The author investigates the likelihood that DB Plan participants will select an annuity rather than a lump-sum benefit given the performance of the US stock market. Additionally, the author examines LIMRA (Life Insurance Market and Research Association) data regarding the demand for purchase of annuities as a function of stock market performance. After controlling for a host of variables including interest rates, education, age, gender, and so forth, the study finds a strong and statistically significant negative relationship between annuitization and stock returns: "More precisely, the correlation between the two time	Citing Milevsky and Young's thesis that the ability to defer the option to annuitize has a measurable value, Previtero calculates that annuitizing after the market drop of 2009 reduced retirement welfare by as much as 10%. A main driver of a retiree's decision to annuitize a pension benefit in lieu of electing a lump sum is stock returns during the six month period prior to the time of election.

	Available at: http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1787123	series is equal to -.509 in panel A [S&P 500 and DB Plan sample] and -.748 in panel B [S&P 500 and quarterly fixed annuity sales]. " For individual annuity sales, "Over the period 1985 to 2009, after controlling for interest rates and business cycles, I document that an increase of one standard deviation in the average stock market return decreases the sales of total fixed annuities by more than 25 percent." The highest weight assigned to stock market returns by retiring employees is assigned to returns occurring just one month before their annuity/lump sum decision: "…for stock market returns six months before their decision date, employees assign a weight about two-thirds of the weight they give to returns one month prior to the decision. The weights are practically zero after about two years."	
2011	"Determining Optimal Withdrawal Rates: An Economic Approach" Duncan Williams and Michael Finke The Retirement Management Journal vol. 1 no. 2 (Fall 2011), pp. 35 – 46.	The authors assert that the most common approach to determining an optimal retirement income withdrawal strategy is to identify allocations that minimize shortfall risk: "The typical approach is to treat the portfolio as a synthetic annuity that can consistently spin off inflation-adjusted income without running out of money." This safety-first perspective requires that consumption—especially in early retirement—occur at a depressed level. A "utility maximizing model," by contrast balances "shortfall minimization with the satisfaction that a client would receive from an increased level of consumption." The model tests Arrow-Pratt risk aversion coefficients of 1,2,5, and 10 with "non-portfolio" income levels of $20 and $65 thousand. "…the appropriate portfolio allocation in retirement is the one that minimizes consumption variance given the chosen withdrawal rate." A high withdrawal rate increases the probability that the portfolio will run out of money with the result that the client must live only on "non-portfolio income." The authors contend: "…the more that is consumed from portfolio withdrawals in retirement, the higher the variance due to increased shortfall risk. A more holistic approach to distribution planning would attempt to design a distribution strategy that optimizes consumption given the strategy's shortfall risk and client's aversion to variance…."	The authors use a bootstrap model with constant 3.2% inflation. The client retires with $1 million at age 65. They use a two asset portfolio [S&P 500 and long-term Treasury index] with returns taken from the period 1926 through 2010. Mortality is based on the general population table provided by the Social Security Administration. They test withdrawal rates of 3% to 12%. Utility maximization—"In this context, it means that the advisor is using specialized knowledge to help the client achieve the most happiness possible from available resources." The risk metric of interest in this essay is consumption variance over the planning horizon. The best combination of allocations and withdrawal rates is the combination that maximizes utility given the investor's risk aversion to budgetary uncertainty. Some investors may apply a high subjective discount rate to the stream of periodic consumption. However, a high initial withdrawal rate may result in a significant drop in consumption

		Using the Arrow Pratt concept of relative risk aversion, the utility of consumption is expressed as: $$U(C) = \frac{c^{(1-\gamma)}}{(1-\gamma)}$$ The model incorporates two states: (1) the good state in which the client has not outlived the portfolio; and (2) the bad state in which the client is forced to live from only non-portfolio income. The probability that the client continues to consume from the portfolio [P] combined with the fall off in income in the bad state [1-P] x [$C_{good} - C_{bad}$] gives the certainty equivalent for a client at the given level of risk aversion: $$CE = \left\{(1-\gamma)\left[P * \frac{C_{good}^{(1-\gamma)}}{(1-\gamma)} + (1-P) * \frac{C_{bad}^{(1-\gamma)}}{(1-\gamma)}\right]\right\}^{(1/1-\gamma)}$$ Investors with higher risk aversion coefficients are more sensitive to consumption variance caused by shortfalls. The greater the non-portfolio income stream relative to the dollar amount of target income, the lower the variance term's value. They point out that longevity risk may result in a time-preference discounting rate that differs from the general risk-free rate: "…a utility maximizer will discount future consumption based on the probability of being alive for each year in the future. If the discount rate is 4% per year, the expected utility from consumption at age 82 will be only approximately half the utility from consumption at age 65. In order to maximize expected lifetime utility then, a retiree would consume more in the early years of retirement and less in the later years when the probability of being alive is lower." Furthermore, "A person might be incented to defer some consumption until a later time if the expected return were high enough. Both the discount and expected return rates are inversely related to risk aversion. A person with a high RRA will have a relatively low expected rate of return due to conservative portfolio choice, but will also not discount future consumption much because he is	at a later time. The optimal choice depends primarily on the risk aversion function—not on the probability of portfolio depletion. [Note: analysis assumes supplemental income from outside of the portfolio].

		not willing to accept much variability in consumption. If these rates are equal, then holding real consumption constant is utility maximizing. We maintain this assumption of rate equality as a baseline in our model….” The article presents tables of results for various asset allocations, withdrawal rates, and coefficients of risk aversion. Once a retirement distribution policy is selected and implemented, it remains static throughout the economic lifetime of the portfolio. For example, for a moderate risk investor with $20,000 of non-portfolio lifetime inflation-adjusted income, utility is maximized with a 40% stock / 60% bond allocation with a 5% withdrawal rate. Not surprisingly, allocations and withdrawal rates tend to increase for utility maximizing retirees with $65,000 of real lifetime income. At different withdrawal rates and risk aversion levels the authors calculate the certainty equivalent values. The extra dollars [consumption risk premium] required to increase the potential variance of future consumption also indicate the degree of welfare improvement for retirees with differing risk aversion levels. “…for each withdrawal rate chosen, the optimal portfolio allocation is the one that is expected to minimize the percentage of bad years…, or variance of consumption for that withdrawal rate.” “These findings are in contrast with the studies on shortfall risk that prescribe the same aggressive portfolio and low withdrawal rate for everyone, regardless of the client’s level of risk aversion and other resources.”	
2011	“Estimating internal rates of return on income annuities,” Nathan Zahm and John Ameriks Vanguard Research Paper (November 2011).	The authors begin by reminding readers that annuities are insurance contracts that will only have a high return in the event that the annuitant is long lived. They define IRR as “the rate the annuity payments are discounted to equate them to the annuity purchase price.” The data is from a sample generated in 2011 through Vanguard Income Solutions—the Vanguard annuity shopping division. They employ the RP-2000 gender-distinct annuitant mortality table projected generationally with Scale AA. The article considers three types of immediate annuities: Male only, Female only, and Joint and 100% Survivor. If purchased on October 27, 2011 the initial monthly payout for an inflation-adjusted benefit purchased with a $100,000 single premium at age 65 is $4,964 / $4,481	Contrast the analysis presented in this article to “The False Promises of Annuities and Annuity Calculators,” David Marotta, Forbes August 8, 2012

and $3,849 respectively. Similarly, a nominal annuity benefit is $6,771, $6,320 and $5,660 respectively. The authors note that the CPI-adjusted annuity payout is in the range of 3% to 4% typically recommended as a safe withdrawal rate by many commentators.

It is difficult for consumers to know the cost of an annuity contract: "Purchasers simply see an all-in annuity quote as a single 'net yield' offering and must assess the attractiveness of the annuity arrangement on that all-in basis…." An annuity load consists of four elements: (1) conservative pricing reflecting anti-selection risk to the insurer; (2) cost of maintaining a reserve against the risk that the annuitant population may realize greater-than-expected mortality improvements; (3) administrative costs; and, (4) profit. "Costs arising from adverse selection in the insurance market and from administering the annuities are substantial." The IRR evaluation metric can help consumers determine if annuitization is an attractive retirement income strategy.

Given current annuity pricing, the article calculates IRR for both nominal and inflation-adjusted contracts issued at various ages. The IRRs are based on the median life expectancy and are based on either 10 or 20 year US Treasuries or, for CPI-linked payouts, on 10 or 20 year TIPS. The 10 year securities are used for the 75 and 85 year old purchaser; the 20 year security for the 65 and 70 year-old purchaser.

The *nominal* IRRs at median life expectancy are:

Age	Treasury Rate	Male IRR	Female IRR	Joint IRR
65	3.02	3.27	3.27	3.18
70	3.02	2.37	2.69	2.74
75	2.28	0.52	1.81	2.04
80	2.28	-4.03	-0.55	0.45

The *inflation-adjusted* IRRs is are:

Age	Treasury Rate	Male IRR	Female IRR	Joint IRR
65	0.74	-0.05	-0.08	-0.07
70	0.74	-0.95	-0.63	-0.38
75	0.19	-2.66	-1.36	-0.88

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<tr><td>80</td><td>0.19</td><td>-7.75</td><td>-3.95</td><td>-2.54</td></tr>
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The authors compare results at the median of the longevity distribution to results for the top quartile. These are the individuals who end up living longer than 75% of the annuitant population and who will realize a better insurance payoff.
The table of nominal IRRs for this sub-group is as follows:
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<tr><th>Age</th><th>Treasury Rate</th><th>Male IRR</th><th>Female IRR</th><th>Joint IRR</th></tr>
<tr><td>65</td><td>3.02</td><td>4.88</td><td>4.64</td><td>3.99</td></tr>
<tr><td>70</td><td>3.02</td><td>5.07</td><td>4.86</td><td>3.99</td></tr>
<tr><td>75</td><td>2.28</td><td>5.25</td><td>5.34</td><td>4.08</td></tr>
<tr><td>80</td><td>2.28</td><td>4.69</td><td>5.59</td><td>3.95</td></tr>
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The table of inflation-adjusted IRRS for this subgroup is as follows:
<table>
<tr><th>Age</th><th>Treasury Rate</th><th>Male IRR</th><th>Female IRR</th><th>Joint IRR</th></tr>
<tr><td>65</td><td>0.74</td><td>1.96</td><td>1.68</td><td>0.99</td></tr>
<tr><td>70</td><td>0.74</td><td>2.22</td><td>1.98</td><td>1.17</td></tr>
<tr><td>75</td><td>0.19</td><td>2.59</td><td>2.65</td><td>1.48</td></tr>
<tr><td>80</td><td>0.19</td><td>1.62</td><td>2.77</td><td>1.34</td></tr>
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The authors conclude: “The internal rates of return for this group are all above current yields available on investments with comparable investment risk over similar horizons.”
Finally, the authors illustrate how a change in interest rates can dramatically impact the annuity payout amounts offered to contract buyers. For example, for a Male Age 65, the annuity payout amount decreased by 8.85% on the nominal benefit contract and 6.07% on the inflation-adjusted payout contract between April 20, 2011 and October 27, 2011.
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<td>2011</td>
<td>Research and Reality – A Literature Review on Drawing Down</td>
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The paper addresses three questions:

How DO retirees draw down their financial savings?
How COULD retirees draw down their financial savings?

</td>
<td>As a counterpoint to the Fisher hypothesis regarding retiree utility, the monograph asserts: “…if significant numbers of retirees draw down their</td>
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	Retirement Financial Savings. Sponsored by Society of Actuaries Pension Section Bonnie-Jeanne MacDonald, Bruce Jones, Richard Morrison, Robert Brown, Mary Hardy. Society of Actuaries (2011) available at: https://www.soa.org/research/research-projects/pension/research-literature-review.aspx	3. How SHOULD retirees draw down their financial savings? This SOA monograph states: "Despite the relatively recent surge in the topic of drawing down retirement savings in academic circles, it has not penetrated the retirement planning process….clear, unambiguous and disinterested guidance on how best to drawdown individual retirement accounts and manage the large associated risks has not been widely disseminated. In large part, individuals have been left to decipher conflicting and potentially self-serving advice from financial advisors, or to follow social norms that may or may not fit their personal circumstances and objectives." The authors note: "Researchers have generally approached the choice among drawdown strategies by determining the optimal strategy through maximizing an objective function (such as the expected discounted utility), or applying risk measures to ascertain the tradeoff between the consumption, security, and bequests generated among alternative strategies. As we explain throughout this paper, however, researchers are increasingly recognizing that individual preferences are not easily captured in simple models…." Such empirical evidence indicates that retirees are very slow to draw down retirement savings. "For instance, examining changes from 1998 to 2006 in asset holdings of persons at least 60 years old from the Health and Retirement Survey, Smith et al (2009) found that individuals were very slow in spending down their retirement savings. In fact, those in the top income quintile actually accumulated wealth!" On the issue of how retirees COULD manage the decumulation of financial assets, the authors group strategies into three categories: 1. The purchase of an annuity 2. Discretionary management of retirement wealth, where the retiree controls the level and frequency of withdrawals 3. Hybrid strategies combining annuities with self-management of wealth. The monograph reviews the concept of an annuity's 'mortality premium:' "…if wealth W is invested in a bond with a rate of return of R,	wealth during the earlier years of retirement when they are better able to enjoy it, and are consequently unable to sustain themselves during the later years of retirement, the hardship will not be limited to the individual since society will end up sharing the risks that the retirees were unable to manage." "The meaning of 'additive separability' is that the utility of one period is not affected by a change in the utility of another period. Davidoff et al. (2005) relaxed this feature of the standard utility function by assuming that individuals may exhibit an 'internal habit' – meaning 'it is not the level of present consumption, but the level relative to past consumption that matters for utility'." "…the main issue is that financial planners do not base their advice on sound economic theory, but give simplified advice to help speed clients through the planning process." Rationale for monitoring policy: "Consumption smoothing requires that individuals make the necessary adjustments so that wealth does not run out. Financial advisors should educate their clients as to the types of spending adjustments that should be made, both in terms of level and timing." "Sound retirement planning includes sound and pre-planned strategies of when and how to adjust spending to avoid a severe reduction in later-life lifestyle, rather than relying solely on intuition." [Compare to Gordon Pye <u>The Retrenchment Rule</u> published in 2012]

		then it will grow to W(1+R) after one period. If the same wealth were instead used to purchase an actuarially fair life annuity, it would grow to W(1+R)/(1-q) if the consumer survives (where q is the probability of death in that one year period) (Brown et al., 2008). Consequently, a surviving annuitant's total annual return equals the rate of interest at the time of purchase plus a mortality premium (less any transactions costs)." Given that the capturing of the mortality premium would enhance the welfare of a retiree, the monograph reviews various hypotheses regarding the reasons for the annuity puzzle: • Loss in liquidity • Loss of bequest: "When an annuity is purchased, the consumer is essentially trading the bequest potential of his/her financial wealth for a mortality premium and longevity insurance." • Benefit to Delay • Low risk aversion" "Risk aversion is the degree to which an individual is not willing to take on financial risk (volatile income stream in this case) in return for a potentially greater return." • High personal discount rate (or personal rate of time preference): "Gustman and Steinmeier (2005) found that 60% of their sample has a time preference rate of over 5%." • Short life expectancy • Ability to pool risk with families • Confidence in personal financial abilities • Other sources of guaranteed income • Illiquid wealth • Expensive pricing: "Annuities are overpriced from an actuarial perspective in that the actuarial present value of the premiums is greater than the actuarial present value of the benefits (Mitchell et al., 1999; Orszag, 2000). This is owing to the insurer's administrative costs that are built into the premiums to cover marketing costs, corporate overhead, income taxes, regulatory compliance, contingency reserves, and profits, as well as the	

		expensive mortality assumptions arising from adverse selection." "Mitchell et al. (1999) calculated the additional expense from an annuity's transaction costs to be 18 – 25% of the value of the benefit for an individual chosen at random from the population, and over 10% was owing to the effect of adverse selection. Calculations done by Babbel and Merrill (2006) suggested that the rates reported by Mitchell et al. (1999) are now lower." • Poor financial market environment: "Retirees could be dissuaded from annuitizing because of current poor financial conditions (low underlying interest rates or a drop in the value of their accumulated wealth)….an American who reached retirement age 65 in 2007, and annuitized his portfolio of 40% bond and 60% equities, would have enjoyed a replacement ratio equal to 24% while the identical, but less fortunate, American reaching age 65 and the end of 2008 would have had a replacement ratio of only 15%." • Incomplete annuity market: "…a mismatch between their desired consumption path and the payment stream of available annuities…." • Access (Many retirement plans do not offer an annuity option) • Distrust of annuity providers • Sex-distinct mortality assumptions • Tax treatment. • Decision framing: "instead of viewing the question of annuitization through a 'consumption frame (focusing on the end result of what can be spent over time), many consumers adopt an investment frame (focusing on the intermediate results of return and risk features when choosing assets, and not considering the consequences for consumption')" "…Webb (2009) suggested that households are disinclined to annuitize because they 'misunderstand the nature of risk' in that they are overly concerned with losses and gains in the short-term and neglect the more important goal of being able to sustain a smooth consumption over the long-term."	

		• Longevity gamble: "People think of annuities as a gamble, where the odds favor the insurance company…" • Mental accounting: "…will I live long enough to make back by initial investment." • Perception of insurance: "…insurance is only for 'bad' events…." • Absence of comprehensive plans • Control • Buyer's remorse • Regret aversion • Misinformation • General financial illiteracy • Individuality • Default options • Discouraging level of income • Historical view on personal retirement savings • Other: e.g., procrastination. Self-managed wealth strategies suffer from "mortality drag" because they forfeit the mortality premium available through annuities. Self-managed withdrawal strategies fall into two general categories: (1) fixed withdrawals, or (2) variable withdrawals. Following a literature review for both strategies, the authors assert: A weakness in past literature is the complete absence of drawdown strategies that respond to personal circumstances and associated financial needs that vary period by period, such as replacing a car, moving, refitting a home because of a disability, or an impending death when health status changes." Of particular interest is the review of hybrid strategies—a mix of annuitization and self-management of wealth. These include: • Annuitizing some assets while managing the remainder; • Self-management early in retirement followed by annuitization at a pre-specified age or event—e.g., wealth dropping below or rising above a particular level;	

		• Purchasing a deferred annuity and self-managing the remainder of wealth during the interim; • Laddered annuitization. There has been considerable discussion in the literature on the topic of the optimal time for annuitization. It is difficult to model effective drawdown strategies because of five variables: 1. Future investment performance 2. Future inflation 3. Future unplanned expenses (including changes to circumstances and preferences) 4. Longevity 5. The evolution of government tax and benefit programs. In general, drawdown strategies are evaluated under three different types of models: Dynamic programming that employs the maximization of utility functions to solve for the optimal withdrawal path, investment strategy, time to annuitize and/or amount to annuitize. Among the observations that they make are: “Simulation studies that employ a utility function to represent an individual’s ‘consumption’ preference is the most common approach to quantitatively evaluate the individual welfare generated from different drawdown strategies (Dus et al., 2004). To maximize lifetime utility, the most common approach has been to use dynamic programming….The standard utility function generally used features constant relative risk aversion, exponential discounting at a fixed rate, and additive separability (Davidoff et al., 2005). Some studies use a utility framework to determine the optimal strategy, but they do not employ dynamic programming, but rather use approaches such as stochastically simulating the expected utility for each alternative strategy, or by direct solution.” The authors remark: “…most financial advisors would have difficulty understanding these models enough to rely on and explain to their clients.” Further, “Given the wide range of individuals and their	

		objectives, the use of any single definition of utility is problematic." <u>Minimizing the probability of lifetime ruin</u>. There has been some criticism of this metric because "…it does not follow the standard economic theory of consumer utility maximization." Some commentators note: "…looking only at the probability of ruin masks the additional value of strategies that deliver higher consumption than others (Sun et al., 2006). Results that ignore tradeoffs between the positive factors associated with increased consumption, and the negative factors associated with running out of wealth, could lead to strange conclusions – such as suggesting that households with a high consumption rate should make risky investments to minimize their probability of ruin, while the more rational advice would be reduce[sic] consumption." <u>Risk-return models</u>. The measures of return include: (1) the expected discounted value of withdrawals, and (2) the expected amount of terminal wealth. Risk measures include: • The probability of consumption shortfall—probability of falling below a specified target • Mean excess loss: E[benchmark – withdrawal |withdrawal < benchmark)] • Shortfall expectation • Expected present value of the shortfall: the sum of all future shortfall expectations, each discounted by interest and the probability of survival to time t • The failure rate—probability of running out of money before the end of a specified period • The standard deviation of income • Percentile values including Value at Risk measure • Distribution of return values • Risk/return values by attained age. Models compute risk and return values through deterministic projections, past investment experience from empirical data and Monte Carlo simulation projections. "In general, findings based on any of these three methodologies are very dependent on the assumptions of the	

		researcher, such as the mortality assumptions and the size of the assumed equity risk premium."	
2011	"Trustee Asset Management Elections: Portfolio Performance Evaluation and Preferencing Criteria," Part One The Banking Law Journal vol. 128 no. 2 (February 2011), pp. 136 – 179; and Part Two vol. 128 no. 3 (March 2011), 237 – 284. pp.	Part One of this two-part article examines several modern portfolio theory performance evaluation measures with close attention given to the Sharpe Ratio. It discusses the nature of the Ratio and provides examples of (1) how the ratio is a valid performance measure under only a limited set of conditions; and (2) how it can be abused either inadvertently by those unaware of its underlying assumptions, or by those seeking to present a distorted picture of the return to risk relationship in an investment program. As an alternative to using performance evaluation metrics, the trustee changed with prudent asset management can employ a utility-based portfolio preferencing metric. Part Two, which is most relevant to this bibliography, offers a comprehensive review of the topic of utility. It takes the reader through a derivation of the Arrow-Pratt CRRA risk aversion function commonly used in mathematical models underlying asset allocation recommendations. It then contrasts CRRA with State Preference Utility and illustrates how investors exhibiting strong subjective discount preferences may reject portfolios exhibiting attractive Sharpe Ratios in favor of portfolios delivering safe consumption opportunities during poor economic states. Part Two continues with discussion regarding the merits and liabilities of using stochastic dominance—first and second order— as a retirement income portfolio preferencing metric. The essay defines three fundamental investment tasks: 1. Determine the allowable amount of risk given initial portfolio wealth. This is the initial calibration between the asset allocation decision and personal risk tolerance. 2. Determine, at reasonable time intervals, the allowable amount of risk given the inevitable changes in portfolio value. This is the stay-the-course or make-a-change decision that best reflects each investor's elasticity of marginal utility per change in wealth. 3. Determine if the allowable amount of risk is sufficient to	Asset allocation, especially when determined by historical risk and return data, may sometimes reduce itself to a search for the portfolio with the highest Sharpe Ratio. Thus, modern portfolio theory based performance metrics become portfolio preferencing measures. But performance evaluation metrics look to the past; investors are more concerned with enhancing utility of wealth in the future. This article is part of a series published in The Banking Law Journal. The series begins with "Managing Private Wealth: Matching Investment Policy To Investor Risk Preferences" by Patrick Collins and Josh Stampfli, The Banking Law Journal vol. 126 no. 10 (November/December 2009), pp. 923 – 958 reviews the economic consequences of asset allocation decisions under three portfolio management strategies: (1) Buy and Hold, (2) Constant Mix, and (3) Constant Proportion Portfolio Insurance [Floor + Multiplier asset management]. It is a primer on the impact of differing risk aversion functions such as Constant Relative Risk Aversion and Hyperbolic Absolute Risk Aversion on the choice of portfolio management strategy and on the suitability of an asset allocation to the asset management preferences of the investor. Much of the literature on portfolio sustainability during retirement fails to incorporate a three-dimensional perspective: • Portfolio Management Strategy • Portfolio Asset Allocation

		generate, on a go-forward basis, the return required to achieve economic objectives given the portfolio's current level of dollar wealth. This is the decision that calibrates the investor's continued willingness and ability to assume risk with the return required for keeping the portfolio on track relative to the economic demands placed against it. It presents a case example of dynamic portfolio management in the face of unique preferences and constraints of a retired couple, faced with the competing investment objectives of consumption and bequest. The case example both quantifies and explores the implication of various risk metrics including probability of shortfall, expected shortfall, and the distribution of the length of shortfall over a variety of asset allocations and spending options. Management elections include adaptive spending in the face of emerging portfolio surplus and shortfall. The importance of an effective and rational portfolio monitoring system constitutes an important topic.	• Investor Risk Preferences and Constraints. The two-part "Trustee Asset Management Elections: Portfolio Performance Evaluation and Preferencing Criteria" article provides an integrated discussion in the context of an evolving retirement portfolio of a number of themes that, commonly, find treatment as separately considered topics in other studies.
2011	Dan diBartolomeo, "Asset/Liability Management for the Private Client," CFA Institute (March 2011), pp. 42 – 48. cfapubs.org	This CFA Institute presentation advocates a dynamic Asset/Liability Management [ALM] approach where liabilities are defined in terms of a private investor's consumption requirements as opposed to, for instance, a Pension Plan's benefit payout obligations. diBartolomeo asserts: 1. It is difficult to obtain a precise measure of future liabilities: "…it is not possible to estimate accurately the present value of future consumption liabilities. Investors can choose a discount rate today and assert that the present value of their liabilities is a specific number at this moment. Conditions change, however, and investors will have to change those discount rates along with the changing conditions." 2. A better approach is to estimate a distribution of consumer surplus throughout a series of future periods; and, to allocate the portfolio dynamically to maximize the median value of the surplus (as opposed to maximizing expected value). 3. Whereas the present value of future liabilities is a function of the	diBartolomeo's observations provide a strong rationale for ongoing portfolio monitoring and evaluation. When using an actuarial benchmark, both the benchmark and the portfolio must be constantly evaluated. Note: the incorporation of the Wilcox discretionary wealth hypothesis parallels, in some respects, the household balance sheet approach advocated by the Retirement Income Industry Association. The two approaches differ, however, in that Wilcox employs a more traditional optimization procedure for discretionary wealth—i.e., consumer surplus—management, while the RIIA methodology conforms more closely to a behavioral finance approach. In either case, a credible method for portfolio evaluation is a prerequisite for intelligent asset management in a dynamic context.

		discount rate, estimating a term structure of rates "…helps investors estimate what that [liability] probability distribution looks like not only today but also in future periods." Additionally, to the extent that a change in assets values is correlated with a change in interest rates, the investor can estimate a distribution of future changes in asset price: "Asset cash inflows, however, are priced in financial markets by a yield curve or term structure of interest rates that reflects investor preferences for maturities and expectations about future changes in interest rates." A continuous updating of yield curve forecasts stands in direct contrast to the traditional "actuarial practice" which applies a single static rate—i.e., assumes that the yield curve is flat. diBartolomeo employs a vector autoregressive prediction model developed by Campbell and Viceira ["Consumption and Portfolio Decisions when Expected Returns are Time Varying" Quarterly Journal of Economics (May 1999), pp. 433 – 495]. For a given future time, a binomial tree expressing the distribution of liability values across possible interest rates is subtracted from a binomial tree forecasting a calibrated and correlated distribution of asset values to arrive at the distribution of surplus: "By having two trees, I can find an expected value for the surplus that is the difference between the present value of assets and the present value of liabilities at every node in the tree that represents a possible future state." Having estimated the distribution of assets minus liabilities, diBartolomeo adds two additional dimensions to his model: 1. He employs traditional mean-variance optimization to arrive at efficient asset allocations: "By using Markowitz's mean-variance optimization but allowing state dependent risk aversion that varies both with time and with the relationship between assets and liabilities, I can change asset allocation dynamically over time." 2. He locates the investor's optimal portfolio on efficient frontier by employing Wilcox's "discretionary wealth hypothesis" method	See Jarrod Wilcox, "Harry Markowitz & the Discretionary Wealth Hypothesis," for an explanation of the difference between multiplicative compounding (mean terminal wealth) and additive—i.e., logarithmic—compounding (median wealth).

		[Jarrod W. Wilcox, "Harry Markowitz and the Discretionary Wealth Hypothesis," Journal of Portfolio Management (Spring 2003), pp. 58 -65]. The Wilcox method employs a personal balance sheet approach to calculate the ratio of assets to net worth [L]: "The more leveraged an investor is, the less able she is to withstand losses." The investor's preference curve in the Markowitz asset only risk/return space, is mapped to a comparable risk-aversion preference measure [λ] in consumer surplus space: $\lambda = \frac{1}{2}L$. As the investor's ratio of assets to net worth changes over time, the value of λ changes and the investor can modify asset allocation accordingly: "…investors can maximize the expected median of future surplus rather than the expected value. If future wealth is compounded over many periods, the result is almost certain to be a highly skewed distribution. Because of the skewness in the distribution over multiple periods, investors tend to concentrate on maximizing expected value (or the geometric mean return). It would be far more to their benefit, however, if they maximized the median—that is, the central tendency." The author notes the similarities between his approach and constant-proportion portfolio insurance [i.e., floor + multiplier] portfolio management systems: "Investor's lambda level will be proportional to the leverage in their life. As they become more leveraged, they will become more risk averse. As they become wealthier and their net worth becomes a larger fraction of their total assets, they can afford to be more aggressive."	
2011	"Spending Retirement On Planet Vulcan: The Impact of Longevity Risk Aversion on Optimal	The authors characterize the traditional lifecycle model as one in which "a rational individual seeks to maximize, over all admissible consumption paths, the discounted additive utility of consumption over their entire life." Although dynamic programming approaches are now used to answer the question of how to spend money over time, the Calculus of	"…a simple rule that advises all retirees to spend x% of their nest egg adjusted up or down in some ad hoc manner, is akin to the broken clock which tells time correctly only twice a day." Article discusses the role of a reserve to deal with

	Withdrawal Rates," M.A. Milevsky and H. Huang Financial Analysts Journal vol.67 no. 2 (March/April 2011) pp. 89 – 100.	Variation model, initially developed by Ramsey in 1928, was an early mathematical expression used to solve the optimal spending question. This article assumes an investment program that earns a constant real rate—i.e., a deterministic rate with no variance. The authors justify this simplification in the lifecycle model because they wish to focus exclusive on longevity risk aversion. Thus, the relative risk aversion function is not the traditional aversion to the risk of uncertain future asset price; but, rather, to the risk of the uncertainty of life span. A central thesis is that longevity risk aversion plays an important role in determining the optimal withdrawal from a retirement portfolio. An arbitrary spending rule such as "…start by spending x% strategy has absolutely no basis in economic theory." The expression for utility maximization is: $\max_c V(c) = \int_0^D e^{-pt}({}_tp_x)u(c_t)dt.$ Where D is the time of death, p is the subjective discount rate, c is consumption, and ${}_tp_x$ is the probability of survival where the probability distribution is parameterized by the Gompertz law of mortality. The expression for longevity risk aversion is the CRRA form: $u(c) = c^{1-\gamma}/(1-\gamma)$. The actuarial present value of the consumption is $a_x(\nu,m,b) = \int_0^T e^{-\upsilon s}({}_sp_x)ds$ Where nu is the discount rate, and m and b are the parameters of the Gompertz mortality distribution function. The closed form solution for the above equation is expressed in terms of the incomplete Gamma function. The "wealth trajectory" at time t is denoted by Ft, and its first derivative is: $F'_t = \nu(t,F_t)F_t - c_t + \pi_0$ Where π is a pre-existing real pension income. The initial condition is F_0 = W where W is the amount of investible wealth at retirement. Assuming no bequest objectives, $F_\tau = 0$ where tau is the "Wealth Depletion Time"—	longevity risk. Authors highlight the importance of portfolio monitoring. The import of portfolio depletion is modified by both the time of depletion and the length of time the investor lives beyond depletion. This is an expected shortfall risk metric. See, also, "Optimal retirement consumption with a stochastic force of mortality," Huaxiong Huang, Moshe A. Milevsky and Thomas S. Salisbury, Insurance: Mathematics and Economics, Vol. 51 (2012), pp. 282 – 291. The reader may also wish to compare the updating and recalibration formulae presented in this article to those advocated by Gordon Pye in: The Retrenchment Rule: When It's Too Late to Save More For Retirement, Gordon B. Pye (GBP Press, New York) 2012. The HMS and Pye models yield a host of interesting insights on how a retirement income portfolio monitoring system directs the investor to make mid-course adjustments based on rational assessment of options rather than on rule-of-thumb nostrums.

the point at which the retirement portfolio is depleted and the investor lives only on pension income. Thus, the wealth trajectory assumes a constant real rate of earnings less consumption and plus pension income.

Using the Calculus of Variation [The Euler Lagrange Theorem] leads to the optimal solution to a second-order differential equation. The solution requires two stages. First the optimal withdrawal rate c_t^* satisfies the equation $c_0^* e^{kt}({}_t p_x)^{1/\gamma}$. Whereas the optimal withdrawal rate includes pension income, the initial portfolio withdrawal rate is: $\frac{(c_0^* - \pi_0)}{F_0}$.

The optimal financial capital trajectory is a function of c_0^*:

$$F_t = (W + \pi_0/r)e^{rt} - a_x^t(r\text{-}k, m^*, b)\, c_0^* e^{rt} - \pi_0/r.$$

Where the modified model value of the Gompertz distribution is $m^* = m + b\ln(\gamma)$. “The actuarial present value term multiplying time-zero consumption … values a life-contingent pension annuity under a shifted modal value of $m + b\ln(\gamma)$ and a shifted valuation rate of : $r – (r\text{-}\rho)/\gamma$ instead of r.”

The remaining unknown, c_0^* or the initial optimal withdrawal rate from the portfolio is the present value of wealth divided by the actuarial present value of consumption equation which takes the form of $W \div a_x^\tau$:

$$\frac{\left(W + \frac{\pi_0}{r}\right)e^{rt} - \frac{\pi_0}{r}}{a_x^\tau(r - k, m^*, b)e^{rt}} e^{kt}(\tau Px)^{1/\gamma}$$

Where $k = (r - \rho - \lambda) \div \gamma$.

The optimal withdrawal rate during retirement for a utility maximizing investor decreases with age as the conditional survival probability becomes greater—i.e., the greater your current age, the greater is your life expectancy. Therefore, a financial planning rule of thumb recommending a constant withdrawal rate is not appropriate: “the utility-maximizing retiree is not willing to reduce their initial standard of living simply because of a small probability they will reach age

		105….They deal with longevity risk by setting aside a financial reserve AND by planning to reduce consumption if that risk materializes in proportion to the survival probability and linked to their risk aversion." The authors quote Irving Fisher (1930): "…[the investor has] a high degree of impatience for income because he expects to die and he thinks: instead of piling up for the remote future, why shouldn't I enjoy myself during the few years that remain?" Whenever the investor has pension income, however, he is not as worried about longevity risk: "Basically, the pension acts as a buffer and allows the retiree to consume more from discretionary wealth." Indeed, the utility maximizing investor will deplete all discretionary wealth after a period of time after which they will live only on the pension income. Where a pension does not exist, the wealth depletion time occurs at the end of the planning horizon—i.e., the final age on the morality table. The article also explores the "optimal reaction to financial shocks" over the retirement planning horizon. A 'spend 4% of wealth' rule provides no guidance on updating spending policy in response to a shock to wealth. The authors recommend the following steps: 1. Recalibrate the model from time zero, but with current wealth equal to the lower amount, and compute the new wealth depletion time equation. 2. Compute the new optimal consumption equation which yields an amount that differs from the pre-shock amount. 3. Continue retirement consumption at the new amount. They provide an example of a 30% decrease in portfolio value that requires a 14% decrease in consumption. "…the time zero consumption plan is based on a conditional probability of survival as opposed to an unconditional probability of survival. In other words, there is a difference between planning what to do at age 70 if you survive – an event with a probability (${}_5p_{65} < 1$) – versus actually surviving to age 70 and then formulating a new plan based on the fact (${}_0p_{70} = 1$)."The rational reaction to shocks is "non-linear and dependent on when the shock is experienced as well as the amount of pre-existing income…."	

		The authors assert: "The optimal consumption rate (c_t^*), which is the total amount of money consumed by the retiree in any given year including all pension income, is a declining function of age….The consumption rate is proportional to the survival probability (${}_tp_x$) and is a function of risk aversion, even when the subjective rate of time preferences (ρ) is equal to the interest rate. In other words it simply does not make any sense to target a fixed constant standard of living or constant portfolio withdrawal rate. The rational consumer—planning at age 65—is willing to sacrifice some income at the age of 100 in exchange for more income at the age of 80." In the context of this article, a high coefficient of risk aversion is equal to an increase in the probability of survival. The investor behaves as if his actual life expectancy increases by an amount proportional to ln[1/γ].	
2011	"Asset-liability management under time-varying investment opportunities," Robert Ferstl and Alex Weissensteiner, Journal of Banking & Finance Vol. 35 (2011), pp. 182 – 192.	The authors state: "In this paper we propose stochastic linear programming (SLP) for an asset-liability management (ALM) problem of a pension fund when investment opportunities are time-varying." To address the problem, the authors create a "first-order unrestricted vector-autoregressive process VAR(1) to model asset returns and other state variables." The model is parameterized by using historical log equity returns (S&P 500) and log dividend-price ratios using quarterly data from Q3 1987 through Q4 2007. These estimates are combined with the Nelson/Siegel yield curve parameters (level, slope, and curvature) in a multi-stage optimization problem characterized by a VAR(1) process. Bond returns are U.S. spot rates provided by the Federal Reserve Board. "The idea behind a vector-autoregressive process is that an economic variable is not only related to its predecessors in time, but also depends linearly on past values of other variables." Innovations (the model's error term) are assumed to be homoscedastic and independently distributed. The matrix of slope coefficients for the three Nelson/Siegel parameters, the state variable (log dividend-price ratio) and log equity returns evidences a stable process—i.e., all eigenvalues have modulus less than one. The authors use the term "shareholder value" [SV] for the difference between the fair market value of plan assets and the present value of	Although this study discusses stochastic linear programming as a tool for management of a Defined Benefit pension plan, it provides interesting insights into in a general asset-liability management context under time-varying investment opportunities. The research is valuable for understanding and utilizing "decision-support models." The model presented in the study, despite its obvious sophistication, is a highly stylized four-stage—i.e., four quarters—decision tree where return realizations evolve in discrete time. The article begins with the statement: "One of the classical problems in finance is the derivation of optimal dynamic investment strategies over a finite planning horizon, where the uncertainty is modeled with stochastic processes driving asset returns and state variables." The authors sketch a brief history of thinking on this issue: Conventional Wisdom: The longer the horizon, the more risk should be taken (e.g., the life-cycle risk

		future liabilities. The preferred asset management strategy minimizes the conditional value at risk of SV at the end of the planning horizon (one year) under the constraints that (1) a target value of SV is attained at the end of the planning horizon, and (2) there is a maximum allowable drawdown in SV between periods for each scenario path: "This amount represents the maximum loss a sponsor of the plan is willing to suffer during one period." The "simplest" approach to controlling the difference between the current market value of assets and the present value of future liabilities is to construct a portfolio of zero-coupon bonds with maturities matching the due date of liability payments—a cash matching strategy. However, if time-varying investment opportunities reflect "predictability" in asset returns and state variables, such an approach may be suboptimal. To test this proposition, the authors create "…a scenario tree consisting of the stochastic asset returns. We model the uncertainty by a VAR(1) process incorporating stock returns, dividend-price ratios and level, slope, and curvature of the term structure of interest rates." The objective function minimizes the conditional variance at risk of SV at a designated confidence level. The "recourse" feature of the SLP allows for purchases and sales (e.g., mid-course corrections in asset allocations) over the four-quarter scenario tree. The model reflects transaction costs. "The expected simple return per annum for equities equals 7.29%, while the term structure of interest rates is increasing and concave." The scenario-generating process matches the first four moments of the asset return process, includes the yield curve parameters , and a correlation structure, and incorporates the dividend-price ratio as a state variable within a four-stage decision sequence along a scenario tree. The base-case model designates lower and upper bounds for portfolio values in each stage; and allows for a short position in riskless bonds to create a "prudent version of a so-called '130/30' strategy." Hedging demands within an ALM context lead the plan sponsor to hold the maximum allowable short position in the short-term bond. The optimal asset allocation reflects a large allocation to long-term bonds for low	aversion hypothesis); Early Research: (Samuelson, Merton, etc.) investors with CRRA should maintain a constant-mix asset allocation; Recent Research: incorporates time-varying investment opportunities in strategies designed to maximize investor utility. Although there are a few studies that offer analytical solutions, most recent approaches either offer a numerical solution that "discretizes the state space and solves the problem by backward induction;" or, offer a simulation-based numerical solution method. The authors also provide a brief review of literature on Stochastic Linear Programming. Some studies use a "rolling forward approach…where at each stage the process parameters are re-estimated and a new scenario tree is generated. To predict and exploit returns in such a context, the starting values should coincide with the most current realizations." The drawback of using unique current parameter values lies in the fact that this approach limits the ability to draw general conclusions from model output. By contrast, the other common SLP approach is "… to start the investigation with the unconditional expected values of the estimated process." This is the approach that the authors elect to use not only for return estimates, but also for all other parameter values. It is interesting to compare the Ferstl / Weissensteiner portfolio consisting of a long-term bond hedge, short position in short-term (riskless) bonds, and equity investments, to the cash-matched and immunized portfolios presented in

		target SV, and an increasing shift to equities for higher target SV. The authors explore the impact of various maximum drawdown constraints as well as the results from static (myopic) allocations under a constant-mix asset management approach. Other variations on the base case model incorporate predictability in bond returns only—the equity portion of the portfolio follows a Brownian motion process—this is termed a "partial predictability" model. In this case, the demand to hold equities decreases while the demand to hedge interest-rate risk increases: "…our results indicate that in the worst case a more prudent asset allocation is appropriate by favoring long-term bonds (to hedge the interest-rate risks from our cash flows), while in scenarios with higher wealth more risk can be taken by holding higher equity positions. Due to the predictability in the equity returns, the mean allocation to equities over all decision stages is higher…." At all levels of target ending SV, there is a demand for a short position in riskless bonds with a correspondingly heavy position in long-term bonds. Equity investments come into play only when a high target SV is required. Using a dynamic SLP strategy under the partial predictability model produces an annual return advantage of 1.08%; under a full predictability model, the advantage is 3.02% compared to a constant-mix strategy.	"Optimal investment strategies with investor liabilities," Edwin J. Elton and Martin J. Gruber, <u>Journal of Banking and Finance</u> Vol. 16 (1992), pp. 869 – 890.
2011	"Systematic Withdrawal Programs: Unsafe at Any Speed," Garth A. Bernard Journal of Financial Service Professionals vol. 65 no. 1 (January 2011), pp. 44 – 61.	The paper characterizes the traditional systematic withdrawal program [SWP] as follows: "The traditional program starts with a stated initial percentage withdrawal rate that defines the annual dollar amount of the withdrawal to be taken from the assets in the first year. Subsequent withdrawals are completely unrelated to the then value of the asset portfolio. Instead, the dollar amount of each subsequent annual withdrawal is simply a function of the previous year's dollar amount of withdrawal." The author develops a Monte Carlo model with two asset classes: stocks and bonds. The return and volatility of each asset class is based on average returns over the period 1927 to 2007. The 2008-2009 period is deliberately not included in the analysis "…in order not to appear to	The author states the retirement income risk/return tradeoff as follows: "…for a retiree, using the retirement assets to withdraw adequate income for life and leaving an adequate legacy are competing objectives…." The paper assumes that a depletion risk of 10% is acceptable. One conclusion is that the portfolio's asset allocation is not the critical driver. This parallel's other commentator's observation that many retirees have portfolios that are so meager than any asset allocation strategy will have minimum effect

		unfairly bias the results; this bear market was considered to be a 'black swan' or once-in-a-generation event even though, technically, it would not have been statistically inappropriate to include this period." Taxes are not considered. The article examines an SWP along five dimensions: • Initial withdrawal rate (3% - 10%) • Percentage allocation to equities (0% - 100%) • Annual increase rate on subsequent withdrawals (0% - 5%) • Retirement Period (10 – 40 years) • Total fees underlying the portfolio (50 – 300 bps). The analytical methodology involves varying one variable at a time to observe how other variables respond. The base case is a withdrawal rate of 5%; an annual increase rate on subsequent withdrawals of 3%; a planning horizon of 25 years; a 50/50 allocation; and annual fees on equity of 100bps and 60bps on bonds. With respect to the withdrawal rate variable, "…both the depletion risk and especially the legacy shortfall risk rise in a nonlinear fashion as withdrawal rates increase." With respect to the asset allocation: "…as the allocation to equities declines below 50%..., the risks of both depletion and legacy shortfall increase dramatically." As equity exposure increases both the expected return and volatility of return increase. This means that a portfolio has an increased vulnerability to sequence risk. "This paradox—that increasing the equity allocation (beyond a balanced portfolio) does not reward the investor for taking increased volatility risk—is a distribution dynamic that is exactly the opposite of what advisors have been taught about investment management in the accumulation phase." Additionally, increasing the withdrawal rate above 5% per year results in a substantial risk of shortfall regardless of the asset allocation. With the exception of a portfolio heavily tilted towards bonds, it is the withdrawal rate rather than the asset allocation that is the primary driver of future portfolio success or failure.	on retirement security.

		With respect to fees, results suggest that "depletion and legacy shortfall risks are sensitive to the level of fees but increase at a more linear pace rather than the pronounced increase observed for some other variables." Fees, of course, reduce the effective portfolio yield. The author contends that the fundamental dynamic operating to increase the risk of portfolio depletion is the interaction between sequence risk and the stress of a fixed (possibly inflation-adjusted) withdrawal under all economic conditions: "…the value of the portfolio will be reduced by the withdrawals taken and, furthermore, will be increased or reduced by investment gains and losses on the portfolio; the higher the equity allocation and thus the higher the volatility of the portfolio, the more pronounced the swings in portfolio value….when the value of the portfolio moves down (as in a bear market), the 'effective' withdrawal rate at the time of the withdrawal can become extraordinarily high." [but note that a safe withdrawal rate is an age-dependent variable]. Furthermore, "…it is not simply a matter of market downturns within 5 – 10 years of retirement that present the primary risk of experiencing premature depletion. That appears to be more marketing fiction than analytical fact." The author recommends revisiting and resetting the withdrawal rate periodically in light of the current portfolio value. Finally, the author reviews several improvements to an SWP: <u>A variable dollar program</u>: either a flexible withdrawal rate as a percentage of the portfolio's current value [unitrust distribution formula] or periodic resetting of the withdrawal rate based on the presumed shortening of the planning horizon faced by the investor. <u>A Variable Annuity</u> with Guaranteed Lifetime Withdrawal Benefit Rider. <u>An Annuitization + Growth Strategy</u>: "One compartment of the portfolio would be dedicated to efficiently generating income, while the other compartment of the portfolio could be dedicated to efficiently generating investment returns." <u>Purchase of Longevity Insurance</u>: A contract "…can be used to 'peg' the time horizon over which the SWP must perform."	

		Although the author does not explore combination approaches, he nevertheless advances the argument that the needs of individual investors might best be served by blending the various approaches.	
2012	"Consumer Preferences for Annuities: Beyond NPV," Suzanne Shu, Robert Zeithammer, & John Payne Working Paper available at: http://www.anderson.ucla.edu/faculty/suzanne.shu/Shu%20Zeithammer%20Payne%20annuity%20preferences.pdf	This study focuses on the extent to which certain annuity attributes might increase or decrease consumer preference for annuitization versus self-management of investment funds. Stated otherwise, the authors seek to ascertain if there are attributes that create positive or negative utility beyond the utility generated solely by the contract's expected value as measured by its NPV. The article points out that: "The consumer's risks in consuming saved assets include either spending too quickly, in which case she may run out of money quickly, or spending too slowly, in which case her consumption is severely constrained and she dies with unused funds." Although an annuity is a financial instrument designed to help the consumer mitigate these risks, annuity contracts are often viewed unfavorably. Several explanatory hypothesis are found in the literature including: (1) "people adopt a narrow framing of the problem as a gamble, rather than as an insurance decision, due to the complexity of the annuity purchase task;" and, (2) "Loss aversion…especially when considering the loss of the annuity purchase price due to early death." Additionally, a host of other factors ranging from procrastination to differential discounting for gains and losses may come into play: "Both intertemporal choice and judgment under uncertainty are crucial elements of the decumulation decision….Consumer uncertainty exists for both judgments of future health and economic outcomes (e.g., inflation) and judgments of life expectancy; research on biases in probability judgments can offer substantial insights on these issues." Finally, other predictors of a consumer's demand to hold an annuity may include "…financial knowledge (both objective and subjective knowledge), financial literacy, numeracy, and overall cognitive ability also offer important predictions for how consumers who differ in individual ability may react to annuity offerings."	A good review of factors that may contribute to consumer reluctance to purchase annuities. They construct, based on responses to a consumer survey, a linear regression model to ascertain which contract features and issuer attributes consumers would find appealing.

		The study presents a regression-based utility model in which the independent variables are attributes that are thought to be relevant in explaining the annuity puzzle. Data was developed from responses to choices presented to survey participants ranging in age from 40 to 65: "'If you were 65 and considering putting $100,000 of your retirement savings into an annuity, which of the following would you choose?' They then saw three annuity options and a fourth option that read, 'None: If these were my only options, I would defer my choice and continue to self-manage my retirement assets'." All annuity purchase choices reflect an initial purchase price of $100,000 at age 65. The first independent variable is a combination of (1) estimates regarding the value of the annuity's expected value to the annuitant where cash flows are discounted at a rate of subjective time preference; and (2) estimates of the value of residual income from a period-certain payment guarantee feature where such payments are to be received by a designated beneficiary/co-annuitant. The second independent variable is the financial strength of the issuing insurance carrier measured by preference for an AAA rated carrier vs. an AA rated company. The third independent variable is a vector of attributes reflecting the type and amount of additional annuity features such as the length of the payment guarantee period, and the amount—if any—by which the starting annual payment increases in the future. The last term is the regression error term in which the values are assumed to be independent. Thus the linear regression model ("…a constrained version of the multinomial probit model") is: $$U_{n,k,j} = \alpha_j + \beta V_{n,k} + \gamma AAA_{n,k} + X_{n,k}\theta_j + \varepsilon_{n,k,j} \approx N(0,1)$$ Of the 363 survey participants, 22% (n=80) did not choose any annuity option despite the fact that all annuity options offered an expected payout with an NPV of at least $160,000 and with some options offering an NPV over $200,000 for a $100,000 purchase. "This strong dislike of annuities with a high benefit relative to upfront costs (more than would	

		ever be offered in the market, in fact) suggests some individuals are unwilling to consider annuities regardless of the benefit offered." The remainder of the analysis concentrates on the sample of 283 participants who chose at least one annuity option. Results are examined on both an aggregate ("average or representative consumer") basis and on an individual consumer basis. On the aggregate level, females were more disinclined to select an annuity option; the attractiveness of annuities decreases with the level of household income; and the perceived fairness of the annuity payout is a critical factor in determining a preference for an annuity solution. Surprisingly, risk aversion, loss aversion, numeracy and life expectancy were not significant factors when determining an investor's willingness to purchase an annuity. The regression coefficient on financial strength [γ] is positive with an investor indifferent between an AAA rated company annuity with an expected payout of $100,000 and an AA rated company annuity with an expected payout of $128,000. The coefficients on the vector of individual contract parameters [X] indicate that consumers prefer additive increases in annuity benefits rather than percentage increases, that short period-certain guarantees (5 years) are not valued but that mid-length (10 and 20 years) guarantees are highly valued. Other observations for the aggregate data sample include: • "…the older, more numerate, and higher-saving individuals choose fewer annuities and more 'none of the above,'…" • "…we do not find any significant positive relationships between having children or being married, and preferring a longer period certain." • "…neither risk aversion nor loss aversion increase the preference for a longer period-certain guarantee." • "…people who consider annuities to be fair dislike the period-certain guarantees." • "…people just before retirement tend to like annuities the least." Finally, the authors note that delayed annuities [ALDAs] may be more attractive than immediate annuity contracts.	

2012	A Utility-Based Approach to Evaluating Investment Strategies, Joseph A. Tomlinson Journal of Financial Planning (February 2012).	The aim of the paper is to build and present a utility function that incorporates three retirement plan risk measures: 1. The probability that a portfolio allocated among stocks, bonds [TIPS], and an inflation-adjusted annuity will run out of money during the investor's life 2. The magnitude of the shortfall conditional on the investor's outliving the portfolio; and, 3. The expected bequest amount given the allocation strategy. Utility of bequests is a function of gains and losses as measured from a \$0.00 bequest amount. Running out of funds during life produces a negative bequest (<\$0) with the magnitude measured by time spent alive-without-funds (with the shortfall discounted at the assumed real risk free rate of one percent). A valid utility function reflects the fact that a negative bequest has a larger impact on aggregate utility than a bequest gain (>\$0). The study uses a Monte Carlo simulation model assuming a 20-year life expectancy for a 65-year old investor in good health seeking to provide funds sufficient to meet basic living expenses which, on a constant dollar basis, are matched exactly to the annuity investment option payout. Initial financial wealth is the annuity's single premium cost. Returns for TIPS are based on historical return series; stock returns are based on the estimates of economists with respect to the expect equity risk premium. The first step of the analysis consists of presenting the constant-dollar outcomes of various asset allocations over the investor's stochastic lifespan. Allocations range from 100% stock / 100% TIPS / 100% annuity, to various combinations thereof. With the exception of the annuity—which produces a \$0.00 bequest—all other allocations produce an expected bequest with a positive present value. However, with the exception of the annuity program, there is a substantial failure rate (≥ 20%) for all other allocations. The 100% bond allocation manifests a 45%	The article provides a good example of how small changes in parameter inputs can result in significantly different output. In this case, the author demonstrates how a change in the forecasted equity risk premium changes preference ranking of asset allocation decisions.

		failure rate—i.e. 45% chance of running out of funds while still alive. The present-valued average loss, conditional on the occurrence of failure, is highest for the 100% stock allocation. This suggests that a bond-oriented allocation may have greater failure likelihood but a lower failure magnitude. The utility function employed to rank investor preferences is centered on the bequest "origin"—i.e. \$0.00. A negative bequest expresses the fact that the plan experienced a shortfall during the investor's lifetime that is quantified by the present value of its dollar magnitude. The concept of "loss aversion" differs from the usual meaning in behavioral finance because the planning horizon extends over a potentially lengthy period and the concept of "gain" is expressed in terms of a positive bequest. Using values for the utility function curvature (convex in certain regions and concave in others) taken from both existing studies by Kahneman and Tversky as well as from evidence from a small survey conducted by the author. The 100% annuity allocation serves as a base case against which the utility values of other allocations are measured. A positive utility value indicates that the investor prefers stock and bond asset combinations; a negative value indicates that the investor prefers the annuity strategy. The utility estimates are a function of the degree of investor risk aversion—i.e., the curvature of the function. For all values of risk aversion, the 100% bond allocation produces negative dollar values—i.e., negative utility. The 100% annuity allocation, by construction, produces zero dollar value. At reasonable levels of risk aversion "…a systematic withdrawal strategy using the specified asset allocation or stock/annuity mix produces more expected utility than the 100 percent annuity strategy…." The full annuitization strategy is attractive for investors with high risk aversion coefficients. Although there is no absolute strategy that dominates across all levels of risk aversion, it is evident that the all-bond allocation is not attractive at any risk-aversion level.	

		Finally, lowering the forecasted equity risk premium increases the relative attractiveness of the annuity strategy: "…the choice of equity premium assumption can tip the scales regarding whether to recommend annuities. This uncertainty creates a dilemma for planners, particularly when advising financially constrained clients."	
2012	"A Framework for Finding an Appropriate Retirement Income Strategy," Manish Malhotra, Journal of Financial Planning, Vol. 25, No. 8 (August 2012), pp. 50 – 60.	The article demonstrates how, via a software program developed by the author, multiple strategies and products can be identified and evaluated. Evaluation takes the form of comparing simulated outcomes along several risk and reward metrics. Each outcome is scored across the following categories/criteria: Income: client's "desired income;" Confidence: a probability of success metric; Years of Income in Bad Markets: percentage of trials over which the desired income is successfully generated by the investment program; Fixed Source Coverage: percentage of yearly income not exposed to "investment risk;" and, Average Legacy: dollar value of terminal wealth. The author offers three case studies. Each case study is a combination of investments and strategies designed to produce acceptable risk/reward tradeoffs (based on the above-listed metrics) for the client. An astute client may select the strategy reflecting his or her risk/reward preferences; or, the advisor may simply recommend the strategy that, in the advisor's opinion, best fulfills client needs and preferences. The case studies provide interesting examples of the risk/reward tradeoffs of various strategies. For example, they illustrate the inverse relationship between the attractiveness of high expected returns from stocks and the ability to sustain income during bad market periods: "Generally, as the volatility of a systematic withdrawal portfolio increases from conservative to moderate to aggressive, the confidence of achieving a goal increases but the sequence of return risk of a higher volatility portfolio reduces the years of income in bad markets." Precise comparisons of product/strategy combinations are achievable	The author begins with a surprising statement: "the profession still lacks a comprehensive analysis framework for comparing retirement distribution strategies." At first blush, it seems as if he is unaware of many academic studies which precede his article. These include applications of dynamic programming within a Life-Cycle Model approach the goal of which is to evaluate various methods for optimizing consumption throughout retirement. Other approaches (option valuation, control theory, actuarial based, and so forth) are plentiful. Things become a bit more clear as one progresses through the article's literature-review section—most cited works are 'practitioner-oriented' articles many of which appear in previous issues of the Journal of Financial Planning. As it turns out, the author is arguing that, absent sufficiently comprehensive financial planning software programs, practitioners lack the ability to model, assess, and recommend programs for clients. A comprehensive planning application allows the advisor to incorporate Social Security claiming options, Reverse Annuity Mortgage programs, fixed and variable annuity products, and a variety of other financial instruments (stocks & bonds) into a "comprehensive analysis framework." It is likely that the article reflects the research underlying the author's commercial offering: Income Discovery. "Income Discovery is an online investment advice

		only within a case study—not between the studies (of course, each study exhibits a different fact pattern so direct comparison is not feasible). In a somewhat curious nod to 'model risk,' the author uses different capital market assumptions in each case study. The case studies do a good job illustrating the "cost of safety" tradeoff. Given an irrevocable (myopic) initial decision, the fact patterns reveal substantial sacrifices of expected upside in financial asset portfolios. Presumably, the advisor can re-run the software program in later years to update the risk/reward metrics. However, quite often a strategy calls for implementation of an annuity or bond ladder program that constrains future adaptability. A truly comprehensive analysis seems to require a dynamic component geared towards monitoring actual outcomes and assessing the strategy and distribution options available to investors as time and circumstances (health, investment realizations, family circumstances, etc.) change. The author sagely acknowledges: "The comprehensive model for retirement withdrawals has not yet appeared in our literature, but it may now be possible to outline its general parameters."	software offering from Fiducioso Advisors, SEC Registered Investment Advisor. The software is offered to Financial Advisors (not to general public) to help them build retirement income plans for their clients." [http://www.incomediscovery.com/about-us.html]
2012	"Optimal retirement consumption with a stochastic force of mortality," Huaxiong Huang, Moshe A. Milevsky and Thomas S. Salisbury, Insurance: Mathematics and Economics, Vol. 51 (2012), pp. 282 – 291.	The article begins by observing that introducing a force of mortality (lifetime uncertainty) into a lifecycle model increases investor impatience to consume. However, under deterministic mortality, the mortality rate "…can be added to the subjective discount rate without any impact on the mathematical structure of the problem." [problem = optimal savings and consumption] When mortality is deterministic and the investor lacks a bequest motive, the Yaari model recommends consumption at time 't' equal to [Wealth(t) ÷ an actuarial annuity factor (t)]. The authors expand upon the optimal consumption path model by including a stochastic mortality rate. Concurrently, they restrict the model by holding all other state variables constant in order to isolate the impact of mortality. They compare two hypothetical retirees having identical utility functions [constant relative risk aversion], elasticity of intertemporal substitution [1/γ], subjective discount rate [ρ], financial capital [F], risk free rate [r], probability of survival curve [ρ(s)], health	The Huang-Milevsky-Salisbury [HMS] model explains the Fisher-utility behavior of decreasing consumption for retirees exhibiting higher longevity risk aversion [γ]. Note HMS cites a passage from Fisher's 1930 book, The Theory of Interest: As Determined by Impatience to Spend Income and Opportunity to Invest It. It also provides helpful theoretical insights into empirical data on actual retiree spending patterns. See, for example, the discussion in this annotated bibliography for "Estimating the True Cost of Retirement," David Blanchett, Presented at the Society of Actuaries 'Living to 100 Symposium' (January 8 – 10, 2014). It is also useful to compare the 2012 HMS article to the 2011 article: "Spending Retirement On Planet

		state, age, and initial longevity risk assessment. The retirees differ only in their views about the volatility of their health—i.e., mortality rate volatility: "…the first retiree (1) believes that his/her instantaneous force of mortality (denoted by $\lambda^{DfM}(t)$) will grow at a deterministic rate until he/she eventually dies, while the second retiree (2) believes that his/her force of mortality (denoted by $\lambda^{SfM}(t)$) will grow at stochastic (but measurable) rate until a random date of death. As such, the remaining lifetime random variable for retiree 2 is doubly stochastic." The stochastic nature of mortality rate captures unexpected improvement or deterioration of *future* health status. The evolving assessment of mortality rate becomes a new state variable: "In this paper, we show how the uncertainty of mortality interacts with longevity risk aversion (γ) – which is the reciprocal of the intertemporal elasticity of substitution – to yield an optimal consumption plan. Mortality no longer functions as just a discount rate." Under a deterministic force of mortality conditional survival probabilities from time 'zero' (the present) are known throughout the entire planning horizon. The probabilities are the "textbook life contingencies" found throughout much actuarial literature—for example, an investor currently age 65 has a probability of surviving to age 80 equal to 74.29%; an investor currently age 70 has a probability of surviving to age 80 equal to 78.37% based on U.S. unisex annuitant mortality. However, such probability calculations are not possible when the mortality rate is stochastic. The Huang, Milevsky, Salisbury model adds a Brownian motion component to the drift and diffusion coefficients of a deterministic model. Under the deterministic, Yaari-type, model, the optimal "financial capital trajectory" satisfies a second-order non-homogenous differential equation when there is pension income; and, a second-order homogenous differential equation in the absence of income—i.e., pension or earned income is the "forcing term." Consumption levels are, of course subject to a budget constraint; and, by specifying a terminal wealth condition of zero at the date of death, an investor can solve for	Vulcan: The Impact of Longevity Risk Aversion on Optimal Withdrawal Rates," M.A. Milevsky and H. Huang Financial Analysts Journal vol.67 no. 2 (March/April 2011) pp. 89 – 100.

		optimal initial consumption: $$\frac{(F(0)+\frac{\pi_0}{r})e^{r\tau}-\frac{\pi_0}{r}}{\alpha_x^{\tau}(r-k,m^*,b)e^{r\tau}}$$ Where: π = pension income, τ = the wealth depletion time, k = (r – ρ – λ(t)) ÷ γ, and, m* and b are mortality parameters and m* = m+bln(γ). This is a ratio that adds (current wealth plus 'future-value-of-income' terms) growing at a fixed constant compounding rate in the numerator; the denominator expresses the cost of consumption in terms of the annuity pricing factor, compounded over time [e^{rt}] which multiplies terms for the risk-adjusted mortality distribution with mean 'm*' and dispersion factor 'b,' and the risk-adjusted discounting rate factors [r-k]. The authors provide a helpful case example which solves for initial and future consumption under their expanded "Yaari deterministic model" for an investor with a subjective discount rate equal to the assumed risk-free rate (2.5%), no pension or earned income, age 65, no bequest motive, and risk aversion parameters (γ) equal to 4 or 8. Given a standardized $100 initial financial capital, the optimal consumption is $4.605 and $4.121 respectively. The model suggests that: "As the retiree ages (t > 0), he/she rationally consumes less each year – in proportion to the survival probability adjusted for γ….a fully rational consumer will actually spend less as they progress through retirement. The optimizer spends more at earlier ages and reduces spending with age, even if his/her subjective discount rate (SDR) is equal to (or less than) the real interest rate in the economy. Intuitively, the individual deals with longevity risk by planning to reduce consumption – if that risk materializes – in proportion to the survival probability linked to their risk aversion." The article's longevity probability table clarifies the central	

		point: the longer you live, the longer you're expected to live. The optimal consumption strategy for a retiree seeking to maximize expected discounted utility over an uncertain lifetime "…collapses to the Yaari (1965) model when the volatility of mortality is zero." The authors introduce a series of simplifying assumptions (fixed terminal horizon date 'T,' no income, subjective discount rate = constant risk-free rate), so that the wealth – consumption trajectory is a martingale process [change of expected mean of the distribution of future wealth process from mean of the current wealth process = zero]. Given that consumption subtracts from wealth, the process becomes a supermartingale [decrease in expected mean] under a general choice of 'c.' The authors seek to solve for the optimal plan under a stochastic wealth process given an investor with CRRA utility. The simplifying assumptions allow the authors to solve for the optimal control by using the Hamilton-Jacobi-Bellman [HBJ] optimality equation under deterministic mortality; and, the corresponding control under stochastic mortality. They present the following theorem regarding (1) impact of risk aversion on consumption and (2) impact of stochastic mortality on consumption: Assume that the survival functions for the two models agree: $\rho^{SfM}(t,\lambda_0) = \rho^{DfM}(t,\lambda_0)$ for every $t \geq 0$, and that utility is CRRA(γ): (a) $\gamma > 1 \rightarrow c^{SfM}(0,\lambda_0,F) \geq c^{DfM}(0,F)$; (b) $\gamma = 1 \rightarrow c^{SfM}(0,\lambda_0,F) = c^{DfM}(0,F)$; (c) $0 < \gamma < 1 \rightarrow c^{SfM}(0,\lambda_0,F) \leq c^{DfM}(0,F)$; Where γ is exactly equal to one—i.e., logarithmic utility--there is a bifurcation in the solutions: "There are two possible reactions to that ability to adjust consumption. One is to shift consumption into the future, taking advantage of the ability to adjust consumption upwards later, if the hazard rate should climb more than expected. The other reaction is to opt to consume more now, in the knowledge that one can cut back later if it seems likely that one will live longer than expected. Our message is that either reaction can be rational, and that which one is	

		adopted depends on the person's risk aversion, with the switch occurring at the point of logarithmic utility." In other words, the decision concerning the retiree's initial consumption rate—i.e., consumption relative to wealth—under both deterministic and stochastic mortality assumptions is the same assuming logarithmic utility. However, "…the ability to adapt to changes in health status and new information about mortality rates allows the retiree to be more generous at time zero" assuming $\gamma > 1$. Nevertheless, despite the fact that uncertainty of lifespan increases the investor's impatience to consume, "…the absolute consumption rate at time zero is uniformly higher the lower the coefficient of relative risk aversion. This is a manifestation of longevity risk aversion. Retirees are concerned about living a long time, and therefore they consume less today to protect themselves and self-insure consumption in old age." There is a tug of war between impatience to consume and fear of outliving resources. In a lifecycle model, health improvement generates an instantaneous decrease in consumption; health deterioration increases consumption.	
2012	Wade Pfau "Retirement Floors and Implications for Evensky's Cash-Reserve Strategy," Advisor Perspectives (June 19, 2012), pp. 1 – 5.	The author refers to a Financial Planning Association survey seeking member feedback on three retirement income strategies: Systematic withdrawals: Pfau characterizes this as a "total return" approach. When the target withdrawal rate is sufficiently low—e.g., one test is that the withdrawal rate could have survived a drawdown equal to that experienced in the great depression--this strategy approaches a floor + upside investment strategy. Time-based segmentation: Pfau suggests that this method uses fixed income investment vehicles to guarantee funding for retirement expenses over a short or medium-term planning horizon. The balance of financial assets can be devoted to higher expected return assets to fund more distant future expenses. Such a strategy assumes, of course, that time reduces investment risk; or, alternately, that sequence risk is greatest during the initial retirement years. Essential-versus-discretionary income: This requires investors to identify critical or threshold income needs in order to determine the	This short essay provides helpful distinctions and clarifications for several approaches to providing retirement income favored by practitioner/advisors. Basically, it appears as if financial planners differ in their preferred methods of funding adequate lifetime retirement income. Some argue for locking in "flooring" via bond-ladders (TIPS, nominal, or a combination thereof) or annuities in an effort to secure a minimum acceptable standard of living. Remaining assets, if any, can be invested for growth. Others suggest investing for total return in order to provide the opportunity for higher levels of lifetime consumption. Such an approach puts a greater premium on monitoring current and future wealth level's lest the portfolio's value dip beneath the cost of security an adequate lifetime income floor. Finally, Pfau points to a middle alternative,

		required level of periodic income. The amount needed to fund the essential needs is immediately locked into place so that the investor secures a threshold standard of living. Any funds remaining, a devoted to more growth-oriented assets designed to provide money for discretionary spending. Pfau contrasts the views of financial blogger Michael Kitces and Mike Zwecher, a managing director at Deloitte. Kitces advocates a 4% withdrawal rate from a financial asset portfolio invested for "total return." This enables investors to retain control of assets, and to capture any realized equity premiums. Systematic economic shocks severe enough to threaten portfolio depletion would also be likely to jeopardize counterparty performance for principal-guaranteed investments. Zwecher, by contrast, is less sanguine regarding historical success rates for total-return portfolios: "Basic needs should not be exposed to the risk of wealth depletion, which is more likely to happen with sustained fixed withdrawals from a volatile portfolio…what is most crucial is to foreclose the possibility that essential needs cannot be met." The author asserts that the Retirement Income Industry Association advocates: "first build a floor, then expose to upside." However, Pfau distinguishes between two views of "flooring." The first is an "investment-based" approach to flooring in which there is no immediate mandate to lock in a floor of guaranteed income so long as the value of the financial asset portfolio exceeds the cost of purchasing the floor. Such an approach preserves the option to annuitize a financial asset portfolio at a future date if such an election appears prudent. This approach is advocated by a number of practitioners including M. Zwecher and Richard Fullmer. "Flooring is the sleeve of the investment portfolio intended to provide flooring income. The cost of building a lifestyle floor (the minimum amount required to maintain a desired lifestyle) can be tracked and monitored over time." By contrast, practitioners like Jason Branning and M. Ray Grubbs advocate priority funding for essential expenses to protect retirement income from "…market, inflation, and longevity risk." Practitioners like Harold Evensky, Stephen Huxley and J. Brent Burns advocate a middle position	promulgated by Harold Evensky, which advocates reserving sufficient money to fund essential needs for several years. Pfau characterizes the first strategy as a variant on goals-based investing ("goal segmentation"); the second as time segmentation; and, the third as investing for return with the caveat of the need for vigilance.

		in which flooring is secured for a number of years while remaining assets are invested for growth.	
2012	The 7 Most Important equations for your retirement Moshe Milevsky John Wiley & Sons, Canada (2012).	Introduces several equations that provide the groundwork for the mathematics of retirement income analysis. These include: 1. Fibonacci's time to depletion equation: $$t = \frac{1}{r}\ln\left[\frac{\mathrm{c}}{\mathrm{c}-\mathrm{Wr}}\right]$$ Where r is the interest rate; c is consumption and W is wealth. The intuition is that the time to portfolio depletion is the date at which the present value of portfolio withdrawals exactly equals the initial account value. "…you can invert and solve for the real interest rate (r) you must earn during retirement so your initial retirement nest egg (W) lasts for a desired number of years (t), assuming you're planning to spend or withdraw (c) per year. Likewise, you can invert and solve for the nest egg (W) required so you can spend (c) for exactly (t) years." 2. Gompertz probability of survival equation: $$ln[p] = \left(1 - e^{t/b}\right)e^{\left(\frac{x-m}{b}\right)}$$ Where p is the probability of survival; x is the current age; b is the coefficient of dispersion of life expectancy at age x; m is the modal value of longevity at age x; and, t is the targeted number of years for which you wish to calculate the survival probability. 3. Edmond Halley's Annuity Factor pricing equation: $$a_x = \sum_{i=1}^{\infty}\frac{iPx}{(1+R)^i}$$ Which, for each $1 of periodic income, the annuity pricing factor is the sum of discounted survival probability where the numerator is the Gompertz survival probability and the denominator is the Fibonacci discounting equation. "Think of it as an exchange rate between money today and the promise of income for the rest of your life."	A good introduction to the mathematics underlyhing many research studies. Milevsky notes that "…there is almost a five-year gap between how long annuitants live, as a group, versus the rest of the population." He explains the concept of the subjective discount rate as follows: "…how much interest you would demand to give up your consumption for a year. The rate you would demand hypothetically isn't necessarily the rate you would receive from the market….The two numbers should be pretty close to each other in a well-functioning capital market….Many economists theorize that for most people the two are equals (so ρ = r)." He stresses that Samuelson would not use the probability of shortfall as a preferencing metric for retirement income strategies. Even events with low probabilities can create extreme amounts of disutility. "Kolmogorov's 1931 paper introduced the concept of a *continuous time* and *continuous state-space* Markov process. A Markov process—named after Russian mathematician, Andrei Markov (1856-1922)—is a collection or a sequence of random variables whose outcomes are unaffected by the past….Kolmogorov derived a differential equation for the transition probabilities that such a continuous Markov process—moving from one

4. Irving Fisher’s spending rate equation:

$$ln[c_{x+1}] - ln[c_x] = \frac{r - \rho + ln[P_x]}{\gamma}$$

Which denotes that the change in the consumption rate from period x to x+1 equals [the real rate of return minus the subjective preference rate plus the log of the 1 period survival probability] ÷ risk aversion coefficient. “The survival probability is always less than one, so its natural logarithm will always be less than zero, or negative. So the lower the survival probability, the higher the probability of dying, and the faster your consumption rate should decline.” In this case, risk [γ] signifies longevity risk aversion. “…those with low levels of longevity risk aversion are willing to take a chance that they have to reduce their standard of living quite dramatically later on *if they happen to still be alive*.”

“Irving Fisher the economist was the first to properly formulate how *rational consumers* should adjust their consumption spending over time. This is the intertemporal aspect of economic tradeoffs.” Data from the US Department of Labor suggests that “By 65, retirees are spending between 50% and 70% of what they did at 50. And, by 80 it has dropped to under 60%.”

“Here is what Fisher said in his 1930 classic, which still resonates today: ‘Uncertainty of human life increases the rate of preference for present over future income for many people….Instead of piling up for the remote future, why shouldn’t I enjoy myself during the few years that remain?’”

5. Paul Samuelson’s asset allocation equation:

$$\psi = \frac{1}{\gamma}(HC + FC)\left(\frac{\mu - r}{\vartheta^2}\right)$$

The equation states that Psi—the amount in dollars to commit to equity—is equal to the reciprocal of the risk aversion coefficient times the sum of Human and Financial Capital times the Merton Optimum. The

state to another—would have to satisfy.”

		equation suggests that asset allocation should remain constant because the "…declining probability of loss multiplied by the increasing disutility of loss canceled each other out under most conditions. Ergo the optimal amount of stocks versus bonds was time-invariant." Andrei Kolmogorov's Portfolio Sustainability Equation: $$P\lambda_t = \frac{\partial P}{\partial t} + (\mu w - 1)\frac{\partial P}{\partial x} + \frac{1}{2}\vartheta^2 w^2 \frac{\partial^2 P}{\partial w^2}$$ The differential equation states that the product of the Probability of Ruin times the instantaneous death rate equals the change in the probability of ruin with respect to time plus the product of (the dollar value of wealth multiplied by expected return) and the change in the probability of ruin with respect to the change in wealth plus one/half [variance times wealth squared] times [the second derivative of the probability of ruin with respect to wealth]. The equation cannot be solved directly.	
2012	"Should Households Base Asset Decumulation Strategies on Required Minimum Distribution Tables?" Wei Sun and Anthony Webb Center for Retirement Research at Boston College Working Paper WP 2012-10, (April, 2012), pp. 1 – 22.	Given the decline in the percentage of employees covered by Defined Benefit Pension plans, the authors note: "The first cohort with substantial amounts of unannuitized pension wealth is now entering retirement. They face the challenge of converting that wealth into lifetime income." When confronting this challenge, households "…trade off the risk of outliving their wealth against the cost of unnecessarily restricting their consumption." Creating an optimal retirement income plan, according to the authors, "…requires the application of numerical optimization techniques that are beyond the abilities of households and their financial advisers." Faced with the lack of sufficient analytical skill, the plan is often a product of "…rules-of-thumb that are clearly sub-optimal." Perhaps the most pervasive of these rules is the consume-4-percent-of-initial-wealth rule. One alternative to this rule is a decumulation strategy based on the IRS Required Minimum Distribution [RMD] tables. Although such a strategy may also be less than optimal, the authors point out: "…a strategy based	The model incorporates a factor for the complementarity of consumption between husband and wife. This is comparable to the feature in the model presented in "Retirement Income Strategies in Expected Utility and Loss Aversion Frameworks," Gaobo Pang and Mark Warshawsky Working Paper (August 26, 2013), pp. 1 – 21. http://reliasllc.com/wp-content/uploads/2013/10/IncomeStrategies.pdf. Although the model makes a series of somewhat simplistic assumptions, it provides valuable insights into the interrelationship of several variables of interest. For example: • "At low interest rates, households with a high intertemporal elasticity of substitution prefer a steeper age-related decline in consumption,

		on these tables does satisfy two important requirements of an optimal decumulation strategy. First, assuming no bequest motive, the percentage of remaining wealth that is consumed each year will increase with age, reflecting decreasing remaining life expectancy. Second, the dollar amount consumed will respond to fluctuations in the market values of financial assets." The research paper compares an optimal decumulation strategy to: • the RMD-based strategy, • the 4-percent rule, • a spend-only-dividends-and-interest strategy (preserve principal), and • a spend-over-life-expectancy strategy. The metric of interest is strategy equivalent wealth [SEW] which is defined as "…the factor by which age-65 pension wealth must be multiplied so that the household is indifferent between adopting the optimal strategy and the alternative….SEW therefore provides a utility-based means of comparing the RMD strategy with both the optimal strategy and plausible rules-of-thumb." The utility-based model sums the discounted utility of consumption over the probability-adjusted lifespan of male and female household members from age 65 until the time of death. The assumed time preference discount rate is 3%; the form of the investor's utility function is Constant Relative Risk Aversion (the reciprocal of which is the investor's elasticity of intertemporal substitution); risk aversion parameters are 2 and 5; portfolios are invested in a risk-free asset only and in a combination of risky/risk-free assets with a log-normal distribution for risky asset returns; Social Security benefits and a constant inflation process are also model features. The study considers a variety of scenarios across levels of wealth, types	reflecting the relative magnitudes of the income and substitution effects. At low interest rates, the strategy of spending the interest is highly sub-optimal at both assumed coefficients of risk aversion because it results in levels of consumption that are far below the optimal." • "The 4 percent rule results in a level consumption path. Households following this rule are not permitted to increase their consumption should they enjoy higher than expected returns. They therefore have no incentive to invest in stocks." • "[…assuming a coefficient of risk aversion of 5 and a bond return of 3 percent] For households consuming the optimal percentage of wealth each period, the optimal stock allocation increases with age, from 66 percent at age 65 to 88 percent at age 100. This reflects the age-related decrease in financial wealth as a percentage of the expected present value of remaining lifetime Social Security benefits. In contrast, the optimal equity allocation of households following the RMD strategy decreases from 78 percent at age 65 to 54 percent at age 100 because financial assets are increasing relative to Social Security wealth." • "…households following the 4 percent rule optimally allocate zero percent of their financial assets to stocks….These results illustrate how choosing a rule of thumb drawdown strategy can distort the household's investment allocation decision."

		of assets (risk-free v. risky asset portfolios), spousal age differences, assumed real returns, interest rates, risk aversion coefficients, expected longevity, several Social Security benefit levels, and, of course, decumulation strategies. The analysis also extends to "typical asset allocations" for households with retired investors over 65 with wealth ranging from $78,000 to $290,000 measured in 2008 dollars. The average allocation of such households was 46% stock, 8% bonds, and 46% short-term deposits. As a general rule, across most scenarios, the RMD decumulation strategy outperforms the alternatives and underperforms the theoretical optimal strategy. As a general rule, the RMD strategy provides too little early retirement consumption.	
2012	"Spending Flexibility and Safe Withdrawal Rates," Michael Finke, Wade D. Pfau and Duncan Williams Journal of Financial Planning vol. 25 no. 3 (March 2012) p. 44 – 51.	Shortfall risk analysis may not be the best preferencing criterion for selecting a retirement income strategy: "By emphasizing a portfolio's ability to withstand a 30- or 40-year retirement, we ignore the fact that at age 65 the probability of either spouse being alive by age 95 is only 18 percent. If we strive for a 90 percent confidence level that the portfolio will provide a constant real income stream for a least 30 years, this means that we are planning for an eventuality that is only likely to occur 1.8 percent of the time. And even that figure assumes that clients are unable to make adjustments to their spending later in retirement. So by relying on standard historical or Monte Carlo simulations to determine a safe withdrawal rate, clients may be unduly sacrificing much of their desired lifestyle early in retirement. The failure to include a client's willingness to adjust is an important shortfall of the shortfall literature. A common thread in the analysis is that all failures are counted the same, without regard to when the failure occurred or what percentage of the client's stated aggregate spending goal was funded. Such an all-or-nothing approach to retirement simulation is inconsistent with the way trade-offs are framed in retirement." It is not enough for a retiree to know that a shortfall might take place. The magnitude of the shortfall is also of importance—especially if the retiree has resources beyond the financial portfolio. Calculations must account for both the probability of	Good critique of the shortfall probability risk metric. However, the problems highlighted by the authors may be attributable to adopting an autopilot distribution strategy. Testing an inflexible spending policy is not the same as invalidating the shortfall risk metric's validity. Some of the article's insights date back to at least Sid Browne's 1999 article; and are developed in detail in "Shortfall-Risks of Stocks in the Long Run," Peter Albrecht, Raimond Maurer & Ulla Ruckpaul. Financial Markets and Portfolio Management vol. 15 no. 4 (2001), pp. 481 – 499. This article employs a utility-based preferencing criteria based on use of a certainty equivalent to consumption streams achieved by various allocations and withdrawal rates. A CRRA utility function underlies calculation expected utility of bootstrapped consumption paths. Maximizing expected utility often results in portfolios that fail to minimize shortfall risk.

		the failure as well as its magnitude should the failure occur. Note: this is a calculations that encompasses both unconditional and conditional probability. This study considers other sources of income such as annuities, pensions and social security benefits. It uses a bootstrap of historical returns (S&P 500 and intermediate-term U.S. government bonds) starting in 1926. The initial portfolio value is $1 million and the allocation is rebalanced annually. The initial withdrawal rate is adjusted annually for inflation; taxes and fees are not considered. The withdrawal strategy preferencing metric is based on expected utility. The combination(s) of asset allocations and withdrawal rates exhibiting the highest utility are identified. The portfolio either will or will not run out of money during the lifetime of the retiree. Thus, there are two consumption states to consider: (1) the good consumption state in which the retiree receives income from both the portfolio and from outside sources throughout his/her entire lifespan, and (2) the bad consumption state in which the portfolio is depleted during the life of the retiree and all end-of-life income must come from sources other than the portfolio. The risk model assumes that the non-portfolio sources of income are default free" "The magnitude of guaranteed income may then be viewed as a client's decumulation risk capacity." "With just two possible spending levels, we can simplify the utility analysis to consider certainty-equivalent dollar amounts. These are the lowest fixed real spending amounts with 100 percent certainty that retirees would be willing to accept to avoid the uncertainty associated with spending more while they still have remaining wealth and spending less when their wealth is gone. Certainty equivalence values are calculated with a formula using the spending amounts when wealth does and does not remain, the probabilities for these two outcomes, and a measure of the retiree's risk aversion. Whichever strategy provides the largest certainty equivalence is the one that maximizes the retiree's utility, providing the most satisfactory balance between the trade-offs. We analyze the situation for withdrawal rates between 3 percent and 9 percent, and stock allocations between 0 percent and 100 percent in 10	

		percentage point increments." For any utility-based metric, the form of the utility function and the coefficient of relative risk aversion are extremely important in identifying the optimal allocation/withdrawal rate tradeoffs. This essay employs a CRRA form of the utility function. Utility of consumption at each age is discounted for the probability of surviving to that age. When the coefficient of risk aversion is 1, utility is simply the log of consumption. The bootstrapped return paths identify the optimal stock allocations for each withdrawal rate. Utility is calculated when the retired couple have both $20,000 and $60,000 in annual non-portfolio guaranteed income. "We use the concept of certainty equivalence to measure the preference for retirement income paths among clients with a given level of risk tolerance." Interestingly, for many values of the risk aversion coefficient, the optimal withdrawal rate / asset allocation strategy involves rates and equity allocations that do not minimize shortfall risk. Even strategies that have a relatively high likelihood of depleting the portfolio during the life of the retirees are preferred over the shortfall minimizing strategies.	
2012	"Making Retirement Income Last a Lifetime," Stephen C. Sexauer, Michael W. Peskin, and Daniel Cassidy Financial Analysis Journal vol. 68 no. 1 (January/February 2012), pp. 74 – 84.	The authors suggest creating a benchmark for evaluating the success of a retirement income investment strategy. In this case, the benchmark consists of a suitable combination of TIPS and an ALDA—advanced life deferred annuity. The benchmark is investible—a retiree can implement this two-asset portfolio; and the income which it generates can serve as a measure to determine how well any investment program designed to "beat the benchmark" is performing. The authors suggest that the best benchmark would be an inflation-adjusted immediate annuity. However, : "…the cash flows from a real, immediate life annuity are unsuitable as a general benchmark for asset decumulation because the illiquidity of such a strategy is so burdensome that almost no one uses it." They call the benchmark a DCDB—defined contribution-decumulation benchmark. The minimum risk portfolio consists of a large weighting to a 20-year laddered portfolio of TIPS. This position provides inflation-adjusted income on a self-amortizing basis. A smaller weighting is made to an ALDA which is geared to provide the annual expected income level achieved in the last year of the TIPS ladder. This income level is paid on a	The authors point out: "Most of the academic research with respect to retirement strategies has focused on the right tail, where the concern is outliving one's assets. In our study, we attempted to bring much-needed attention to the left tail, where the concern is getting as much income as possible while a large majority of retirees are still alive." The simulation-based shortfall literature is only marginally relevant because "…it focuses on minimizing the probability of running out of money while still alive….we do not want to *minimize* the probability of shortfall—we want to *eliminate* it." They make the statement: "Thus far, no annuity money has even been lost by an annuitant." This is

		nominal basis for the remainder of life. Note: the expected inflation rate is assumed equal to the TIPS nominal Treasury breakeven rate for 20-year maturities. E[Inflation] = IRR equating lower coupon TIPS with 20 year nominal coupon treasury coupon. They acknowledge that "the inflation adjustment on TIPS may differ from the inflation experienced by the investor; but this risk is minor compared with not being protected from inflation at all." The remainder of the article compares implementation of the DCDB to other commonly-recommended strategies. These include: <u>Purchasing a constant dollar immediate annuity</u>: The initial payouts and subsequent payouts over the first 20 years are almost identical. However,, after year 20, the annuity continues to adjust for inflation. "Thus, over any life span, the inflation-indexed annuity either dominates or is a breakeven relative to the DCDB if the investor does not care about liquidity or counterparty risk." <u>Purchasing a target date fund</u>: The fund has an initial (age 65) allocation of 40% to global equities and 60% to bonds. Projected initial income is $5,118 for the DCDB and $3,800 for the fund. They state: "The trade-off facing the participant is complex and may be expressed as follows: Target-date funds provide additional utility from holding assets that are expected to grow in value (instead of being depleted), but this higher *expected* return is achieved by assuming the risk that the return will be unacceptably low and that the assets will be depleted within the participant's lifetime." <u>Purchasing an immediate nominal annuity</u>: Initially the nominal annuity's cash flows dominate those from the DCDB. If inflation remains as expected (expectation derived from difference between TIPS and nominal Treasury yield curves), it requires approximately 17 years for the cash flow from the DCDB to catch up to the nominal annuity's periodic payments. If inflation is greater than expected, the inflation-adjustments of the DCDB enable it to catch up more quickly. If inflation is very high, the DCDB "almost completely dominates."	incorrect. The authors point out that realized v. expected inflation is a key factor in determining the relative attractiveness of asset management elections. Compare their analysis with "Revisiting Retirement Withdrawal Plans and their Historical Rates of Return," Chris O'Flinn & Felix Schirripa available at: <u>http://ssrn.com/abstract=1641382</u> [2010]

2012	"The Annuity Duration Puzzle," N. Charupat, M. Kamstra and M.S. Milevsky Working Paper (2012) available at: http://www.ifid.ca/research.htm	A preliminary literature survey states that there are four strands to annuity literature: 1. Portfolio decisions and welfare optimization with and without annuities; 2. The Annuity Puzzle; 3. The "money's worth" of annuities ["…the evidence suggests that mark-ups (or loadings) on life annuity policies do not appear to be excessive"]; and, 4. Application of annuity pricing techniques to the valuation of pension plan liabilities. They define an actuarially fair annuity as one for which "…an insurance company will set the annuity price such that the price and future investment return on it exactly cover annuity payouts that the company expects to make." Using continuous time notation the price of an actuarially fair annuity paying $1 per year for life is: $$A(x,r) = \int_0^{\infty} e^{-rt}\, p(x,t)dt.$$ Where p(x,t) is the probability that an individual now age x will survive to at least age x+t, and r is the (constant) interest rate. To calculate the interest rate sensitivity of the annuity price they utilize the concept of modified duration where duration [D] is defined as: $$D = -\frac{1}{a(x,g,r)} \cdot \frac{\partial a(x,g,r)}{\partial r} \text{ or, } -\frac{\frac{\partial a(x,g,r)}{a(x,g,r)}}{\partial r}$$ Where 'g' is the guaranteed payment period, if any. The paper presents a table of actuarially fair prices and the corresponding duration values of single-life immediate annuities at various ages. The interest rate is the average 10-year swap interest rate during the sample period—4.35% per year. ["The 10-year swap rates are the best match to the average duration of the annuities we look at…."]The annuity pricing calculations also include the probability of survival as based on the Gompertz distribution with a mode of 88.18	The article describes an annuity as a "survival-contingent financial product."

		years and a dispersion value of 10.50 for males age 65; and a mode of 92.63 years and a dispersion value of 8.78 years for females age 65. As survival probability decreases the less expensive annuities become. Correspondingly, the duration of annuities declines with age. The theoretical pricing factors calculated from the pricing model for an actuarially fair annuity are compared to the observed pricing factors for actual annuities taken from an annuity price database. The values are very close: Theoretical factor for a 65 year old man = 12.6786 per $1 per year of lifetime income v. an average actual annuity factor of 12.784. At age 70, the values become 11.0530 and 11.190 respectively. Using the pricing model, the authors also calculate theoretical values for duration. For example, the theoretical duration value for a male age 65 for an immediate annuity (no guarantee period) is 9.65; and, for a male age 70, it is 8.33. The authors calculate the sensitivity of the annuity pricing factor by regressing the negative of the percentage change in the factor against changes in interest rates. This calculation results in empirical durations. The authors try a variety of independent variables such as the one week change in the 10-year swap rate, the five week change, and so forth. The empirical duration values (dependent variable) are, however, very low when compared to the calculated theoretical values. The highest R-squared values are attained when the 30-year mortgage rates are used in place of the 10-year swap rates. This suggests that annuity pricing factors are not particularly sensitive to interest rate changes. The authors speculate that annuity providers have a "…desire to smooth out the price changes and/or wait in order to get a better sense of the trend of interest rates. Another possible reason is that annuity providers from time to time have unbalanced books (i.e., mismatches of durations of assets and liabilities). As a result, they may deliberately offer uncompetitive annuity prices or delay price adjustments in order to allow themselves time to rebalance their books. For these reasons, annuity quotes can *in the short run* be unresponsive to interest rate changes." Finally, the authors note that annuity pricing factors adjust more rapidly to increases in interest rates than to decreases in rates.	

2012	"Revisiting the '4% spending rule'," Maria A. Bruno, Colleen M. Jaconetti and Yan Zilbering Vanguard Research (August 2012).	A short essay published by Vanguard Research. The authors employ a simulation model with inputs based on financial forecasts generated by the Vanguard Capital Markets Model. The model provides data to update the probability of long-term portfolio sustainability given a lower yield / lower real growth projection for the forthcoming decade. The question at issue is whether the anticipated change in expected returns materially impacts the viability of the 4% spending rule: "Vanguard's analysis supports a general initial withdrawal rate of roughly 4% for an investor with a 'moderate' allocation" [50% stock / 50% bonds]. The authors assert that the most important factor in determining a sustainable withdrawal rate is the planning horizon. For example, the moderate allocation can sustain [at an 85% or better success rate] a withdrawal rate of 9.6% for a retiree with a 10 year horizon. Lengthening the horizon in 5-year increments reduces the sustainable spending rate to 6.6, 5.2, 4.4, 3.9, 3.5, and 3.3 (at a 40-year horizon). The model makes no adjustments for mortality. The authors rerun the model assuming annual expense ratios of 0%, 0.25% and 1.25% for the 50-50 allocation for a 30-year planning horizon. A zero-cost portfolio can sustain, at an 85% confidence level, a constant-dollar spending rate of 3.9%. At a 25 basis point cost decrement, the portfolio's sustainably rate slips to 3.8%; and, at a 125 basis point decrement, it slips to 3.3%. "Although a 1% difference in annual portfolio costs may not sound that significant, the impact on a retiree's spending could be substantial. "	Updates conclusions regarding portfolio sustainability given projections for modest growth and low yields in the capital markets. Of primary interest is the presentation of how fees and expenses change sustainability rates.
2012	Walter Woerheide and David Nanigian, "Sustainable Withdrawal Rates: The Historical Evidence on Buffer Zone Strategies," <u>Journal of Financial</u>	The article opens with a list of the "four variables" of retirement portfolio decumulation strategy: 1. The portfolio's required value at retirement date; 2. The optimal allocation given projected withdrawals; 3. The initial withdrawal rate; and, 4. The withdrawal strategy. "Unfortunately, as much as many people would like to think so, there is no such thing as the correct or optimal answer to what each of these	This article provides a counter-point to advocates of the portfolio "bucketing" approach. The gist of the argument is that, for most allocations and planning horizons, the cash drag of the investment reserve account generally overwhelms the benefits of portfolio variance reduction. One of the central assumptions of the Woerheide/Nanigian argument is that equity

	Planning Vol. 25, No. 5 (May 2012).	four variables should be…." The article directs attention to a commonly recommended withdrawal strategy that includes "…use of buffer zones as a component of a decumulation portfolio." The buffer zone strategy involves putting an amount equal to one or more years' income needs into a cash account. In any year in which the portfolio decreases in value, the investor takes the withdrawals from the cash account; in any year in which the portfolio increases in value, the withdrawals come from liquidating investment positions. This strategy avoids selling securities in a down market. Portfolio value increases are also used to restore the cash account to its target level. The authors test this strategy by replicating the historical back test method of Cooley, Hubbard, and Walz. They test buffer zones extending from one to four years' income. Allocations range from 100% bonds to 100% stocks in increments of 25%. They examine overlapping periods of 15, 20, 25, and 30 years across a database of returns on the S&P 500, Lehman Brothers Long-term Bonds and One-Month T-bills from 1926 through 2009. Withdrawal rates range from 3% through 9% of initial portfolio value. "The key to determining whether there is value in the strategy of maintaining a buffer zone is whether such portfolios allow withdrawal rates to be sustained a higher percentage of times than if one does not use a buffer zone." The first test involves an examination of withdrawal rates for various allocations over the designated planning horizons. The grid of portfolio success percentages assumes a four-year income buffer zone. Empirical results indicate that "Regardless of the size of the buffer zone, the CHW strategy clearly dominates the buffer zone strategy." A second grid presents results across buffer zones ranging from one to four years income at various asset allocation weightings. Again, the preponderance of evidence suggests that, except in the case of a 100% bond allocation, using a buffer zone produces relatively unfavorable results in the majority of outcomes. Finally, a third grid demonstrates the outcomes over allocations, time horizons, and withdrawal rates: "As soon as equities are introduced into the portfolio or the withdrawal rate goes	returns are not autocorrelated—i.e. exhibit no statistically significant tendency for mean reversion: "…the results from regressions of our time series of stock and bond return s on their respective lagged one-year values provide no evidence of autocorrelation in market returns. The results from both ordinary least square and Yule-Walker estimation methods...show that lagged returns did not have an impact on current returns at even the 20 percent significance level. This implies that mean reversion, at least at the annual frequency, is simply an illusion in clients' 'minds." This assertion contrasts with several other studies. For example, see: Noel Amenc, Lionel Martellini, Vincent Milhau and Volker Ziemann, Asset-Liability Management in Private Wealth Management EDHEC-RISK Asset Management Research (September, 2009). The authors of the EDHEC monograph use a VAR (vector autoregression) model to explore the merits of portfolio optimization in an asset/liability management context. The outputs of their VAR model imply that stocks are not particularly risky for the long-term investor: "This effect is explained by the presence of implied mean-reversion in stock returns." By contrast, the volatility of T-Bill investments increases with the planning horizon: "…due to the uncertainty involved in rolling over short-term debt in the presence of stochastic interest rates." The question of mean reversion in equity markets is still an open one.

		above 3 percent, the use of a buffer zone strategy of any sort—one year, two years, three years, or four years—is more likely than not to leave the investor worse off than if he or she simply set up an investment portfolio with a static asset allocation….the results from this study show that a static asset allocation strategy is superior to a buffer zone strategy at minimizing longevity risk….”	
2012	The Retrenchment Rule: When It's Too Late to Save More For Retirement, Gordon B. Pye (GBP Press, New York) 2012	The book's premise is that many, if not most, investors fail to generate a sufficient amount of capital to finance sustainable lifetime income at a level that can permanently preserve their standard of living. Additionally, many investors do not have the option to continue in their jobs past normal retirement age. Given the economic reality which these retirees face, "…the key question in spending retirement assets [is] not achieving sustainable withdrawals, but when to retrench." There is no doubt that spending cuts must occur. However, if the initial cut in spending is so great that it forces an unacceptably high level of immediate economic pain, then a more gradual schedule of reductions may be the preferred alternative provided that the reductions do not result in future withdrawals the amount of which become too low. Pye advances the following argument: "The initial withdrawal should be the one required to provide the prior or desired standard of living subject to a limit. That limit should be the largest withdrawal that could be made without increasing too much the risk of low withdrawals later in retirement. Moreover, this limit should be based on the discount rate that gives the best results in a series of simulated withdrawals." The discount rate that gives the best results—i.e., balancing sufficient current income against the probability-adjusted likelihood of lower future cash flow—is the *Retrenchment Discount Rate*, the application of which is the "Retrenchment Rule." Pye's approach to retirement withdrawal strategy is aimed at retirees who exhibit a strong preference to maintain their current standard of living but who own resources insufficient to support the required cash flows throughout long planning horizons: "…the present value of the withdrawals required to sustain their existing standard of living exceeds the value of their initial investment. Thus,	Pye's central focus: what do you do when you retire and find that a 4% distribution from your does not pay for necessary expenses? A primary control variable for dynamic asset management is the withdrawal amount. This puts the emphasis on retirement income monitoring. The feasible withdrawal amount is also a function of age: "Reasonably sustainable annual withdrawals for those with a maximum remaining lifetime of 20 years are 2 percentage points or more above the sustainable withdrawals for those in good health at the beginning of their retirement." Pye advances a preferencing criterion for risk models: "a probability distribution for the investment returns that is both realistic and reasonably conservative." Pye's model suggests that investment in fixed income is suboptimal compared to annuitization when the initial retirement withdrawal amount is unsustainable. However, for portfolio withdrawals that are reasonably sustainable the flexibility offered by fixed income investments may make them preferable to annuities. "A simple way to undertake a limited amount of annuitizing is to defer the start of Social Security." Social Security

		these retires will have to retrench." Drawing on observations similar to those made by Irving Fisher [Note: Fisher is not mentioned in the book], Pye states: "The Retrenchment Rule weighs whether this immediate pain is justified given that the future funds provided may turn out to never be needed." Given the book's premise, following the four percent initial withdrawal strategy will not work for many retirees because "…many individuals have to begin retirement without having saved nearly enough to cover essential expenses with a 4 percent withdrawal." Furthermore, even if the beginning nest egg is large enough to entertain application of the 4 percent rule, the rule itself is suspect because it is based only on a small number of historical observations. Pye illustrates the difficulties with the 4 percent rule. He conducts two tests assuming an annually rebalanced portfolio of 75% S&P 500 and 25% intermediate US government bonds. The first test applies the 4 percent rule to an age 65 retiree, in good health, beginning in 1991. Despite the bear markets in stocks both early and late in the first decade of the 21st century, the retiree was able to sustain a 7.5 percent withdrawal rate through 2010—the time of the book's composition. However, if retirement started in 1966, an initial withdrawal strategy of 7.5 percent would have required gradual reductions so that the portfolio would avoid depletion. By 1982, the retiree could have withdrawn only 2.1 percent of the initial portfolio value. As it turns out, 1966 was the worst year to retire of all years since 1926. By contrast, 1991 was the most favorable year for retirement. Pye's point is this: Following the 4 percent rule would result in a significant and unnecessary permanent reduction in income for the 1991 retirees. For 1966 retirees, the 4 percent rule would have curtailed early retirement income substantially [7.5% - 4.0%] and would have fully depleted the portfolio by 1996 [30 years]. "Those who withdrew 7.5 percent in 1966 eventually had to retrench somewhat more. But they had 10 years [before they hit the 4 percent withdrawal target level] over which to plan and make these adjustments….Also, with	deferral is a painless annuitization strategy. "For a reasonably sustainable initial withdrawal allocating about a quarter of the portfolio to fixed incomes seems to be the better strategy. As the initial withdrawal becomes less sustainable, however, annuitizing clearly becomes the better choice. And the portion of the portfolio that is annuitized should be increased to 50 percent." However, "…annuities may be somewhat less attractive than fixed incomes for those who have saved enough to achieve a sustainable standard of living." This puts Pye's work squarely in the annuity-as-safety-net camp. Information on current annuity payouts is available at brkdirect.com or immediateannuity.com Unlike many retirement income risk models, Pye's model treats inflation as a random variable: "…the volatility of inflation and its persistence has been based on averages over 1960-2009. Thus, in the simulations there is a possibility of an episode of high inflation….In particular, the annual rate of inflation in the simulations is the sum of a weighted average and a random drawing. The weighted average is .75 of inflation in the prior year and .25 of its expected normal value in the long run of .03. The random drawing is from a normal distribution with a zero mean and standard deviation of .02. With this relation whatever the current rate of inflation, inflation is tending to gradually revert to its expected normal value in the long run of .03. The weights of .75 and .25 are those of the weighted

		good health, and perhaps some pent-up enthusiasm for activities such as travel, increments of spending early in retirement are likely to provide more satisfaction than equal increments later on." Bottom line—the 4 percent rule mandates an immediate draconian reduction in many retirees' standard of living. The Retrenchment Rule requires a more modest type of "glide path" reduction and, of crucial important, this reduction is implemented only if an unfavorable sequence of returns unfolds. "This downside risk is the cost of somewhat lower withdrawals later in retirement….But this cost is limited if retirees do not survive very long. And in any case this cost is offset by significant benefits earlier in retirement." Over all historical periods since 1926 withdrawals beginning at the 7.5 percent rate stay above the 4 percent rate in 70 percent of the cases. Cuts should be made when the withdrawal rate is unsustainable. But retirees do not know *ex ante* the future sequence of returns, inflation, health status with which they will be faced. Therefore, the utility maximizing retiree will avoid making immediate and economically painful cuts when such cuts may, in fact, prove to have been unnecessary. Any retiree with a positive time preference for consumption will wish to avoid painful retrenchment forced upon them at the beginning of their retirement by a four percent withdrawal rule. "Future investment withdrawals of the same real value are worth less the further in the future they are expected to occur. One reason is that it becomes increasingly less likely that retirees will survive that long. Another is that many have pent-up plans for activities after they retire such as travel, but these desires become satisfied. Also, later on withdrawals of the same real value are likely to provide less satisfaction as lifestyles slow and mobility declines even for those who remain in as good health as can be expected. To reflect this decreasing value the Retrenchment Rule discounts future withdrawals by a constant rate of interest for each year in the future until they will be made. The value of this discount rate is selected by simulating the use of the rule with	average that best fits the observed annual inflation rates over 1960-2009. This is when the expected inflation rate in the long run is assumed to be the average inflation rate over 1960-2009 of .041. The standard deviation of the random drawing of .02 is the standard deviation of the error by which the estimated weighted average fits the observed inflation rates each year." "…unexpected inflation…is assumed to have a negative effect on the expected return on stocks equal to 3.0 times its value….each year in the simulations the return on stocks is set equal to the sum of two components. One is a deduction equal to 3.0 times that component of inflation for the year that is a random drawing. The other component is another drawing from a normal distribution….This distribution has a mean of .07 and a standard deviation of .170." "…since the early 1960s one percentage point of unexpected inflation has been associated with a 2.2 percentage point reduction in the real return on the intermediates. To reflect this relationship in the simulations the return on the intermediates each year is set equal to the sum of two components. One is a deduction equal to 2.2 times that component of inflation for the year that is a random drawing. The other component is another drawing from a normal distribution set to keep the mean and standard deviation of the annual real return on intermediates equal to the base case values of .03 and .07. The mean and standard deviation of this latter distribution are respectively .03 and .0536."

		different rates over a hypothetical retirement period. This is to see which rate gives the best performance. The discount rate selected is called the Retrenchment Discount Rate, or RDR." Although Pye does not use the term "feasibility condition" when discussing decumulation strategies, he advances an argument that is compatible with this concept: "To reflect their declining value future withdrawals are discounted by a constant rate of interest for each year in the future until they occur. [Note: the vector of discounting is not based on the current yield curve—as would be the case for pricing annuities] This discounting gives the present value of a future withdrawal based on the discount rate that has been used. Adding up these discounted values for each of the withdrawals gives the present value of the stream. This is the present value of the future withdrawals in real terms needed to provide the existing standard of living. The present value of the funds available to make these withdrawals is the current value of the investment portfolio. Suppose the present value of the stream of withdrawals required to provide the existing standard of living is less than the value of the portfolio. Then retrenchment is not required….there are sufficient funds to provide the existing standard of living in the future. On the other hand, suppose that the present value of the stream to provide the existing standard of living exceeds the value of the portfolio. Then retrenchment is required." A key issue is how to determine the appropriate discount rate. The best discount rate is the RDR; and this rate is determined by calculating the present value of the stream of future withdrawals required to provide a given standard of living—the desired discount rate—limited, in turn, by the discount rate that gives the "best" results—where best is defined as the most appropriate tradeoff between current cash flows and possible lower future cash flows—across the distribution of simulated results. Simulations assume a maximum limit on life expectancy of age 110. Note: Compare Pye's choice of discount rate to Hughen, Laatsch & Klein	In modeling emergencies, Pye notes that they have "…the same effect on the investment portfolio as a financial bear market. But unlike a bear market there is no possibility that such losses will later be reversed." "In particular, suppose each year that there is a .05 chance that an emergency will occur. Over a 20 year period the chances that a given number of emergencies will occur under this assumption can be calculated using the binomial distribution….There is a .36 chance that there will not be any emergencies. There is a .38 chance of one, a .19 chance of two, a .06 chance of three, and a .01 chance of more than three. On average over the 20 years one emergency is expected." "It will be assumed that the outlay for the emergency is equal in real terms to 20 percent of the initial value of the investment. …When an emergency occurs the Retrenchment Rule may require a significant reduction in the withdrawals that are allowed and in the standard of living. On the other hand, if investment returns have been favorable a large extra withdrawal for an emergency may be accommodated with little difficulty." "When the risk of the emergencies is included there is no way that a 4 percent withdrawal can still be considered as reasonably sustainable. There is now a .37 chance that the withdrawal will be less than .9 of its initial value at age 95. And there is a .22 chance that it will be less than .5 of its initial value….With the emergencies only close to 3 percent can be withdrawn to achieve the same sustainability as 4 percent without the emergencies." Pye discusses the role of valuation in determining

		[“Withdrawal Patterns and Rebalancing Costs for Taxable Portfolios (2002) which develops an “equivalent payment rule” by using the total return on the portfolio over a particular time period] For example, suppose that a group of retirees in good health require an initial withdrawal rate of 8% from their portfolio. At this rate, they can avoid making a painful cut in their standard of living. If the discount rate with the best tradeoff between current cash flow adequacy and the likelihood of future painful income reductions is 6%, however, then some immediate retrenchment must occur: “Suppose that the value of the investment is 100 so that the required withdrawal is 8. Input n=45 [110-65=45], i=6.0, PMT=8, and FV=0. Calculate the present value of this stream getting 131.1, which exceeds the value of the investment of 100….When the withdrawals have declined from 8 to 6.1 their present value has fallen to 100. As this is the value of the portfolio this withdrawal is allowed. It is the largest withdrawal allowed by the Retrenchment Rule with a discount rate of 0.6.” Another way of looking at this calculation is that “It is the largest fixed annuity that can be obtained each year from the investment for the longest that the retiree might live. This is when the investment earns a return equal to the assumed discount rate of .06. A larger stream of withdrawals than this annuity will have a present value that exceeds the value of the investment and will require retrenchment.” [Note: the discount rate for this hypothetical annuity is not the current term structure of interest rates. It is also not the expected portfolio rate of return. Rather, it is a discount rate derived from a trial and error process over the entire distribution of simulated portfolio returns]. Here is how Pye’s example plays out in a dynamic portfolio monitoring and withdrawal setting: “For making the simulations it is assumed that 6.1 is withdrawn at the beginning of the year and spent over the year. Another withdrawal is not made until the beginning of the following year. Suppose the real return on the portfolio over the coming year is	the expected return of a portfolio: “…the sustainable amount that could have been withdrawn from a given portfolio in the early 1980s was more than 50 percent higher than in the late 1990s. Thus, more than a 50 percent larger portfolio would have been required in the late 1990s to sustainably cover the same living expenses.” Pye asserts: “…it is prudent to accumulate significantly more savings before retiring at a time when valuations are high.” The issue then becomes one of (1) identifying valuation metrics and (2) predicting how the market will perform given the level of these metrics, the magnitude of change, and the direction of change. Unfortunately, predicting market response to changes in valuation metric is fraught with difficulties: “…changes in prior valuations account for only a very small portion of the changes in subsequent realized investment returns. Allocations based on valuations are therefore frequently going to be wrong after the fact….” Pye identifies a series of valuation metrics (e.g., price/earnings ratio, dividend yield, etc.) and, when the metrics evidence an abnormal valuation level, expected return is adjusted accordingly. Although it may be tempting to change the stock/bond weighting because of changes in relative valuation, this is often counterproductive: “This is not surprising as changes in valuation account for only a very small portion of changes in realized investment return.” Irrespective of how well valuation metrics act as predictors of future security returns, it is doubtful that S&P 500 returns come from a process that can

		.03. The value of the portfolio at the beginning of the following year before a withdrawal is made is then (100-6.1)(1.03), or 96.72. This is in real terms because the return of .03 is in real terms. The 96.72 is 96.72 percent of the initial investment because the initial investment is equal to 100. The Retrenchment Rule is now applied just as it was initially to get the next withdrawal. All of the inputs have changed, however, except that the discount rate is still .06. To calculate the largest allowed withdrawal input n=44, PV=96.72, i=6.0, and FV=0. Calculate PMT getting 5.9 as the largest withdrawal allowed by the Retrenchment Rule. As 6.1 must be withdrawn to sustain the prior standard of living some retrenchment is required. Note: the formula explicitly amortizes terminal wealth to a value of zero over the planning horizon [FV=0]. This is an income-oriented strategy where no value is assigned to bequests. Suppose that a severe bear market occurs over the coming year and that the real return is -.25 instead of .03. In this case the value of the investment at the beginning of the following year is (100-6.1)(.75), or 70.42. Calculating PMT in this case the largest allowed withdrawal is 4.3. Now a major retrenchment is required. The withdrawal must be reduced from 6.1 to 4.3. Suppose only 4.0 had been withdrawn initially instead of 6.1. In this case a major retrenchment is required initially, but no retrenchment is necessary at the beginning of the following year. The value of the investment is now (100-4.0)(.75), or 72.00. But this is only slightly higher than the 70.42 obtained with the 6.1 withdrawal. Thus, suppose a severe bear market occurs in the coming year. It then makes little difference if the Retrenchment Rule is used with a .06 discount rate or there is a 4 percent withdrawal….The advantage of the higher withdrawal is the very strong chance that a much better return will be earned over the coming year. In this event major retrenchment may never be necessary….If only 4 percent is withdrawn initially a major retrenchment occurs for sure immediately.”	be characterized by a normal distribution: “This is because the available observations are not independent drawings from the same distribution each year.” [i.e., the equity risk premium is dynamic]. Pye notes: “…given the high volatility of the annual returns on the S&P 500 any average of these returns is an unreliable estimate of their true expected annual return.” This, in turn, suggests that the Central Limit Theorem is not applicable. Pye also points out that standard deviation might also be modeled as a draw from a probability distribution as opposed to taking on a pre-specified parameter value. Here is Pye’s stylized example: “Suppose, for instance, each year that the chances are .25 that the standard deviation is .32, and the chances are .75 that the standard deviation is .095. Beforehand, the standard deviation of the returns each year is .18….This is because $(.75(.095)^2 + .25(.32)^2)^{.5} = .18$. But now the chances of low returns have increased….Instead of being expected only once every 137 years a return of -.37 is now expected once every 47 years. And a calculation using the binominal distribution shows that the chances of 3 or more such low returns over 84 years is now equal to .26 instead of only .02.” In modeling taxes, Pye acknowledges “…it may be necessary to realize capital gain or loss by selling stocks to obtain cash for making withdrawals. This gain or loss affects the tax owed for the year. The size of the tax in turn affects the size of the withdrawal necessary to pay that tax. In simulating taxes it is convenient to avoid this simultaneity by determining the size of the tax payment each year

		The process of recalculating the allowable withdrawal amount continues each year through a maximum age of 109. This includes simulation of the distribution of future returns and recalculation of the most appropriate discount rate. "Suppose next that the discount rate is .08 instead of .06. In this case the largest allowed initial withdrawal is the value of the annuity obtained when the investment earns a return of .08 instead of .06. The largest allowed initial withdrawal in this case rises from 6.1 to 7.6 percent. If the required withdrawal to avoid initial retrenchment is still 8 percent the initial withdrawal increases to 7.6 percent." This is the limit imposed by the annuity calculation for n=45 and FV=0. If a major health crisis occurs, planning over the long horizon may be sufficiently conservative so that funds will be available to provide the needed liquidity. [Note similarities to the AIR assumption used to calculate withdrawal rates from variable annuities. The higher the initial AIR the less likely it is that future withdrawal amounts will grow.] The gist of Pye's observations is that a higher discount rate [RDR] will provide higher withdrawals early in retirement. However, the high withdrawals may deplete the portfolio more rapidly than a withdrawal strategy calculated under a lower RDR. Selection of a high RDR results in a higher probability of future retirement retrenchment. [Note: Assuming a normal return distribution with stocks having an expected return of .07 and a standard deviation of .18, all retirees will select a discount rate of .08 because the expected withdrawals at this rate are higher than those for any lower RDR. Thus, .08 exhibits a property of dominance at each age. For example, a discount rate of .10 offers a higher probability of a lower withdrawal at age 90 than does an RDR of .08. However, a retiree may still select an RDR of .10 if his or her time preference factor ("impatience") is high. If, however, either of the two parameters estimates are inaccurate—e.g., lower mean—then .08 may not be the optimal RDR. Pye suggests that .08 is probably not too far	using a similar procedure to that in the tax rules. Each year an estimated tax is paid that is independent of the transactions at the beginning of that year and any tax owned for that year. In addition to this estimated tax each year there is another tax payment or rebate to allow for the difference between the tax owed for the prior year and the estimated tax pad for the prior year. Generally, the estimated tax paid each year is assumed to equal the tax owed for the prior year." Note—this is a payment of estimated taxes with a retrospective adjustment. Net capital losses are carried forward after deducting $3000 from AGI.

		from the optimal value; and, therefore, investors may still prefer to stick with it because they will be adequately compensated by the opportunity to avoid severe retrenchment early in their retirement]. When simulating future investment returns, Pye asserts that it is important to distinguish between markets within the range of "normal valuations" and markets that exhibit an "abnormal valuation." The valuation that exists at the time retirement withdrawals begin can significantly affect the size of sustainable withdrawals. Likewise, the risk model must account for variations in the standard deviation of returns. "This is because in reality the volatility of the annual returns is likely to vary significantly over time. Such random variation causes an increase in the risk of large negative returns, and the risk of severe bear markets….Random variation in the standard deviation of the returns has a similar effect as random variation in the expected return. Either type of random variation reduces the sustainability of investment withdrawals." Assuming a market with normal valuation, Pye makes a case for choosing a discount rate of at least .08. This rate is based on simulations assuming that the expected return on the S&P is 7% with a standard deviation of 18%. Pye assumes that returns for the S&P 500 and intermediate government fixed income securities are independent drawings. The valuation of fixed income issues is calculated by the real return available from rolling over a portfolio of short-term T-Bills over the prior year. Based on data from 1960 through 2009 the distribution of real returns has a mean of 3% and a standard deviation of 7%. "When the discount rate increases above .08 the expected withdrawals increase earlier in retirement, but decline later on. Some retirees might be willing to accept the lower expected withdrawal later on to get the higher expected withdrawals earlier in retirement. Making such a choice depends on their willingness to accept future costs so as to achieve earlier benefits…Instead of needing to quickly make some major decisions about retrenchment there is likely to be a number of years to decide how to cut so as to retain as much satisfaction as possible."	

		"…presumably almost everyone will choose an RDR of at least .08. A discount rate of .06 offers a small chance of higher withdrawals late in retirement when investment returns have been unfavorable. But this benefit appears too small to compensate for the lower withdrawals that a discount rate of .06 provides otherwise." Taxes are also incorporated into Pye's retirement income risk model. For taxable accounts, "It is assumed that the cost basis of any stocks sold is the same proportion of their total cost basis at that time as the proportion of stocks sold is to their total value." Tax payable depends on the specific economic situation of each retiree. Withdrawals can come from both taxable and tax-favored accounts, the level of other income will differ from taxpayer to taxpayer, etc. Taxes, however, have little impact on the choice of RDR. Finally, Pye discusses the importance of the subjective time preferencing / discount rate. He notes that a key objective of retirees "…is to avoid low withdrawals at any time over their retirement. But retirees are likely to be more averse to low withdrawals early in retirement than later on….If retirees are less averse to low withdrawals later in retirement this difference should be reflected in the utility function. To do so later withdrawals can be multiplied by an equivalence factor to make their effect on utility comparable to that of earlier withdrawals." This means that the retirement income goal is to maximize time-adjusted utility. For a simple utility function, that each future time-adjusted withdrawal generates utility equal to that of the initial withdrawal divided by the time equivalence factor for the applicable future year. Because the sequence of withdrawals must also satisfy the budget constraint, the initial withdrawal must be divided by (1 + the risk free rate)(1 + subjective time preference rate)(the small cross product). Retirees wishing to spend more in the early years of retirement will have a larger subjective time preference rate. When the rate is equal to zero, (1+0)=1 and the discounting is simply 1 + risk free rate. This is a retiree that prefers a constant standard of living. Often, however, retirees "…prefer	

		some downward slope in their withdrawals."	
2012	"Annuities vs. Safe Withdrawal Rates: Comparing Floor-with-Upside Approaches," Michael Kitces Private Wealth Management Conference CFA Institute (2012).	An annuity strategy provides a floor on income that cannot be outlived. However, the 4% withdrawal rule is also effectively a floor strategy. Although it does not guarantee lifetime income, it has never failed in market history. Thus an annuity may be seen as an alternative way to achieve safe withdrawals but with a loss of liquidity as well as upside return potential. The safe 4% withdrawal rate is calculated based on a 30-year planning horizon whereas the annuity—which is backed by an insurance company guarantee, pays periodic income irrespective of the annuitant's actual lifespan. "To truly fail, the couple need to be unlucky enough to live through an investment environment worse than any found in history (i.e., no principal left at the end) and be the (approximately) one couple in six who are still alive at the 30-year time horizon. When you combine low-probability investment disasters and low-probability longevity scenarios, you end up with some astonishingly low-probability scenarios, many of which could be further 'saved' by small midcourse corrections." Kitces argues that "…extraordinary investment shocks that could destroy a 30-year safe withdrawal rate could also threaten an insurance company….the failure rate of a 4% withdrawal rate is about the same as the failure rate of an insurance company rated AA or better….Simply put, the tail risks are correlated." "The bottom line is that choosing between immediate annuities and safe withdrawal rates is not a decision about *whether* to use a floor-with-upside approach; it's about choosing *which* floor is preferable in light of the tradeoffs the decision entails."	Implies that a 4% withdrawal strategy ["…has never failed in market history"] is almost as conservative a strategy for producing safe and sustainable income as is the purchase of an annuity. This is especially true when the probability of insurance company default is considered. Both a 4% withdrawal rate strategy and an annuity offer "floor protection," but the annuity truncates upside potential.
2012	"Is The Retirement Plan with the Lowest 'Risk of Failure' Really the Best Choice?" Michael Kitces Blog Post	Although many Monte Carlo simulations demonstrate the probability of running out of money, most clients will not blindly adhere to a spending policy that is obviously unsustainable. "…if the plan is clearly heading for ruin, clients begin to make adjustments." [Note: it is not always evident when a plan is headed for ruin]. Kitces directs attention towards a "risk of adjustment." "…the plan with the lowest risk of adjustment may not	An argument that tempers the somewhat pervasive enthusiasm for high equity weighting in the retirement portfolio. High equity allocation improves portfolio sustainability on average—however, it also increases the likelihood of having to make a severe downward adjustment on periodic

	(2012) available at: https://www.kitces.com/blog/is-the-retirement-plan-with-the-lowest-risk-of-failure-really-the-best-choice/	be the ideal plan for the client….” A plan with a relatively low risk of adjustment may require a draconian adjustment conditional on the adjustment requirement. Alternately, a plan with a higher risk of adjustment may require a less disruptive change in retirement income: “In other words, it may be better to follow the plan that leads to a slow failure – which can be easily fixed with mid-course adjustments –than a fast failure.” In Kitces opinion the investor must consider both the risk of adjustment and the potential magnitude of the adjustment. These observations temper the advice to tilt the retirement portfolio towards equity: “Due to the heavy exposure to equities, a severe bear market is a rather destructive event….failure may be relatively uncommon, but if it does occur, it requires a *big* adjustment.”	income.
2012	“Recent Developments in Life Annuity Markets and Products,” Mark J. Warshawsky Chapter 2 in <u>Retirement Income: Risks and Strategies</u> Massachusetts Institute of Technology (2012), pp. 1 – 36.	The author develops a “pricing simulation model” for SPIAs based on annuitant mortality, gender, and the term structure of interest rates on Treasury securities. The author estimates that antiselection adds “…about 10 percentage points” to the cost of annuities. “This is aside from the usual marketing and sales costs and margins for profits and reserves….” For example, “…at the end of May 1984, a $100,000 premium bought a monthly payout of $1,134 for a couple. By the end of June 2003, however, as interest rates fell to secular lows, the same $100,000 bought only$503 in fixed monthly lifetime benefits….By December 30, 2008, the fixed monthly lifetime payment on newly issued SPIAs had dropped to only $417 before recovering throughout 2009 to the $500 level.” He then compares the pricing outputs against the actual quotes available to annuity contract buyers. The ratio gives the “money’s worth” of actual annuity contracts where such contracts contain embedded loads. Over the period February 1, 2002 through December 31, 2009, the ratio averaged just over a value of one for the particular insurance company under consideration. However, this includes the highly volatile period at the end of 2008 / beginning of 2009. During this time the ratio value spiked much higher than one. “By the end of 2009, more typical pricing	This is from a collection of essays all of which were authored or co-authored by Mark Warshawsky. The book’s Introduction and Overview chapter makes the following observations: • “…the utility gain from obtaining access to an actuarially fair inflation-indexed life annuity for a 65-year-old male with quite modest risk aversion would be equivalent to his getting a 50.2 percent increase in wealth.” • “…a fixed income flow from an annuity frees the retired household from having to manage the size of withdrawals from its pool of savings. It also blocks the opportunity to entertain rash temptations for large, imprudent expenditures. It is also likely that simplicity is more highly valued and becomes more appropriate as the retired household ages and its cognitive abilities decline.” • “PPA [Pension Protection Act] also removed

		relationships were returning." Over the period February 2, 2005 through December 31, 2009, the money's worth ratio for inflation-indexed annuities "…has averaged at 0.98, but there has been a lot of volatility around that…." Note: see "The Annuity Duration Puzzle," N. Charupat, M. Kamstra and M.S. Milevsky Working Paper (2012) available at: http://www.ifid.ca/research.htm for a discussion on the "stickiness" of insurance company discounting factors in volatile interest rate periods.	a stringent fiduciary requirement from the Department of Labor on DC plans offering annuities, mandating the 'safest available' annuity." The Introduction also provides some nice definitions of economic terms: Risk aversion—a consumer or investor values a certain income or wealth holding more than an equal amount that involves risk or uncertainty. Elasticity of intertemporal substitution—measures the degree of substitutability of consumption across time. A lower value means that the consumer is more concerned about consumption smoothing from year to year because fluctuations are painful. Vector autoregression—an econometric model that estimates the evolution and interdependence between multiple time series. All the variables in a VAR are treated symmetrically by including for each variable an equation explaining its evolution, based on its own lags and the lags of all the other variables in the model. A VAR is a relatively theory-free way to estimate economic relationships. Monte Carlo simulation—a technique used to approximate the probability of certain outcomes by running multiple trial runs, called simulations, using random variables.
2012	"Longevity-Insured Retirement Distributions: Basic Theories and Institutions," Mark J. Warshawsky and Jeffrey R. Brown	The article reviews the Annuity Equivalent Wealth [AEW] calculations from Brown, Mitchell and Poterba (2001). For a 65-year-old male with log utility [CRRA = 1], the AEW is 1.502. The article notes: "With CRRA utility, the reciprocal of the risk aversion parameter is the elasticity of substitution in consumption across periods. A low risk aversion coefficient corresponds to a high willingness to substitute consumption intertemporally."	The appendix to this chapter provides a step-by-step guide to the calculation of Annuity Equivalent Wealth. Intuitively, since annuities relax the budget constraint—less wealth is needed if an actuarially fair annuity is available to finance a target amount of lifetime income, the annuitant will realize enhanced utility. Assuming that the utility of

	Chapter 3 in Retirement Income: Risks and Strategies Massachusetts Institute of Technology (2012), pp. 57 – 84.	The article asserts that annuities are not attractively priced for the general population: "First, insurance companies selling annuities need to cover administrative and marketing expenses and earn a competitive accounting profit. Second, to the extent that individuals who choose to annuitize have longer life expectancies than the general population, insurance companies need to adjust their prices to reflect this fact." They estimate that "…administrative costs account for a 3 to 5 percent reduction in annuity payouts….adverse selection is responsible for an 8 to 12 percent reduction in annuity payouts." The article identifies a list of factors that may serve to reduce the value of a nominal annuity. For example, the welfare gains from annuitization are not as great for a married couple as they are for individual annuitants. Additionally, nominal annuities are particularly vulnerable to persistent inflation. The authors calculate that the purchase of a nominal joint and 50% survivor annuity in a 3.2 % annual inflation environment by a household with a 65-year-old man and 62-year-old woman who have 50% of wealth pre-annuitized generates an AEW of 0.88. "…that is, the couple faces a 12 percent load factor on their annuity purchase….In fact, for a risk-aversion coefficient of one, complete annuitization would actually lower utility."	consumption can be expressed by a one-period function incorporating the Arrow-Pratt coefficient of relative risk aversion [β]: $U(C_t) = \frac{C_t^{1-\beta}}{1-\beta}$ and that 1/ β is the intertemporal substitution in consumption, the optimal consumption can be determined. Assuming log utility and no difference between the personal discount rate and the risk free rate, "In the case with no annuities, consumption declines over time due to falling survival probabilities, whereas with annuities, the consumption profile is level for the rest of the individual's life. Thus consumption with annuities is greater than or equal to consumption without annuities in all periods, and thus utility is higher." [Note: utility is higher only under the assumption of CRRA—which has been criticized as an unrealistic assumption]
2012	"Good Strategies for Wealth Management and Income Production in Retirement," Mark J. Warshawsky and Gaobo Pang Chapter 7 in Retirement Income: Risks and Strategies Massachusetts Institute of	This article focuses on a combination strategy of mutual fund investing with systematic purchases of SPIAs during retirement. "The success criterion for the strategy search is to minimize the shortfall risk, which is defined as a weighted probability of real income and wealth balances falling below certain thresholds, in a stochastic model. The objective somewhat departs from the conventional analytical assumption that investors maximize their expected utility over consumption and bequests." Consumption shortfall is defined as a distribution that falls below a pre-established threshold amount. A shortfall in wealth is defined as lacking sufficient funds to cover uninsured contingencies or to leave a targeted level of bequests. Mitigation of consumption shortfall risk competes with mitigation of wealth shortfall risk.	Model incorporates insurance company failures. Preferencing criteria based on shortfall risk minimization. The shortfall risk metric is extended to include the risk of falling below a threshold income target in any period throughout the planning horizon.

	Technology (2012), pp. 163 – 178.	Mutual fund investment enables the retiree to improve the budget constraint if returns are good, but exposes the retiree to significant declines in consumption if returns are poor. Annuity purchases in low interest rate environments also constitute a risk. However, delaying an annuity purchase may risk a loss in the mutual fund portfolio with the result that the investor may lack sufficient funds to purchase the desired about of future annuity payments. Note: Compare with <u>Ruin Problem in Retirement Under Stochastic Return Rate and Mortality Rate and its Applications</u>, Feng Li MS Thesis, Dept of Statistics and Actuarial Science, Simon Fraser Univ. (Spring 2008). The authors employ a vector autoregressive model using returns of the S&P 500, ten-year government bond total return, and 90-day T-bill return based on 1962-2009 quarterly data. Economic shocks are incorporated based on a model developed by Robert Barro ["Rare Disasters and Asset Markets in the Twentieth Century"]. Insurance carriers are assumed to fail with an annual probability of 0.15 percent. In disasters, the probability of insurance company failure is twice that of government bond default. The authors report that "…incorporation of rare economic disasters significantly lowers the expected asset returns and increases their volatility." Mutual fund positions are charged with an annual 1.2 percent fee regardless of their asset composition. The discount rate for annuity pricing is based on the yields on ten-year U.S. Treasury bonds. Annuity pricing uses the annuitant population mortality table with an additional load of 10 percent to cover administration and marketing costs. Survival of retirees, however, is simulated using general population unisex mortality tables. Taxes are not considered. For the base case of retirement at age 65 with a $1 million portfolio invested in a 50-50 stock / bond allocation and a two-thirds preference rate on income for a real consumption floor of $45,000 and a real wealth target of $250,000, the optimal strategy is to immediately annuitize 10%	

		of wealth and annuitize up to 100% over the course of 20 years while making a fixed 5% annual withdrawal from the remaining mutual fund wealth. The mutual fund allocation to stocks increases to an eventual 100% at age 74 to reflect the increasing weight of annuities within the financial asset portfolio. Without modeling disasters, the shortfall probabilities for wealth and income are 41.3% and 23.8% respectively; with disasters the probabilities increase to 44.1% and 32.2%. Absent any annuitization, the income shortfall probability is significantly higher (55.7%) although the wealth shortfall is lower (15.1%). No change is made to the withdrawal policy—i.e., the model assumes a static withdrawal policy. For most of the variations on the base case—i.e., differing asset allocations—a phased annuitization program generates better results: "…systematic withdrawal strategy, however, requires certain risk tolerances and presumes the ability of an investor to weather storms such as market crashes. This downward risk is in turn mitigated by the annuity layer, even though the annuity contract itself is subject to some bankruptcy risk from the insurer."	
2012	"Be Kind to your Retirement Decumulation Plan—Give it a Benchmark," Daniel Cassidy, Michael Peskin, Laurence Siegel and Stephen Sexauer Risks and Rewards No. 60 (August 2012) pp. 25 – 28.	The article begins with the observation that an actuarial firm working with an "institutional mandate" for a DB plan, must first understand the schedule of cash flow requirements. This liability schedule is itself a benchmark in the sense that "…the return on the liability can be calculated using market interest rates and other data, and the return on assets held for the purpose of paying the liability can be compared to the benchmark return." However, a personally owned retirement income portfolio also shares important characteristics with a DB plan because it holds assets that must defease targeted cash flow liabilities. Given the context outlined above, the authors pose several questions: "Since the purpose of a benchmark is to capture the overall goals and characteristics of an investment strategy while avoiding active bets and other difficult decisions, we can ask: what is the appropriate benchmark that does all these things? And after deciding on a benchmark we have additional questions: Can we invest directly in the benchmark, in an approach akin to indexing? Can investors beat the benchmark?"	The argument advances the concept of a liability-relative benchmark that is investible and that matches the cash flow requirements of the retired investor. The investor can implement the minimum risk portfolio by investing in the benchmark; or, can try to beat the benchmark by implementing a variety of other retirement income strategies. The empirical risks and performance of alternative strategies is compared to the benchmark in order to assess their success or failure.

		The authors list several candidates for a retirement decumulation benchmark: • Cash flow matching benchmark (liability-driven investing); • A conventional 60% stock / 40% bond portfolio; • A portfolio consisting only of TIPS; • A portfolio based on nominal or real annuity payouts; or, • Target Date mutual fund glide path allocations. They claim that any choice for a decumulation benchmark must minimize "…the four dominant decumulation risks: longevity, investment (including inflation), counterparty, and liquidity. It should also be an executable and indexable portfolio." For a retiree age 65, the authors nominate a benchmark consisting exclusively of TIPS and a deferred annuity. "This benchmark has minimal risk. It provides inflation protection through age 85, does not contain any equity risk or fixed income duration-mismatched risk, and only the deferred annuity cash flows starting at age 85 have any credit risk. To further reduce inflation risk would require annuitizing the whole investment balance in a real—i.e., constant dollar—life annuity, but this would expose the whole portfolio…to credit risk, and would be unacceptable to most investors because of the liquidity loss."	
2012	"Annuities and Your Nest Egg: Reforms to Promote Optimal Annuitization of Retirement Capital," Norma L. Nielson C.D. Howe Institute Commentary No. 358 (2012) available at: http://www.cdhowe.org/pdf/Commentary_358.pdf	This paper studies the Canadian retirement system, and has the aim of making a variety of public policy suggestions. Among the observations that the author makes are: There is a classic agency problem because the professionals offering retirement planning advice either urge clients to buy annuities because of commission-driven motives or urge clients not to buy annuities in order to generate fees from continuing asset management. Most annuities are sold by firms with large blocks of life insurance. Some of these companies may be able to offer better than expected annuity returns because "The sale of an annuity reduces ther overall risk faced by the firm and produces a corresponding reduction in the needed (risk-adjusted) rate of return."	"It is a fact that the sale of life annuities by life insurance companies actually improves the risk profile of those companies – to the point where they have offset the mortality risk present in their portfolio of life insurance contracts."

2012	"Optimal Withdrawal Strategy for Retirement-Income Portfolios," David Blanchett, Maciej Kowara, and Peng Chen Retirement Management Journal vol. 2 no. 3 (Fall 2012), pp. 7 – 18.	The article introduces a new preferencing criterion: Withdrawal Efficiency Rate [WER]. "The Withdrawal Efficiency Rate compares the withdrawals received by the retiree by following a specific strategy to what could have been obtained had the retiree had 'perfect information' at the beginning of retirement." If the retiree knew with certainty future investment returns and life span, the portfolio could be managed so that there would be zero possibility of either a shortfall or a surplus. The authors note that "Most research on retirement portfolio withdrawal strategies has centered on the ability of a portfolio to maintain a constant withdrawal rate or constant dollar amount (either in real or in nominal terms) for some fixed period…." Although some research considers dynamic strategies for asset allocation and withdrawal policy, "…up until this point there has been no measure to evaluate the effectiveness of these withdrawal strategies (other than probability of failure, which has significant limitations). The first step in calculating the WER is to determine the Sustainable Spending Rate [SSR] under perfect certainty of investment returns and life span. A Monte Carlo simulation generates many paths incorporating return and longevity variables. For each path, there is a constant amount withdrawal strategy that results in zero wealth at the exact time of death; and, for each path, this amount can be determined with precision. Thus, the SSR is "…the constant amount that it is feasible to withdraw for a given combination of market returns and death scenarios…" when bequest motives are not considered. But inflation-adjusted withdrawals can exhibit substantial variance and, therefore, the next step is to calculate the Certainty Equivalent Withdrawal [CEW] based on the assumption of a CRRA utility function. "The CEW is the constant payment amount that a retiree would accept such that its utility….would equal the utility of the actual cash flows realized on a given simulation path. The sum of all the CEW payments is smaller than the sum of all the realized cash flows – by the nature of the CRRA utility function, a retiree would give up some of the potential cash flow amount to ensure a stream of unchanging cash flows." CEW for a path of N-period cash flows is	The authors note: "Milevsky and Robinson 2005, incorporate the stochastic character of both the mortality and market returns, but are focused more on finding the 'constant-dollar' probabilities of success or failure rather than finding the 'best' strategy; the two are not equivalent." This article, like the one authored by Sharpe et al applies a "penalty function" to both shortfalls—inadequate lifetime income—and to surplus—missed consumption opportunities. Authors seem not to be aware of the 2008 paper: "Following the rules: Integrating asset allocation and annuitization in retirement portfolios" authored by Wolfram J. Horneff, Raimond H. Maurer, Olivia S. Mitchell, Ivica Dus. The Horneff *et al.* paper compares the expected cash-flows from three distribution rules. They use a certainty equivalent income stream from each of the distribution rules to compare the certainty equivalent amount to an annuity. The utility-adjusted income stream produced by the risky asset portfolio is then directly comparable to the annuity income stream. Likewise, the approach is comparable to that of Gordon Pye: The Retrenchment Rule: When It's Too Late to Save More For Retirement, (GBP Press, New York) 2012.

expressed as:

$$CEW = \left(\frac{1}{N}\gamma\sum_{1}^{N}\frac{C_i^{-\gamma}}{\gamma}\right)^{-\frac{1}{\gamma}}$$

Therefore, the WER is expressed as:

WER = CEW ÷ SSR

"The WER reflects the utility-adjusted income percentage of income received by the retiree…vs. the maximum potential income." The higher the average WER from any particular set of simulations of a given withdrawal strategy, the better the strategy.

The article evaluates five withdrawal strategies where the Monte Carlo simulation assumes a lognormal return distribution of a two asset portfolio: S&P 500 and Intermediate Government Inflation-Adjusted Total Return Bond Index [TIPS]. Historical returns were reduced by 50 basis points and standard deviations were increased by 200 basis points for four allocations: 0%, 20%, 40% and 60% stock weighting. The five withdrawal strategies are:

- Constant Dollar Amount based on the initial portfolio value;
- Constant Percentage (the "endowment approach");
- Changing Percentage based on maintaining a constant probability of failure approach;
- Changing Percentage (one divided by life expectancy = 'RMD approach"); and,
- Changing Percentage using updated survivorship expectations and constant probability of failure metric.

Constant dollar strategies exhibit the lowest WER because the utility function penalizes these strategies when they deplete the investment portfolio. The WER maximizing strategy for a constant dollar withdrawal scheme is 3.5% constant withdrawal for 0% equity or 4% constant withdrawal for the other equity weightings.

The Endowment [unitrust] approach maximizes the WER at 5% withdrawal for the 0% and 20% equity allocations and 5.5% for the other

		equity weightings. The constant probability of failure approach allows withdrawal rates generally to increase as the time horizon shortens—"it's a duration-based measure." "The Constant Failure Percentage approach effectively creates a 'distribution path' the retiree can follow each year with respect to how much retirement income can be achieved from a portfolio." Equity allocations in the low 70% range generated the most favorable WERs. The Changing Percentage approach incorporates mortality into the withdrawal strategy. A commonly used method is to divide one by the remaining life expectancy to derive the fraction of current wealth to distribute—this is the well known RMD approach for qualified plan distributions. The final strategy tested in the article combines the Failure Percentage and the RMD approaches. The authors name this approach the "Mortality Updating Failure Percentage" approach. This approach generated the optimal withdrawal strategy. The Constant Failure Percentage approach yielded the second best results. For three of the equity allocations, the worst result under the WER metric was the Constant Dollar strategy: "…the primary method employed by many practitioners, where a constant real dollar amount is withdrawn from the portfolio until it 'fails'…is often the least efficient approach to maximizing lifetime income for a retiree."	
2012	"Adaptive Investing: A responsive approach to managing retirement assets," Sam Pittman & Rod Greenshields Russell Research (March 2012), pp. 1 – 10.	The authors contend: "Retirees consistently express three primary needs concerning retirement wealth management: they want low risk of outliving their assets (sustainability), consistent income (predictability), and financial flexibility (liquidity)." In order to promote better management of retirement income portfolios, advisors should focus on "funded ratio management"—the ratio of assets to liabilities. Focusing on the funded ratio determines (1) if the client's wealth can support his desired lifestyle, and (2) the benefits of exercising an option to purchase an immediate life annuity. By definition, tracking the funded ratio is a form of asset/liability	Although this article recommends an asset/liability approach to retirement income portfolio management via the concept of "funded ratio," neither the calculation of asset value nor liability value is based on current observables—e.g., (the current market value of assets) ÷ current cost of a guaranteed lifetime income stream. Assets include the present value of income benefits (Social Security) which is stochastic because of the constantly changing discount rate. Liabilities are

		management: "…assets need to be managed considering the liabilities they will fund." The authors define assets as the value of liquid financial assets plus the present value of guaranteed income streams such as Social Security and guaranteed pension benefits. Liabilities are defined as the "…present value of future spending needs." Although exercising an annuity purchase option is an effective way to manage longevity risk, it may increase vulnerability to general inflation or to specific economic shocks like health care expenses. The authors recommend maintaining assets sufficient to fund a real target income level for a window of time—e.g., ten years at age 65. Beyond the window of time, future cash flows are discounted by the probability that a retiree is alive to receive them. "Planning for your client to live 10 years and then have enough wealth to buy an annuity at the end of those 10 years is a useful way to address longevity concerns without significantly overstating the spending liability. As your clients continue through retirement, you should always plan that they will be alive for a window of time to develop investment plans and to manage longevity risk." The article expands on the implications of an A/L management approach as it points out a need for an adaptive asset allocation. The authors provide an example of an asset-only approach through a discussion of an income-oriented investment portfolio: "Many believe that a retirement portfolio should be invested in securities that explicitly provide regular payments, such as bonds, dividend paying stocks, covered calls, etc. However, simply collecting income producing investments to meet retirement income is a naïve approach. Focusing on dividends emphasizes the income characteristics of these assets over their investment characteristics." They advance the proposition that "To effectively manage retirement income, it's important to consider the actual liabilities, including their timing, duration, risk, and correlation to other assets, so that the income strategy is truly optimized to meet them….ass allocation should be set according to the relative size of assets and liabilities, their relative risk, and their correlation." In short, "as the investor's situation changes through time the asset allocation	based on projections of future annuity costs which depend on future interest-rate levels and on trends in annuitant population mortality. Bottom line: the authors' definition of funded ratio is subject to model risk. The planning issue is whether the client wants to exercise the option to annuitize when it approaches or nears the "at the money" point. However, the "at the money" point, unlike a financial option, is a projection based on modelling assumptions and parameter values. The A/L Management approach is a variation on the "household budget" funding strategy. An asset only approach focused primarily on generating accounting income is akin to the pyramid approach commonly found in financial planning literature—lock in the funding for critical goals evaluated and quantified in isolation before moving on to fund lower-priority goals. This approach ignores the "timing, duration, risk, and correlation" factors.

		needs to respond." When investor resources are insufficient to fund critical goals, the authors recommend: (1) revising spending targets downward, and (2) continued investment in financial assets with a conservative asset allocation lest portfolio volatility makes the funded ratio (<100%) value worse. The article designates four critical metrics to assess the viability of a retirement plan: 1. Current funded ratio 2. Probability of success 3. Magnitude of failure 4. Expected surplus The key to retirement income portfolio management under an A/L management approach is "…to develop a regular monitoring procedure and review contingency plans with clients." The authors are squarely in the annuity-as-safety-net camp: "…we handle longevity by preserving the option to annuitize. Rather than prematurely transferring assets to an annuity provider, we emphasize monitoring client funded ratios."	
2012	"The False Promises of Annuities and Annuity Calculators," David Marotta, Forbes August 8, 2012 available at: http://www.forbes.com/sites/davidmarotta/2012/08/27/the-false-promises-of-annuities-and-annuity-calculators/	A short opinion piece that uses an IRR approach to value annuities. In the hypothetical fact pattern, a couple aged 64 and 62 purchase an immediate annuity that pays a 6% annual rate. However, the author argues that the "…annuity begins with the immediate loss of 100% of your original investment. So for the first 15 years, the annuity company is simply giving you back your original purchase price….But when analyzed, the purchase price is a loss you can never recover from. We can analyze this annuity purchase like an investment and calculate an internal rate of return (IRR). For the first 15 years, the IRR is 0% because the annuity company simply hands you back your own money. If Thomas and Martha die after 16 years, the IRR would be 0.92%. If they live to age 85, after 23 years the IRR will finally have risen to 3.47%. If they both live to be 100, the IRR would still only be 5.57%. Even if [they] lived forever, the IRR couldn't exceed 6% because they lost their original [principal]."	Note: valuing annuities based on an IRR metric may raise problems similar to using IRR to value life insurance. Under an alternative perspective [e.g., "Generating Guaranteed Income: Understanding Income Annuities," by Ameriks and Ren [2008] annuities are insurance—against the risks of longevity. Thus the costs of an annuity represent an insurance expenditure not an investment return. Compare, also to "Estimating internal rates of return on income annuities," Nathan Zahm and John Ameriks Vanguard Research Paper (November 2011).
2012	"Sustainable	The article begins with some general observations on the nature of	Argues against establishment of a cash buffer in a

	Withdrawal Rates: The Historical Evidence on Buffer Zone Strategies" Walter Woerheide and David Nanigian Journal of Financial Planning vol. 25 no. 5 (May 2012), pp. 46 – 52.	retirement income planning. The authors remind us that decumulation planning involves four variables: (1) the amount of assets at the time of retirement, (2) the optimal asset allocation to sustain a desired series of withdrawals, (3) the initial withdrawal rate, and (4) the withdrawal strategy. Unfortunately, "there is no such thing as the correct or optimal answer to what each of these four variables should be…." For example, "…clients differ in both their risk tolerance and risk capacity. With respect to risk tolerance, some clients may be willing to take on more risk in the hopes of achieving higher withdrawals and a larger estate. Risk capacity will be based on the client's projected living expenses, and to what extent these are necessities versus luxuries as well as fixed versus variable in nature. In addition, risk capacity is also a function of the amount of other income the client has, and the fixed versus variable nature of that income." A common way to evaluate different strategies is to determine "the portfolio success rate"—the percentage of times the portfolio sustains a time-series of withdrawals. Some commentators advocate the use of a cash buffer zone to enhance the probability of portfolio success. "A buffer zone involves the use of money market funds to ensure the safety of withdrawals taken over the near future and to avoid selling in an under-valued market." Under this strategy, if the portfolio decreases in value, the planned withdrawals are made from the money market fund. As the portfolio increases in value, the portfolio becomes the source both of periodic withdrawals and cash to replenish the money market fund. One argument for the use of a buffer zone is that it allows the investor to avoid selling assets in a down market. The authors contend that this "advantage" depends on the existence of mean reversion in the time series of stocks and bonds. However, lagged one-year values for both stocks and bonds fail to exhibit autocorrelation of returns. Yule-Walker estimation methods show that lagged returns fail to have an impact on current returns at the 20 percent significance level. Alternately, one might argue in favor of a cash buffer zone because it reduces the standard deviation of the portfolio. The test of this advantage involves	retirement income portfolio's asset allocation. Methodology used is historical back testing. Concludes that in most markets investors are hurt by the drag of the cash position. This position runs counter to that advanced in "The Benefits of a Cash Reserve Strategy in Retirement Distribution Planning," Shaun Pfeiffer, John Salter and Harold Evensky, Journal of Financial Planning vol. 26 no. 9 (September, 2013), pp. 49 – 55.

		ascertaining whether the reduction in variance is sufficient to offset the reduction in the rate of portfolio return. The authors follow the methodology of the Cooley, Hubbard and Walz study ["Portfolio Success Rates: Where to Draw the Line," Journal of Financial Planning, April, 2011]. They test the portfolio success rate with a buffer zone over various asset allocations ranging from 100% stocks (S&P 500) to 100% bonds (U.S. Long-Term Treasuries). They examine comparative success rates over overlapping periods of 15, 20, 25, and 30 years using data over the periods 1926 through 2009—55 periods for the 30-year planning horizon. Withdrawal rates range from 3% to 9% (inflation-adjusted) of the portfolio's initial value. The value of the money market fund buffer extends from one year's planned withdrawal to four year's withdrawals. Under most all circumstances, the creation of a cash buffer zone results in lower portfolio success rates when compared to portfolios without such a zone. Indeed, under most variations, the success rates for portfolios without a cash money market component are substantially greater. Thus, the authors suggest that a static asset allocation strategy to stocks and bonds is superior to a strategy incorporating cash that acts as a source of withdrawals during bear markets.	
2013	"Individual post-retirement longevity risk management under systematic mortality risk," Katja Hanewald, John Piggott and Michael Sherris, Insurance: Mathematics and Economics, Vol. 52 No. 1 (January, 2013), pp. 87 – 97.	"…post-retirement financial planning…" must take into account inflation risk, investment risk, unsystematic (idiosyncratic) longevity risk—the risk associated with the length of an individual's lifespan—and systematic longevity risk—the risk associated with systematic improvements in life expectancies of a large group or cohort. Systematic longevity risk occurs because of "shocks to population-level mortality rates;" idiosyncratic longevity risk is "individual-specific mortality risk." The authors develop a two-period expected utility model to study the optimal transfer of idiosyncratic and systematic longevity risks and to examine the effects of costs and insolvency risks on investor choice. Initially, the model assumes a complete market which is fully spanned by four instruments: (1) a risk free investment, (2) a life annuity, (3) a longevity bond, and (4) a GSA fund. At the beginning of period 1, the	Note: text cited is from draft copy dated July 11th 2011. The study considers several financial products and strategies including a GSA [group self-annuitization] plan. This is a voluntary association of individuals who agree to pool their mortality risk without the involvement of an insurance company. By construction, such organizations are "mutual," "non-guaranteed," and "voluntary." Insurance companies issue annuity contracts that include "…loadings for expenses, cost of capital and adverse selection and are at the risk of insolvency, which make them difficult for individuals to assess."

		investor is endowed with retirement wealth [W_0], selects consumption [C_0], and implements a portfolio of financial and actuarial instruments to provide consumption [C_1] across possible future states. Depending on the payoff and probabilities of future states, the prices (costs) of the portfolio's individual components are derived through contingent claims analysis. Costs and benefits are then assessed in terms of investor preferences per the expected utility model. Survival states ('dead' in high or low survival regimes or 'alive' in high or low population survival regimes) exist across two population states (high population survival rate or low population survival rate), two individual states (alive or dead), and their accompanying probabilities. This results in a 4x4 matrix in the two-period model. The payoffs of each instrument depend on its consumption value in each state where the state-contingent price is a function of the payoff value and is expressed as either 1 or 0. For example, in the states of dead/high population survival rates and dead/low population survival rates, an individual annuity's payoff value is 0. A longevity bond is an instrument that provides a payoff of 1 irrespective of whether the investor is dead or alive—but only if the population survival state is high. The longevity bond hedges systematic population mortality improvements and pays off to whomever owns the bond. The GSA state-contingent price is 0 in most states and 1 in the low-population-survival-and-alive state. A risk-free bond, of course, pays a guaranteed 1 in each state. The analysis assumes that investors are able to buy and sell these securities to create an optimal retirement portfolio—i.e., they can hold a negative position in a security. In the complete market case, subject to the investor's budget constraints, the study derives the optimal consumption pattern and then determines the portfolio that best replicates this pattern. Initially, this portfolio is determined in the absence of expenses or insolvency risk. "An individual with initial wealth W_0 (i.e., his retirement savings) has to determine his optimal consumption by maximizing his expected utility over future uncertain states. He faces a budget constraint after consuming initial consumption of C_0 that the remaining wealth, ($W_0 - C_0$),	The study is a valuable addition to the literature because it utilizes a state-contingent pricing approach to the retirement income challenge.

must not be exceeded while purchasing contingent claims to consumption in each of the future states." For an individual lacking a bequest motive, the optimal consumption in a "dead" state is 0. His optimal consumption problem is then to maximize utility expressed as:

$$U(C_0,C_1) = u(C_0) + \beta E[u(C_1)]$$

where β is the time preference parameter.

Utility is maximized by considering only the products, prices, probabilities, and preferences applicable to the 'alive' states. First order conditions are determined with a Lagrange multiplier for the budget constraint; the rearrangement yields the marginal rates of substitution between beginning period consumption and ending period consumption as well as the marginal rate of substitution between consumption in the 'alive' states "This is the ratio of the state-contingent prices divided by the state probabilities. That is, the trade-off is determined by the price of transferring consumption between the states." A similar, albeit more expansive, calculation occurs for investors manifesting a bequest preference. Finally, given the 4x4 state-price matrix and the investor's budget constraint, the authors derive the optimal portfolio by solving a series of equations for the number of units of the risk-free asset, the life annuity, the longevity bond, or the GSA plan purchased by the investor.

The study extends the analysis to encompass a bequest motive and assumes financial products charging expenses and exhibiting insolvency risk. The life annuity product's price now includes a load factor; and, to reflect insolvency risk, the annuity pays the full benefit in a low survival population state but only a partial benefit in a high survival population state.

The authors employ a numerical solution with the time preference parameter [β} set to 0.98 and a coefficient of relative risk aversion of 2. The probability of a high survival population state is 0.60 with a conditional probability of individual survival in a high survival population state equal to 0.9 and, for a low survival population state equal to 0.8.

		Given these probabilities, the prices for the state-contingent claims represented by the various financial products are set so that the price of a product paying benefits in high survival states is higher than a product not providing such payoffs: "…the demand for consumption is higher when more individuals are alive. Furthermore, the price of a contingent claim that pays off when the individual is alive is higher than the price of a contingent claim that pays off when the individual is dead." In the complete market, no load, no insolvency risk model, the results differ from those of Yaari. In this model, the optimal no-bequest-motive portfolio is a combination of a short position in the GSA and a long position in a life annuity contract. However, the dominant strategy is annuitization because "…even in the presence of systematic risk…life annuities insure both idiosyncratic and systematic longevity risk." For individuals with a bequest motive in the complete market state, there is a substantial increase in the demand to hold the risk-free bond and a substantial decrease in the demand to own life annuities: "The risk-free bond is the equivalent of self-insurance in the two-period example and portfolio strategies that include self-insurance are optimal in the case of a bequest…." Extending the analysis to include costs and insolvency risk indicates that "Life annuity demand is substituted with holdings in the longevity bond and GSA that incur no loadings….Overall, the inclusion of a loading for the life annuity is welfare decreasing and individuals place more of their retirement wealth into self-insurance and mutual products without guarantee loadings." [Note: assumed annuity loading factor is 0.05]. "Portfolio strategies now dominate full annuitization." The study turns from a two-period utility-maximization model designed to explore longevity insurance strategies to a multiperiod simulation model to compare a spectrum of specific products: • <u>Fixed Life Annuities</u> where the contracts price includes a load and where the issuing insurer may become insolvent. • <u>Deferred Annuities</u> with payout contingent upon survival to age 85.	

		• Inflation-Indexed Annuities. • Group-self annuitization (GSA) where payments to participants depend on the population mortality experience. The GSA plan is a mutual fund with no loading factor or insolvency risk. • Self-annuitization assuming that the probability of depletion is less than 5% based on expected future mortality rates. For funds remaining, the utility of bequests is scaled by the factor 0.15. If an insurer becomes insolvent (probability of insolvency = 0.005) future payments are reduced by 0.95. The investor can form a portfolio from the above instruments using two types of strategies: 1. Horizontal self-insurance or co-insurance. The investor purchases an annuity to fund a portion of lifetime cash flow and uses a savings drawdown program with the remainder of wealth. 2. Vertical self-insurance or a deductible. The investor purchases a deferred life annuity and uses remaining wealth to fund cash flows until the deferred annuity payments commence. The simulation model is a cointegrating vector error correction model that assumes two regimes—a normal state and a high volatility state. Economic time series are generated for the log of GDP, the log of a bond index, the log of a stock index and the log of an inflation index. Series are based on Australian data. Mortality rates are simulated with a model based on Australian Population Mortality Data for ages 65 through 99 from the years 1971 to 2004. “Products are compared using expected utility of future cash flows computed from simulation of economic scenarios and future mortality.” The model adjusts cash flows to inflation (a stochastic variable) and to the investor’s time preference discount factor [0.98]. The utility risk aversion function is CRRA; the model assumes additive utility across each simulation path, and expected utility is the average of the sum of utility generated by the cash flows from selected product configurations—i.e. retirement income portfolios. Portfolios are formed at retirement wealth levels of $75,000,	

		$150,000, $350,000, and $750,000 for investors aged 65, 75, and 85. The model calculates utility in the 'dead' state by multiplying utility by a constant: k = 0.15. All told, the study considers 10 different portfolio configurations. For each portfolio, the model calculates the expected discounted utility values and converts them into certainty equivalent consumption levels: "The certainty equivalent consumption CEC is the fixed yearly consumption level that gives the same utility as the product portfolio we want to assess. The mathematical expression for CEC [$\overline{U}$] is: $$\overline{U} = E\left[\sum_{t=0}^{\omega-x} Pr(survive\ t\ years)\ \beta^t \frac{(CEC)^{1-\delta}-1}{1-\ \delta}\right]$$ where δ is the coefficient of relative risk aversion [2]. The study provides a table of preference rankings based on the CEC for each portfolio owned at various wealth levels and investor attained age. Given the CRRA form of the utility function, values scale linearly in wealth: "Increasing the individual's initial wealth …increases the certainty equivalent consumption levels but, as noted, does not change the relative ordering of the individual's preference-based ranking of portfolios." Increasing age tends to make portfolios containing a deferred annuity become more attractive (the period for the "deductible" is shorter; and, at age 85 a deferred annuity becomes an immediate annuity). Likewise, an investor at age 75 can increase consumption-per-unit-of-wealth markedly because only 10 years of self-insurance need be funded from retirement savings prior to the onset of the annuity payout period. "These results suggest that deferring retirement makes portfolios containing deferred annuities more attractive." The largest change in portfolio preference rankings occurs as the level of loading on annuity products varies. The model changes the load assumption to both 10% and 25%. Not surprisingly, the immediate life	

		annuity preference ranking sinks markedly—together with the inflation-adjusted annuity product ranking—while the self-insurance alternatives are more attractive: Loadings in guaranteed annuity products are clearly one of the most significant factors influencing the demand for these products. The importance of mutual risk sharing arrangements such as GSA funds becomes much more significant in the presence of these loadings." Not surprisingly, introducing a bequest motive also tends to decrease the investor's demand to hold annuity products: "For individuals with a bequest motive, portfolio strategies including self annuitization and GSA's are shown to dominate full annuitization."	
2013	"Examining the Benefits of Immediate Fixed Annuities in Today's Low-Rate Climate," David M. Blanchett Journal of Financial Planning, vol. 26 no. 1 (January 2013), pp. 42 - 50.	The author begins by reminding the reader that the buyer of an annuity should expect to bear a cost because the annuity is a form of insurance contract. However, even in the absence of a positive expected value, an annuity buyer may experience positive welfare. However, today's low interest rates increase the cost of periodic income because the insurance carriers cannot earn substantial amounts on their underlying bond investments. The article presents a table of sustainable withdrawal rates derived from a Monte Carlo simulation model assuming 3% inflation, constant dollar withdrawal rates based on percentages ranging from 3 to 10% of initial value for a $1 million portfolio allocated 25% to equities. Planning horizons range from 20 through 40 years. For example, at a 4% real withdrawal rate, there is a 9% chance of portfolio depletion by the end of 30 years. By adding the condition of survivorship of at least one spouse over the planning horizon, the failure rate for the corresponding withdrawal rate drops to 8% for a 30 year period assuming an age 65 couple. The challenge faced by the retiring household is to determine whether the potential benefits of an annuity outweigh the costs. The author calculates the annuity's IRRs for a male, female and joint couple (100% survivorship benefit) assuming they are both age 65 at the time of annuitization. The calculations indicate that IRRs for all groups are	Calculated annuity IRRs approximate those presented in "Estimating internal rates of return on income annuities," Nathan Zahm and John Ameriks Vanguard Research Paper (November 2011). Article concludes that, in a low interest rate environment, it is not optimal to annuitize a significant portion of wealth until later in retirement.

		positive by approximately age 80. An early death results in a highly negative IRR, a long life may result in a modestly positive IRR. The IRR distribution has a pronounced left side skew: "From a practical perspective, the negative skew associated with IRRs…should be viewed as the 'cost' of offsetting the potential positive skew associated with life expectancy. The annuitant is effectively trading the possibility of dying early (and the corresponding negative IRRs) with the hedge of living a long life and having guaranteed income the entire period." The author supplements the IRR analysis with a utility-based analysis. The model's utility function reflects a CRRA and is the standard von Neumann Morgenstern power utility function. Utility maximization is a function of the total income goal replaced during retirement: This is calculated by dividing the net present value of all payments received over the retiree's lifetime plus the total balance of assets at death, by the net present value of the total income need." "…the utility-maximizing portfolio will be the combination of assets that both maximizes retirement income and minimizes the downside variability associated with generating the income." The portfolio under consideration has a parameterized mean of 7% with a standard deviation of 9%. This is characteristic of a portfolio allocation 40% to stocks and 60% to bonds. Returns are lognormally distributed and inflation is a constant 3% per year. Each retiree (male, female, or couple) has a portfolio balance of $500,000 and has an inflation-adjusted pension of $30,000 per year (100% survivorship basis). If the annuity provides more than the targeted withdrawal rate, the excess is reinvested into the stock/bond portfolio. Results are tested at the currently available annuity rates and for alternatives having rates 50 and 100 basis points higher. The utility-based analysis concludes: "An IFA is not featured for any of the current rate scenarios under age 70, and only with material allocations for those annuitizing at age 80. However, the optimal allocation to an IFA does increase at older ages, and increases considerably should IFA rates improve." For example, at a 4% of initial portfolio withdrawal rate, a male should forgo annuitization until	

		approximately age 75. At that time, 30% of the portfolio should be annuitized. The female retiree should defer any annuitization until approximately age 80 when 50% of wealth should be annuitized. A couple should annuitize 20% of wealth at approximately age 80. At a 50 bp rate increase, the numbers become 20% of wealth annuitized for a male at age 70; 30% for a female age 75; and, 50% for a couple age 80. At a 100 bp rate increase the numbers become 30% of wealth annuitized for a male at age 65; 30% for a female age 70; and, 40% for a couple age 75. "Given today's annuitization rates, which are currently near all-time lows, many retirees are likely better off waiting until interest rates and subsequent annuitization rates improve, or delaying the IFA purchase decision to an older age. Even with today's low rates, IFAs remain an attractive longevity hedge for retirees age 80 or older…."	
2013	"What Makes Annuitization More Appealing?" John Beshears, James, J. Choi, David Laibson, Brigitte C. Madrian and Stephen P. Zeldes, National Bureau of Economic Research NBER Working Paper No. 18575 (June, 2013), pp. 1 – 28.	The study combines two surveys each of which presented "hypothetical annuitization choices" to individuals aged 50 to 75. The project focuses "…on the elasticity of annuity demand with respect to annuity product design and choice architecture." Specific issues under examination are: 1. What factors are important in an individual's annuitization choices; 2. How outcomes are altered when offering an individual a choice to partially annuitize wealth; 3. Preferences regarding the "intertemporal slope of annuity payouts;" 4. The effects of framing on an individual's annuitization choices; and, 5. The demand for 'bonus' annuities that offer a higher payment in a specified month of the investor's choosing. Three considerations emerge as having the greatest impact on an investor's decision about whether to annuitize: (1) making sure that there is sufficient income later in life; (2) spending flexibility; and (3) concern about the financial viability of the carrier underwriting the annuity guarantee. The ability to partially annuitize contributes	The authors acknowledge: "hypothetical choices must be interpreted with caution, since they may not closely correspond to the choices people would actually make…." The article has an interesting link between the option to annuitize and state-preference utility--i.e., "…the attractive state-contingent payment properties of an annuity." They point out that annuitization enables investors to shift resources from "death states to survival states." Absent bequest motives, income and wealth during the death state has no value. But this is exactly the state in which the annuity loses its value. Alternately, an annuity gives the investor the ability to raise the rate of return during the survival state—i.e., achieve "an implicit survivorship bonus." An important implication is that "…an agent with access to an annuity should choose a higher rate of consumption growth relative to an agent without access to an annuity." An Appendix derives this

		significantly to the attractiveness of an annuitization election. Although empirical evidence demonstrates that retired investors follow a declining consumption path, investors in these surveys preferred flat or rising retirement income. Additionally, framing changes can reduce or increase the demand to hold annuities; and, finally, an ability to choose a “bonus payment” month increases annuity demand. Of particular interest is the examination of investor preferences for the slope of the annuity payout throughout their retirement: “…19% preferred the declining real annuity (-2% per year), 32% preferred the flat real annuity, and 50% preferred the rising real annuity (+2% per year).” Given the empirical data documenting consumption declines in retirement, the authors offer a variety of possible explanations for the survey participants’ preference for rising payments. They speculate that actual investors may underestimate retirement costs such as home repairs and health-related expenses. As these costs eat into their wealth, consumption may reduce not because of consumer preferences; but, rather, because of budget constraints—“Households may not anticipate the extent to which consumption and health status are complements.” Alternately, households at the start of retirement may be overly optimistic regarding future investment returns; or, they may misestimate life expectancy (“…respondents may believe their life expectancy to be longer than the life expectancy that equates the present value of the various payment streams. Such a belief would lead them to expect to collect a greater present value of payments from the rising annuity than from the other two annuities.”); or, finally, households may exhibit “money illusion” with respect to their choice of retirement income stream trajectories.	result from the Euler equation that presents a two-period utility of consumption model for an investor with constant relative risk aversion. The marginal utility of consumption in period one must equal the marginal utility of consumption in period two where period two consumption depends on investment earnings discounted by a time preference factor and survival probability. Likewise, given the CRRA assumption, the risk-aversion for variance [EIS] in period one consumption level must equal that for period two. In an economy without the option to acquire an actuarially fair annuity, by the property of logarithms: $\ln(c_2) - \ln(c_1) = 1/\gamma(\text{earnings} - \text{time discount} - \text{mortality})$ The Euler equation for the growth rate of the optimal consumption path with annuities, however, becomes: $1/\gamma(\text{earnings} - \text{time discount})$ The annuity raises the growth rate by: (mortality rate / γ). “Intuitively, this effect arises because the annuity raises the effective rate of return for claims on consumption in the survival state….”
2013	“Retirement Income Strategies in Expected Utility and Loss Aversion Frameworks,” Gaobo Pang and Mark	This study compares several strategies for producing retirement income: • immediate annuities, • deferred annuities, • systematic withdrawals from a financial asset portfolio, and • mixed strategies.	The study concludes that the economic benefits of ALDAs can be achieved more efficiently by combining ongoing withdrawals from financial asset portfolios with a gradual purchase of immediate nominal annuities. Much of this result is explained by higher loads for both inflation-adjusted

	Warshawsky Working Paper (August 26, 2013), pp. 1 – 21. http://reliasllc.com/wp-content/uploads/2013/10/IncomeStrategies.pdf	The analysis considers social security, stochastic investment returns, mortality, social security, and fees for various financial products. The authors further assume that the selected strategy stays constant throughout retirement. Strategies are evaluated on the basis of a utility metric—'utility of consumption and bequest'—and on a shortfall risk metric—'loss aversion.' The expression for the additive lifetime utility function [V_τ] follows a standard form that incorporates terms for discounting [β], consumption [c_τ], and bequests [b_τ]. There are, however, additional terms for the effective number of adults per household [[h_τ], and for the ratio of per capita consumption [c_τ / h_τ]. The function sets the value of 'h' to one for a single adult household, and to a value of $\sqrt{2}$ for couples. This value takes into account the economies of scale in a two-person household. The value equation takes the following form: $$V_\tau = \Sigma\left[\beta^\tau h_\tau u\left(\frac{c_\tau}{h_\tau}\right)\right] + \beta^\tau v(b_\tau)$$ The utility function is constant relative risk aversion for both consumption and bequest. The bequest function operates only above a designated level of threshold consumption. Finally, in order to compare strategies more precisely in terms of their welfare gain or loss, the authors calculate the Certainty Equivalent over each path and the Average Certainty Equivalent [ACE] over all simulated paths. The preferred retirement income strategy is the one that achieves the highest ACE. Both the utility-based and loss aversion models assume: • An initial wealth endowment of $250,000 per household at the time of retirement, • Consumption equal to the sum of annuity payouts, Social Security benefits, and systematic withdrawals from the financial	immediate annuities and ALDA contracts: "Welfare measures using ALDA strategies are lower. A cheaper, less risky and more transparent strategy is the combination of systematic withdrawals with laddered purchases of immediate life annuities." For a counter-argument on the issue of ALDA loads, see "Better Financial Security in Retirement? Realizing the Promise of Longevity Annuities," Katharine G. Abraham and Benjamin H. Harris, Economic Studies at Brookings (November 2014), pp. 1 – 20. Abraham and Harris argue: "We have seen little empirical support for the claim that longevity annuities are more susceptible to adverse selection concerns—indeed, intuition suggests the opposite may be true…."

asset portfolio,

- An initial asset allocation of 50% bonds / 50% stocks with bonds proxied by the U.S. 10-year Government Bond Total Return Index, and stocks by the S&P 500,
- Inflation measured by the periodic change in the urban CPI, outputs are adjusted for realizations of a stochastic inflation process,
- A small probability of insurance carrier bankruptcy,
- A 25 basis point fee for index fund investments,
- Fixed loads that differ for nominal immediate annuities, real immediate annuities, and Advanced Life Deferred annuities [ALDAs] if purchased through age 75, with loads increasing by 5% if the investor purchases an annuity contract at ages 76 through 85,
- Single adult households purchase single life annuities; two-person households purchase joint and survivor annuities with a 75% survivor benefit. ALDAs assume a one-time purchase at age 65 with survival-contingent payments commencing at age 85.

Innovations to asset returns and inflation are modeled as a vector autoregressive process with coefficients and variance/covariance matrix estimated on quarterly data from 1962 through 2011.

The authors to evaluate the following retirement income strategies under both shortfall and utility-based models:

1. Fixed Constant Dollar Withdrawals
2. Withdrawal of a Fixed Percentage of the Current Period's Portfolio Value
3. Immediate Nominal Single Premium Life Annuities
4. Immediate Constant Dollar Single Premium Life Annuities
5. Advanced Life Deferred Annuities
6. Combinations of the above.

		The study reports results for both single and married households. For each strategy / combination strategy, the authors detail inflation-adjusted income and terminal wealth outcomes at the 5th, 50th and 95th percentiles as well as the mean return and standard deviations. For example, for single retirees in good health the optimal strategy produces an ACE value of 26.3. This strategy consists of no initial annuitization of wealth, a withdrawal of 8% per year from the balance in the risky asset portfolio, and gradual annuitization of 45% of wealth over the next 20 years. Other strategies combining systematic withdrawals, ALDAs and real and nominal immediate annuities produce ACEs in the range of 25.2 to 25.8. Strategies employing ALDAs tend to produce lower-range ACEs: "Among the many options we considered, a new strategy stands out: combining laddered purchases of immediate annuities (that is, dollar cost averaging) with systematic withdrawals from a dynamically changing retirement investment portfolio. The combination strategy outperforms the alternative using ALDA." Not surprisingly, for single individuals in poor health the optimal strategy when measured in terms of an ACE metric avoids annuitization. Similar results occur for married couples. The optimal strategy is to withdraw 8% per year from the balance in the risky asset portfolio, zero percent initial annuitization, and a gradual annuitization of wealth of 50% over 25 years. A higher risk aversion parameter increases annuitization slightly; a stronger bequest motive reduces annuitization by roughly an offsetting percentage. Results derived from the utility-based model are then compared to a shortfall-minimization strategy. The question of interest is: to what extent do the optimal strategies differ under each risk/reward evaluation metric? A household prefers the strategy that minimizes the weighted risks of income and portfolio balance falling below a target real-dollar threshold:	

		α*Prob(income < C_0) + (1-α)*Prob(balance < W_0) The income target [C_0] is the amount required to sustain a threshold income, the wealth target [W_0] is defined as an amount sufficient "to cover uninsured contingencies and leave a large bequest." The thresholds are set at 4.5% of initial wealth income withdrawal and 25% of initial wealth fund balance. "The preference weight on income shortfall is two-thirds." Although there are some differences in the optimal strategy, the authors remark: "Overall…the resulting optimal strategies are remarkable similar." The risk minimizing strategy for single person households is a fixed real dollar annual $10,000 withdrawal from the risky asset portfolio, an initial conversion of 10% of wealth into an immediate nominal annuity, and a gradual annuitization over 25 years with annuities equaling 20% of wealth. For couples, the optimal strategy is the same. The weighted shortfall probabilities under the optimal strategy are 9.4% and 11.4% respectively (data from table in article appendix).	
2013	"Liability Investment with Downside Risk," Andrew Ang, Bingxu Chen & Suresh Sundaresan, National Bureau of Economic Research Working Paper 19030 http://www.nber.org/papers/w19030 (May 2013).	The authors explore a liability-driven investment [LDI] approach for managing a Defined Benefit [DB] Plan. The study considers how best to manage a DB Plan's surplus; and it includes a penalty cost parameter that is a function of a plan's risk aversion parameter. Shortfall (or, downside) risk aversion (the penalty 'cost') is a function of a plan's asset allocation, current funded status, and likelihood of a potential future shortfall. Downside risk occurs whenever assets are insufficient to meet plan liabilities. The authors simultaneously calculate the 'cost' of a failure to meet plan liabilities and a portfolio's optimal asset allocation. Their model considers fund liabilities as a state variable and specifies an objective function that incorporates both plan assets and liabilities. Optimal asset allocation decisions take into account asset/liability covariance—with special attention to downside covariance— in a LDI framework. The article starts by reviewing the surplus optimization model developed	This is one in a series of articles discussing portfolio management strategies within a Defined Benefit Plan context. Earlier articles focus primarily on asset management techniques to (1) avoid a shortfall in a plan's funded status while (2) optimizing the management of plan surplus. This article presents an option valuation approach that asks the following question: Given (1) a DB plan's current funded status [assets ÷ liabilities], and (2) a plan's asset allocation, how much would the plan sponsor pay for holding a put option the payoff of which is the plan's future shortfall amount, if any. [Payoff equals (Maximum of end-of-period Liabilities) – (end-of-period Assets), 0)]

		in 1990 by Sharpe and Tint. Surplus is the value of [(Assets) – (Liabilities)]. Surplus return [z] divided by assets (i.e., normalized return) equals: $$z = \frac{S_1}{A_0} = \left(1 - \frac{L_0}{A_0}\right) + \left(r_A - \frac{L_0}{A_0} r_L\right)$$ Where r_A is the return on assets, r_L is the return on liabilities and $L_0 \div A_0$ is the inverse of the funding ratio. [Note: the L/A term is also a key term in the leverage-adjusted duration gap measure used to determine first-order sensitivity of existing capital and surplus to interest rate changes: LADG = Change in Net Worth = $D_{assets} - D_{liabilities}$(L/A). If a change in interest rates results in a decrease in liabilities greater than assets—i.e., a positive duration gap, net worth increases. Risk—i.e., the rate of increase or decrease—depends, in part, on the leverage ratio's value]. In the Sharpe & Tint model, the objective function is to maximize surplus expected return while taking into account the correlation of plan asset and liability returns: $$max_w E(r_A) - \frac{\lambda}{2} var(r_A) + \lambda cov(r_A, r_L)$$ In order to work within an LDI framework, the authors (Ang, et al.) assume a simplified asset allocation choice between risky stocks and a risk-free asset: Risk-free Bond return period 1 = B_0exp(r_f); Expected Stock return period 1 = $E_0 exp\left(\left(\mu - \frac{\sigma_E^2}{2}\right) + \sigma_E \varepsilon_1^E\right)$. Liability Return is: $L_1 = L_0$exp$\left(\left(\mu_L - \frac{\sigma_L^2}{2}\right) + \sigma_L \epsilon_1^L\right)$	This question can be generalized to consider the value of a "shortfall" put option for an individual's retirement income plan where the value of the lifetime cash flow liability is measured by an actuarial (annuity) benchmark. Given the current funded status of an individual's retirement plan, what is the current market value of a put option calibrated to eliminate future shortfall risk? The option valuation approach provides an interesting way to compare and contrast a variety of planning methodologies including behaviorist-oriented "flooring" recommendations to establish a bucketing/laddering strategy via bonds; or, to secure annuitized cash flow for a threshold periodic income target level. What is the intrinsic value of a "protective put" where the strike price is based on a stochastic lower-bound terminal wealth level v. what is the "opportunity cost" of locking in a permanent budget constraint via a flooring strategy. [See also, the Meder/Staub article where the liability hedging portfolio for a DB Plan consists of larger stock positions in order to mitigate liabilities calculated under the PBO—Projected Benefit Obligation—actuarial method which incorporates future wage growth factors.]

		where the shock to equity return is a standard normal variable and the correlation between equity and liability shocks is ρ. The simplified model operates in a one-period setting where the value of the portfolio is a function of its allocation to equity. The value of a put option on the terminal shortfall is a function of the terminal value of assets where the option strike price is *unknown* at time 0. Downside risk depends on the funding ratio and the asset allocation at time zero. The value of the put option is: $P(\omega, L_0, A_0)$. The objective function is the mean/variance optimized portfolio plus a downside risk penalty based on a shortfall relative to the plan's liability: $$max_w E(r_A) - \frac{\lambda}{2} var(r_A) - \frac{c}{A_0} P(\omega, L_0 A_0)$$ Where c is the penalty cost of a shortfall (i.e., a shortfall risk aversion parameter), ω is the asset allocation, and P is the option price. The initial value of the option is the price a plan sponsor would pay today to insure against the possibility of a future shortfall in plan assets. [Note: 'c' is a cost-of-risk measure]. The above expression replaces the covariance term in the Sharpe and Tint model with a shortfall penalty term. [See Sharpe & Tint 1990] This means that the option's value is endogenous because the manager can reduce its value by increasing the correlation between the pension's investment portfolio and the plan's liabilities. The optimal asset allocation [ω] and the value of the put insurance are computed simultaneously: "…option valuation endogenously depends on the optimal portfolio strategy, and the optimal strategy simultaneously depends on the cost of the shortfall risk." As the penalty cost increases, the effects dominate the LDI effects noted in the Sharpe and Tint model. Although the authors note that the shortfall process does not follow a log-normal process, it can be interpreted "…as a spread option due to the stochastic evolution of both pension assets and pension liabilities." There are no closed-form solutions for option valuation. There are, however, approximations of the shortfall put option's value. Both the optimal portfolio allocation [ω*] and the cost of shortfall insurance are	

		implicitly defined in the following expression: $$\omega^* = \frac{1}{\lambda}\sigma_E^{-2}\left[(\mu - r_f) - \frac{c}{A_0}P_\omega\right]$$ If the penalty term 'c' approaches zero then the portfolio assumes the characteristics of a mean/variance optimized portfolio with the Lamda term quantifying investor risk aversion. If the penalty term approaches infinity, the portfolio assumes the characteristics of a pure liability hedging portfolio [ω^{LH}]. Assuming a small time step, the authors employ a log-normal approximation to value the shortfall put option: $$P(\omega, L_0, A_0) = L_0N(d_1(\omega)) - A_0N(d_2(\omega))$$ where N is the cumulative normal probability density function and d_1 and d_2 are as follows: $$d_1(\omega) = \frac{\ln\left(\frac{L_0}{A_0}\right) + \Omega^2(\omega)/2}{\Omega(\omega)} \text{ and}$$ $$d_2(\omega) = \frac{\ln\left(\frac{L_0}{A_0}\right) - \Omega^2(\omega)/2}{\Omega(\omega)}.$$ $\Omega(\omega)$ is the portfolio variance formula: $(w^2\sigma^2_{stocks} + \sigma^2_{liabilities} - 2w\rho\sigma_{stocks}\sigma_{liabilities})^{½}$ where variance is measured relative to plan liabilities—i.e., surplus variance. The optimal weight to equity in the portfolio's asset allocation is a weighted average of the mean/variance efficient portfolio and the liability-hedging portfolio: $$\omega^* = \frac{\lambda}{\lambda + \frac{cP\Omega}{A_0\Omega}}\omega^{MV} + \frac{\frac{cP\Omega}{A_0\Omega}}{\frac{cP\Omega}{A_0\Omega}}\omega^{LH}$$ Where $P_\Omega = A_0n(d_2)$ is the Vega [Δ in price sensitivity to a 1% change in the underlying volatility] of the option and n(·) is the normal probability	

		density function. A large value of 'c' tends to motivate the investor to move towards a liability-hedging portfolio. The weight on the LH portfolio [Θ] increases as 'c' grows large: $$\frac{\delta\Theta}{\delta c} = \frac{n(d_2)}{\Omega} > 0$$ and the first derivative of theta with respect to the funding ratio is: $$\frac{\delta\Theta}{\delta(\frac{A_0}{L_0})} = \frac{cn(d_2)}{(\frac{A_0}{L_0})\Omega^4}$$ The authors note that there is a nonlinear relationship between the funding ratio [A÷L] and risk aversion [λ]. This is evident in the denominator of the equation for the optimal portfolio in which the denominator adds $(\frac{A_0}{L_0})$ to the λ term. A high funding ratio indicates that the put option's value is close to zero. The investor ignores plan liabilities and sets an asset allocation close to the mean-variance efficient allocation even if 'c' is large. As the funding ratio approaches a value of one, the option's value becomes very sensitive to portfolio volatility [i.e., Vega of the option] which motivates the investor to move towards the liability-hedging portfolio. If the funding ratio is substantially below a value of one, the option's value is not sensitive to portfolio variance because only a high-variance portfolio can 'cure' the shortfall [see Sid Browne's observations that leverage must grow large as time to cure a shortfall grows short. "The Risk and Rewards of Minimizing Shortfall Probability," The Journal of Portfolio Management vol. 25 no. 4 (Summer, 1999), pp. 76 -85.]. The article makes no suggestions on how an investor should select a value either for 'λ' or 'c.' The article continues by applying empirical stock/bond returns to both the 1990 Sharpe-Tint LDI model and the current authors' LDI model. The	

		2013 model suggests that the fund investment manager is most risk averse when the funded ratio is in the neighborhood of one: "For c = 1, the maximum of effective risk aversion is achieved when the funding ratio is 1.03." The option price sensitivity "…is concave in A_0/L_0, and reaches a maximum when $A_0/L_0 = 1.04$. The option's sensitivity is highest when the pension plan is close to fully funded because the probability of the shortfall risk is highest at this point." The model generates similar results when both risky bonds and stocks are incorporated within a LDI model context. The latter part of the article calculates optimal equity weightings for a range of 'c' parameter values for a stock/bond/cash portfolio based on monthly returns of the S&P 500 and the Ibbotson U.S. long-term corporate bond index over the period 1952 through 2011. The return on cash is an assumed 4% per year. The liability portfolio has an assumed correlation of 0.98 with bonds and 0.35 with equities. [Note: see, Aaron Meder & Renato Staub "Linking Pension Plan Assets to Liabilities," on the Society of Actuaries website (https://www.soa.org/library/.../apf-2007-10-meder-staub.pdf) for an argument that some pension obligations are highly correlated with equity. In general, however, the Meder/Staub plan liability decomposition suggests that the optimal portfolio for managing surplus variance will have a higher weighting to fixed-income assets]. In general, the presence of a shortfall penalty motivates a reduced equity exposure compared to that recommended under the Sharpe-Tint model: "The LDI with the downside penalty recognizes that although equities are positively correlated with the liabilities, there can be instances of substantial underperformance when investing in equities. This is costly, and reflected in the value of the put option." Additionally, "As c increases, the optimal weight asymptotes to the liability-hedging portfolio, which holds 24% in equity." [Note: when the analysis includes only stocks and bonds—i.e., no risk-free asset] the portfolio asymptotes to 4% equity / 96% bonds.] Further analysis considers the interrelationship between the value of the funding ratio and the value of 'c'. The put option is most price sensitive	

		around a funding ratio value of one. "When the plan is very underfunded, the shortfall option is deep in the money. As a result, the objective function starts to put less weight on the shortfall risk because there is less ability of the manager to alter the portfolio choice to meet the liabilities….There is an overall U-shaped equity weight as a function of the funding ratio, with a minimum weight on equities at a funding ratio close to one." Not surprisingly, the value of the put option decreases as the funding ratio increases. "The sensitivity is concave in A_0/L_0, and reaches a maximum when $A_0/L_0 = 1.04$. The option's sensitivity is highest when the pension plan is close to fully funded because the probability of the shortfall risk is highest at this point." The Appendix provides technical details on Spread options and on valuation of the shortfall option. In a nutshell, a spread option is the difference in valuation of two options where option #1's value is a function of the liability portfolio underlying; and option #2's value is a function of the asset portfolio underlying. The authors acknowledge that neither underlying satisfies the assumption of a log-normal diffusion; but, at relatively short time intervals, a European exchange option valued under the Black-Scholes pricing model is a reasonable approximation of the spread option's value. The structure of the shortfall spread put option is a put and a call on European options for the asset portfolio underlying and the liability portfolio underlying: "The spread option is thus equivalent to compound exchange options on two calls and two puts."	
2013	"Safe Withdrawal Rates, Optimal Retirement Portfolios, and Certainty-Equivalent Spending," Druce Vertes, http://papers.ssrn.com/sol3/papers.cfm?a	Using the historical back testing method with data for a two-asset class portfolio consisting of the S&P 500 and Intermediate-term US Government bonds over the period 1926 through 2012, the author explores the interrelationships between asset allocation, spending during retirement, and investor risk aversion. The main objective is to quantify the tradeoffs between the level of initial spending, the amount of achievable lifetime spending (defined as the percentage of initial portfolio wealth that is consumed), the probability of a shortfall (defined as likelihood of a decline in spending from the initial spending level—a	Although the questions posed by the author are most interesting, the text does a poor job in helping the reader distinguish between levels, percentages, rates, etc. As a result, tables and graphs must be painstakingly examined if their intended meaning is to be made clear. In this article, a picture is often not worth anything close to 1,000 words. For example, "shortfall severity" is not a level; rather, it is a percentage decline from a level of initial

	bstract_id=2263998 (May 11, 2013).	downside variability risk metric) and the severity of a shortfall (the percentage decline from the initial spending level). The base case is the 4% spending rule. Historically, given a 50-50 stock-to-bond portfolio allocation, this rule exhibits no shortfall risk over a planning horizon based on life expectancies based on general population tables for various ages developed by the Social Security administration. As a technical note, the no-shortfall-strategy—i.e., a fixed amount of spending throughout retirement--is a 3.9% spending rule because the Vertes' portfolios take the full withdrawal amount at the beginning of the portfolio year. However, as Vertes notes, such a spending election results in a tradeoff that many retirees would consider unacceptable—to achieve no variability whatsoever in spending, the investor follows an autopilot rule that does not take advantage of either better-than-expected portfolio returns, or of changes in investor life expectancy because of aging ("A 90 year-old retiree with a $1 million portfolio can safely spend more than a 65 year-old retiree with the same portfolio.") Furthermore, adopting the 4% spending rule is likely to result in a relatively low amount of aggregate lifetime spending: "since the rate is never adjusted, these cohorts spent significantly less than they might have." Portfolios intended to maximize initial constant-dollar spending subject to a no-downside-spending-variance constraint are attractive only for highly risk-averse investors. On the other hand, retirees are concerned about the consequences of fixed spending when confronted with possible future return sequences worse than any yet historically realized: "If, on the other hand, future returns fall short of the worst case in the historical sample, the outcome is catastrophic: you run out of money." Using historical data, a simple but tractable risk aversion function (Constant Relative Risk Aversion [CRRA]), a simple two-asset class portfolio, Vertes modifies the single-parameter 4% rule by adding additional elements. Specifically, his spending function incorporates a parameter for a floating percentage of spending depending on the future value of the portfolio [a 'unitrust' distribution formula], a smoothing parameter [akin to a 'unitrust distribution formula with an 'x' year	spending. For example, it is often unclear, whether the author is defining a 'magnitude' as a percentage decrease relative to an initial level or as a dollar value shortfall from a targeted retirement portfolio objective. This is unfortunate because the article does an excellent job discussing the nature of the risk/reward tradeoffs faced by the investor throughout retirement. The author acknowledges the limitations of various model assumptions. For example, "While CRRA is a strong assumption about investor preferences over large changes in wealth and income, it's more reasonable to assume risk aversion is relatively constant over smaller changes in income below 50%. If we didn't assume constant relative risk aversion, we would have to make more complex assumptions about the slope and form of the risk aversion function, and our outcomes would become sensitive to the arbitrary level we choose for the starting portfolio." Additionally, "A portfolio outcome, like a movie, is an experience good, which can only be judged successful after experiencing the whole thing…An a priori estimated γ will not match realized γ, which may not be consistent over time, but may increase abruptly at market extremes." The author makes some general comments regarding investor utility: "A retiree will often have a discontinuous utility function with critical breakpoints. If income goes below one value he may need to downsize painfully, if it goes above another value his offspring can go the college of their choice. The CRRA utility function has useful properties, but it is unlikely to correspond perfectly

		average value element—however, Vertes applies the smoothing rule as an exponential moving average of portfolio value], and an age factor (spending can accelerated as life expectancy decreases). The final factor is similar to the minimum distribution withdrawal factor provided by the Treasury department for over age 70 ½ investors owning tax-qualified accounts. Whereas the Treasury rules are set up to force spending funds so that tax revenues can be recaptured, Vertes adds a mortality modifier to his age factor in order to prevent a too rapid withdrawal schedule for late-in-life investors. Using brute force, the author cranks various allocations through historical return data where the spending parameters are systematically adjusted. Millions of combinations of multi-parameter spending formulae are back tested. The data is organized into graphs that depict the allocation/spending parameter choices with shortfall risk metrics pertaining to both periodic spending variability and aggregate achievable lifetime spending. The data clouds produced by the numbers crunching permit Vertes to isolate the scatter-plot locations of shortfall probability and shortfall severity percentiles. The intersection of initial spending levels and shortfall constraints are mapped to charts and tables in order to illustrate the "shortfall severity frontier," the "shortfall frequency frontier," ad the lifetime spending "shortfall severity" and "shortfall frequency frontiers." Vertes then turns his attention to exploring the characteristics of a portfolio that maximizes investor utility at various values of the CRRA risk aversion coefficient. A utility maximizing or optimal portfolio looks at the tradeoffs between spending levels, lifetime spending and shortfall risks. Given the variance in spending, the allocation risks, the shortfall probabilities and severity, a certainty-equivalent of a risky income stream is the guaranteed income stream that would be exchanged for it without a change in investor utility: "By converting each income stream to a certainty-equivalent cash flow, we can quantify the desirability of a variable and uncertain cash flow. By multiplying each CE outcome by the probability the retiree remains alive to experience it, we can calculate expected lifetime certain-equivalent spending. More variable and risky	to a retiree's experience, and it depends on parameters which cannot be observed. A CE analysis, like the Kelly criterion, may be a useful framework, within which one must err on the side of conservatism, since the consequences of excessive risk may be disastrous." These excerpts demonstrate nicely that the article does not fall into the category of facile historical data mining the purpose of which is to extract retirement planning spending and allocation rules. It also links the analysis to papers on investor decision making in the face of uncertainty/ambiguity. In this case, when the investor is not sure about the economic models presented to him, he tends to weight heavily the model producing the worse outcomes as a precautionary measure. Irrespective of parameter values, the author suggests that "…retirees should err on the side of moderate initial spending, embrace the volatility they can tolerate as the key to unlocking maximum lifetime spending, and accept that their retirement trajectory is ultimately dependent on hos the timing of their retirement intersects long-term economic and market trends."

		cash flows will have lower CE values." The author tests portfolio allocations and spending rule parameters using historical data to outline the certainty equivalent frontiers: e.g., maximizing lifetime income subject to a shortfall severity constraint; maximizing initial spending subject to a shortfall severity constraint. The study concludes: "if our retiree is completely risk neutral, he will want to maximize lifetime spending. If he is completely risk averse, he will aim for the highest sustainable fixed spending, similar to the Bengen '4% rule'. If his aversion is between these two extremes, he will seek a strategy with volatility and lifetime spending between these two extremes. As you increase risk aversion, the variable component of spending decreases and constant spending increases, the portfolio swings from all equities to a diversified mix, and initial spending decreases toward the maximum sustainable fixed spending."	
2013	Life Annuities: An Optimal Product for Retirement Income. Moshe A. Milevsky The Research Foundation of the CFA Institute (2013)	The first two sections are in the form of Q and A. Here is a list of observations: "…like any interest-sensitive financial product, its price, or value, is inversely related to interest rates. When interest rates move higher, the annuity's value falls, and vice versa; in periods of (very) low interest rates, the value of a life annuity bond is (much) more expensive." "…when the holder of the bond (the creditor or buyer) dies, the issuer (the debtor or seller) is no longer obligated to make coupon payments." "The more guarantees, refunds, and options you add on to the life annuity, the more you water down the benefits of *longevity pooling*." "Whether you view the cost as price paid or payout received, the ratio between your premium (what you paid) and your annual income (what you get) will be the same and is called the 'annuity yield'." "So, today's retiree, if he or she chooses to buy a life annuity, enjoys 20% less income than was available 6 years earlier and almost 50% less income than 15 years ago. One can only sympathize with those who are annuitizing today versus a few years ago." "…life annuity payout rates are an average of 3.5 percentage points	This is a CFA Research Foundation monograph. Milevsky's annuity yield calculation formula assumes no changes in interest rates, insurance company reserving requirements, profit objectives, etc. However, the future price of an annuity is, itself, a distribution of possible costs. See, e.g., Ruin Problem in Retirement Under Stochastic Return Rate and Mortality Rate and its Applications, Feng Li MS Thesis, Dept. of Statistics and Actuarial Science, Simon Fraser Univ. (Spring 2008). The concept of "lifetime ruin probability" closely parallels the feasibility condition—wealth must be greater than liabilities. However, under the closed form expression, the evolution of wealth is governed only by the first two moments of a normal distribution. The observations on Fisher utility clarify how mortality drives a wedge between the risk free

		more than 10-year U.S. T-bonds, although this difference tends to be variable over time, especially during times of financial stress. So the extra 3.5% is not a bad rule of thumb." "…commission can be anywhere from 0.5% to as much as 5% of the premium investment, depending on the company and product." "More than 165,000 policyholders had purchased high-yield annuities from Baldwin-United, and the money was frozen for more than three years while regulators and the courts picked up the pieces." "Another saga that has been ongoing for 20 years…is Executive Life Insurance Company of New York." "NOLHGA has been actively recently in the following instances: when Golden State Mutual Life Insurance was shut down by regulators in California in September 2009; when Shenandoah Life entered receivership in Virginia in February 2009; when Standard Life Insurance Company of Indiana was taken over by Indiana regulators in December 2008; and when London Pacific Life & Annuity Company was liquidated in July 2004." "…the concept of diversification applies not only to stocks and bonds but also to insurance policies, including life annuities." Milevsky defines a life annuity as a type of bond—"one in which the coupons are higher than those of government or corporate bonds, partly because of the annuity's illiquidity. This unique and personal longevity-linked bond is subject to default (i.e., your death), at which point only a small fraction of the original investment will be recovered by creditors (i.e., your heirs)." Because you cannot trade an annuity, there is no market value. However, it has a theoretical value which declines with age. "…this longevity-contingent claim is likely to be the best hedge for their longevity risk. It is asset/liability management on the personal balance sheet." "The fact that life annuities are priced in a competitive market to account for healthier, longer-lived individuals implies that an adverse selection cost is built into these insurance products. It is not a mark-up	interest rate and subjective time preference interest rate. Note also that the Epstein-Zin utility function separates the utility of consumption and the time discounting preference rate. Under these utility functions the EIS ≠ 1 / coefficient of CRRA.

		or loading, per se, but a reflection of the clientele who are interested in acquiring life annuities." The "money's worth ratio" is the expected present value of the lifetime annuity payments divided by the market price (i.e., premium) of the annuity. For an actuarially fair annuity, the ratio value should be close to one. However, "…the numerator involves *a model with particular assumptions* and the denominator is a *snapshot of a price* at a given point in time. Both quantities are subject to biases…." "Longevity-risk aversion is about the fear of living longer than expected and having to reduce your standard of living in retirement as a result. Individuals who are longevity-risk averse will probably consume less of their wealth early in retirement and allocate more of their next egg to annuity products to protect against this risk." Milevsky defines the concept of 'Implied Longevity Yield" as follows: If a retiree at age 65 could purchase an annual lifetime income of \$6,204 beginning today for a premium of \$100,000, this is an annuity yield of 6.2%. If a retiree at age 75 could purchase a lifetime annuity income of \$8,452 today for \$100,000 this is an annuity yield of 8.45%. Dividing \$6,204 by 0.0845 equals \$73,400 which is the amount that the retiree would need in ten years in order to secure the same amount of lifetime annuity income that \$100,000 could secure today. Using an Excel IRR formula, the Implied Longevity Yield is RATE(T,1,-A,B,0,guess), where A is the annuity premium cost per \$1 of periodic income. The current yield on a risk free government bond is lower than the calculated IRR and the difference constitutes the implied longevity yield. If the implied longevity yield is close to risk-free bond rates, the annuity is not providing much benefit." "…for people at older ages, the implied longevity yield is almost impossible to beat." Milevsky defines the concept of "lifetime ruin probability" [LRP] as follows: "It is the probability that your *biological* lifetime will be longer	

		than the *financial* lifetime of your portfolio." The mathematical expression for the LRP formula is: $$\text{LRP}(w,x,\mu,\sigma) = \Pr\left[\sum_{t=1}^{\omega-x} \frac{p(x,t)}{\prod_{j=1}^{t}(1+R_j} > w\right]$$ Where w equals wealth; x is the retiree's initial age; p(x,t,) is the probability of survival curve; ω is the maximum age; and R_j is the realized portfolio return in period j. The return is a random variable dependent on the portfolio's asset allocation. Return is determined by the first two moments: μ and σ. When R is constant, the left-hand term in the brackets is the present value of a lifetime annuity. The entire expression is the annuity factor. "And if the annuity factor is greater than the initial sum of money, w, available to finance spending, the individual is ruined." "The LRP value…can be computed in a number of ways. A relatively easy methodology is to simulate a vector of R_j portfolio returns and assume a particular mortality table, p(x,t), then count the number of scenarios in which the mortality-weighted present value is greater than w. This is the Monte Carlo approach to retirement income simulations." Increasing exposure to 100% equity will also increase expected future returns. However, it will not lower the LRP because of the corresponding increase in the variance term. "At low spending rates, the LRP declines with increasing exposure to safe assets. At higher levels of spending, the LRP is U shaped as a function of asset allocation….The intuition here is that further increasing exposure to stocks does not necessarily improve the odds of success because of the higher shortfall risk embedded in the portfolio." "The LRP is a summary risk metric that can help measure the sustainability of a retirement plan" "I want to be careful not to advocate LRP minimization, however, as a dynamic portfolio strategy. Rather, it should be viewed as yet another way of quantifying the benefit of annuitization." The next section of the CFA Institute monograph is a review of the scholarly literature. Among the noteworthy observations are:	

		Irving Fisher (1930): "The shortness of life thus tends powerfully to increase the degree of impatience, or rate of time preference, beyond what it would otherwise be. This is especially evident when the income streams compared are long…But whereas the shortness and uncertainty of life tend to increase impatience, their effect is greatly mitigated by…solicitude for the welfare of one's heirs." Yaari: "…his paper was intend as 'positive' (to explain observed behavior) as opposed to 'normative' (to provide financial advice). Yaari (1965): "This paper was the first to offer a recommended allocation to an insurance product—that is, 'product allocation' in contrast to investment 'asset allocation'." Davies (1981): "In the absence of annuities, after an initial period influenced by borrowing constraints, under constant relative risk aversion, uncertain lifetime depresses consumption by a proportion increasing with age if the elasticity of inter-temporal substitution in consumption is 'small." Davidoff, Brown, and Diamond (2005): They "examined demand for life annuities with market incompleteness. They found that some annuitization remains optimal over a wide range of preference parameters but complete annuitization does not. They also argued that utility need not satisfy the Von Neumann-Morgenstern axioms and need not be additively separable for the Yaari (1965) result to hold. Furthermore, annuities need not be actuarially fair; they only must offer positive net premiums (i.e., mortality credits) over conventional assets." Milevsky and Young (2007b) used preference-free dominance arguments to develop a framework for locating the optimal age (time) at which a retiree should purchase an irreversible life annuity. In this framework, the selection of time is a function of current annuity prices and mortality tables….annuitization prior to age 65 or 70 is dominated by temporary	

		self-annuitization even in the absence of any bequest motives." Dellinger (2011): "To the extent one's objective is to maximize retirement income with the potential to keep pace with inflation while minimizing the probability of outliving that income, delaying income annuity purchase can be suboptimal." Gupta and Li (2007): "…found that high insurance charges can make the net return from the annuity less than the return from other available investment assets—for example, the risk-free asset."	
2013	"Optimizing Retirement Income: An Adaptive Approach Based on Assets and Liabilities," Yuan-An Fan, Steve Murray, and Sam Pittman, The Journal of Retirement Vol. 1 No. 1 (Summer, 2013), pp. 124 – 135.	The authors characterize their research in terms of "…a dynamic, multiperiod asset-liability model (called 'the adaptive model') that incorporates retirement spending liabilities in the asset allocation model." This structure sees retirement planning in terms of surplus optimization: stochastic optimization in a multiperiod framework to address intertemporal liability management decisions (where liability is defined as retirement spending needs in excess of those met by social security and guaranteed pensions). Spending is funded by inflation-adjusted withdrawals from an investment portfolio from the assumed retirement date (age 65) to age 85. The investor purchases an immediate life annuity with wealth remaining at age 85 to continue target distributions for the remainder of life. "The objective of our model is to maximize the expected wealth at the final time, less a penalty for spending shortfall." Specifically, "Missed distributions caused by running out of money during the model horizon, and final wealth outcomes where assets are insufficient to purchase an immediate life annuity in the final time period, are penalized at a quadratic rate." The risk metric of interest is 'shortfall' rather than 'volatility.' Initially, the article presents a simplified three-time-step model during which the spending target is an inflation-adjusted $46,000 per year from a portfolio with an initial value of $1 million. At age 65, the investor (married couple) sets aside $460,000 to fund withdrawals until age 75. The remaining $540,000 is invested until age 75 when an additional sum	Retirement spending needs are quantified in terms of a fixed periodic payment throughout retirement: "…we find that retirees favor consistent, predictable spending, similar to the way they spent the paycheck they received prior to retirement." Retirees revise spending only when such a revision is necessary. The most important factor in determining optimal equity exposure is the relative size of the present value of the liability to current wealth. But this is the reciprocal of the Wealth/Consumption Ratio first introduced by Milevsky et al. The "adaptive strategy" is a variation on the Constant Proportion Portfolio Insurance [Floor + Multiplier] approach to asset management. In this case, the floor is defined in terms of periodic cash flow requirements. As the research study moves from a consideration of a simple model with a stated risk aversion parameter and a quadratic penalty for shortfalls to a more complex adaptive strategy model, the risk metric of interest becomes simply expected shortfall. Although the author's do not place their

		is set aside to cover expenses until age 85. Assume a fixed and deterministic immediate annuity cost of \$381,800 for joint income of \$46,000 for life with a 100% survivorship benefit. The model considers two portfolio allocations: 70% stock / 30% bond and 30% stock/ 70% bond. Investment results unfold sequentially between each time step in either a "good" market or a "bad" market state. In a good state, the 70/30 portfolio's realized annual return is 10.6%; the 30/70 portfolio's realized annual return is 7.4%; in a bad state, the returns are 0% and 1.8% respectively. Thus, the outcome tree starts with a \$1 million portfolio at node A; and depending on the investor's asset allocation choice, arrives at one of four nodes at the end of the first time step (age 75) and at one of 16 nodes at the end of the second time step (age 85). The investor purchases a lifetime annuity if age 85 wealth is sufficient and uses the surplus to invest either for additional future income or to satisfy a bequest motive. The expected objective value at each of the 21 nodes is expressed as: $$\text{Max } E\{V_2\} - \lambda^* E\{\max(381.8 - V_2, 0)^2\}$$ where V_2 is age 85 wealth and λ is a risk aversion parameter value. The problem of finding the best sequence of investment allocation decisions at each node (time/state combination) is solved recursively using the Bellman equation. This is straightforward in the simple model because all calculations are deterministic and the value of the objective function is known at each node. The best path through the asset allocation decision tree is the one that leads to the highest age 85 objective value. Paths leading to relative suboptimal outcomes are eliminated; and the best path is the 30/70 portfolio at age 65 and, assuming a good market, to switch the allocation to 70/30 at age 75. If, however, the investor confronts a bad market during the initial time step, the age 75 allocation remains conservative. A sequence of 2 "bad" market periods results in an age 85 shortfall; but the shortfall magnitude is lower than that realized along other possible allocation paths.	recommendations within a utility-of-wealth/consumption framework, their approach raises some interesting questions. In the utility-based models, to what extent do pre-specified functions—e.g., CRRA—introduce weaknesses comparable to assuming pre-parameterized return variables (mean / variance)? The track followed by the authors suggests that it might be profitable to swap out a consideration of a utility function (a risk aversion parameter value) in favor of an empirical test of investor response to the opportunity set which he or she currently faces.

		In the adaptive model, investors can change allocation yearly. The full model considers 2,000 outcomes over the decision tree and allows for a variety of allocations with a maximum equity exposure of no more than 80%. Portfolios consist of three investment choices: stocks, fixed income, and cash. Each asset class, as well as the inflation process, has fixed parameter values for return and standard deviation. There is a single static correlation matrix. For a given level of expected shortfall, when compared to a fixed-mix or a buy-and-hold strategy, the adaptive strategy evidences a higher expected ending surplus at age 85, and a lower magnitude of shortfall. In terms of shortfall risk, the adaptive strategy's efficient frontier is superior—highest surplus per expected shortfall amount. Equity exposure changes as a function of wealth and age as the investor progresses through the steps of the model: "the minimum equity exposure occurs approximately where the present value of future spending (the total liability value) equals the wealth in the portfolio." However, if a shortfall materializes, then equity allocation increases: "When liabilities are larger than assets, investing in fixed income that has the same rate of return as the discount rate on liabilities produces certain failure." Thus equity allocation traces out a U-shaped curve—increasing exposure at the extremes of surplus and shortfall, decreasing as current wealth nears the present value of future spending because "…market declines can cause an investor to become underfunded." Advisors can implement the adaptive strategy by "…periodically measuring the funded status of their clients and then adjusting the asset allocation to what is appropriate."	
2013	"The 4 Percent Rule is Not Safe in a Low-Yield World," Michael Finke, Wade D. Pfau and David M. Blanchett Journal of Financial Planning vo. 26 no. 6 (June 2013),	The authors test the sustainability of portfolios operating under a variety of stock/bond asset allocations under the traditional 4 percent withdrawal rule. The paper's initial section questions the assumption that bond returns are mean reverting as suggested in the 1977 Vasicek model. This model assumes that interest rates exhibit (1) a drift component which tends to move them towards the mean and (2) a diffusion component which tends to create a degree of randomness in their periodic movement. The authors assert, however, that the	

	pp. 46 – 55.	empirical evidence from the U.S. bond market shows little evidence of reversion in a "predictable manner." By contrast, the current yield on bonds is a good predictor of subsequent bond returns: "…the yield on bonds today can describe 92 percent of variation in the average annual 10-year compounded bond return. If we assume a current bond yield of 0.7 percent…the average annualized bond return is expected to be 1.05 percent over the next five years, and 1.4 percent over the next 10 years…." The article proceeds to contrast Bengen's 1994 data on the 4-percent withdrawal rule. Bengen's conclusion—the 4-percent rule exhibits a 0-percent failure rate—is based on historical back-testing of 30-year historical periods for a portfolio allocated 50% to stocks and 50% to intermediate-term bonds. A Monte Carlo simulation using updated historical data exhibits an approximately 6% failure rate. If historical averages are adjusted downwards by 2.6%--i.e., assumes a time-invariant equity risk premium—failure rates skyrocket to 33 percent. Calibrating bond returns to the 2013 real-yield on five-year TIPS , sends the portfolio failure rate to 57 percent assuming a constant historical equity risk premium. Finally, the authors develop a Monte Carlo risk model with inputs for stock and bond returns based on current market conditions as opposed to historical averages. The model tests various constant-dollar withdrawal rates over a thirty-year planning horizon. The goal is to determine, under a low-yield interest rate environment, the maximum sustainable withdrawal rate. Their model assumes a lognormal return distribution, and a constant equity risk premium at its historical level of 6 percent above bond returns. The model tests for sustainability under both a permanent low-yield environment and interest-rate reversion to the mean after periods of 5 or 10 years from the start of retirement ('The purpose of doing this is to investigate the role of sequence of returns risk…."). The retirement portfolio allocations are 30, 5- and 70 percent stock. For example, a Monte Carlo simulation using unadjusted historical returns for a portfolio allocated 50-percent to stocks manifests a failure rate of 6%. Adjusting the returns under the various market scenarios	

		considered by the authors generates failure rates under the same Monte Carlo methodology ranging from 18 to 57 percent. "…even a 3 percent withdrawal rate has a more than 20 percent failure rate for all asset allocations. The retirement risk model suggests that the probability of failure for a portfolio operating under the 4 percent withdrawal rule is minimized by a 100 percent allocation to stocks. Under current market conditions, the model indicates a maximum sustainable withdrawal rate of 2.5 percent with a 100 percent equity allocation. They conclude: "…when financial planners recalibrate assumptions for Monte Carlo simulations to market conditions facing retirees in 2013, the 4 percent rule is anything but safe."	
2013	"Simple Formulas to Implement Complex Withdrawal Strategies," David M. Blanchett Journal of Financial Planning vol. 26 no.9. (September, 2013), pp. 40- 48.	The author introduced, in 2012, a withdrawal formula preferencing criterion that he and his co-authors termed the 'Withdrawal Efficiency Rate' ["Optimal Withdrawal Strategy for Retirement-Income Portfolios"]. In this paper, he revisits the WER to test the efficacy of several formula derived from regressing formula-based withdrawal guidelines against simulated evolutions of retirement income portfolios. The higher the value of the WER, the better it captures the available income projected across the aggregate simulation paths. The author uses a the certainty equivalent income—the amount of constant, utility-adjusted income that an investor would accept across any individual simulated path—and compares it to the income generated by following the withdrawal formula: "The goal is to select the strategy with the highest WER value." The author notes that the best way to determine the optimal withdrawal amount from a retirement income portfolio would be an annual assessment "…through a comprehensive analysis of that retiree's facts and circumstances." Assuming such a process, the issue becomes one of finding the most efficient withdrawal formula. For planning horizons less than 15 years, the author contends that the IRS formula for calculating the required minimum distribution from a qualified retirement account	Compare to Utility Evaluation of Risk in Retirement Saving Accounts," James M. Poterba, Joshua Rauth, Steven F. Venti and David A. Wise [2005].

		is close to the optimal. For horizons greater than 15 years, he regresses a set of dependent variables—number of years, equity allocation, desired probability of success, and fees against the initial withdrawal rate. Data is generated by 10,000 Monte Carlo simulations. The optimal withdrawal formula for periods of 15 years or greater is: $.195 – 3.701\% \ln(\text{Years}) + 1.255\%(\text{equity}\%)^{½} - 4.471\%$ target success probability + .507Fees. "For example, using the dynamic formula, if a planner were to assume a retirement period of 30 years, an equity allocation of 40%, a 90% probability of success, and an alpha of -1.0 percent (i.e., total fees of 1.0 percent) the estimated withdrawal percentage according to the dynamic formula would be 3.15 percent…." The author concludes that annual application of the dynamic formulae results in a significantly higher WER value than is achievable by application of a static withdrawal approach such as the 4% of initial portfolio value formula.	
2013	Woo Chang Kim, John M. Mulvey, Koray D. Simsek and Min Jeong Kim, "Longevity Risk Management for Individual Investors," Stochastic Programming Applications in Finance, Energy, Planning and Logistics (Chapter 2) http://www.worldscientific.com/doi/pdf/10.1142/97898144075	A primary focus of the study is on finding the optimal investment policy for a retired couple faced with allocating resources to term insurance, stocks, bonds, and cash. The authors create a stochastic programming model and numerically solve for maximum expected utility over their unknown joint lifetime by dynamic allocation of insurance and investment assets at periodic (10-year) steps. Within each step-period, the allocation of financial assets remains constant—i.e., the couple follows a fixed-mix allocation policy with short-positions disallowed. Within each step-period, the amount of term insurance coverage (death benefit) and premium cost remains level. The 'base-case' model makes several underlying assumptions: • Both husband and wife are age 65; expected lifespan for the husband is 19 years and expected lifespan for the wife is 21 years. • Lifespans are independent. • Only during the husband's lifetime does a pension plan pay an	The stochastic model demonstrates that it is decidedly suboptimal to follow a fixed, "rules-based" policy for determining the optimal amount of insurance to keep in force during retirement. Indeed, the model produces several strikingly counterintuitive examples regarding the couple's demand to own term insurance. In general, the authors' conclusions are compatible with those of Patrick Collins & Huy Lam ["Asset Allocation, Human Capital, and the Demand to Hold Life Insurance in Retirement," Financial Services Review (Winter, 2011), pp. 303 – 325] who suggest that the insurance retention/surrender decision for retired couples is largely facts-and-circumstances with the spread between spousal ages an important factor in the analysis.

	19_0002 (January, 2013), pp. 9 – 41.	annual $40,000 constant dollar pension income. No benefit is forthcoming beyond the husband's date of death. • Annual expenses also equal a constant-dollar $40,000. • Initial savings amounts to $200,000. • All input and output values are in constant dollars • The wife, if living, ceases paying insurance premiums at the husband's death. Joint-life longevity risk can be mitigated either through investments in financial instruments or acquisition of life insurance coverage. The model employs a Conversion Ratio [CR] by equilibrating expected premium payments to the expected death benefit according to the applicable mortality rate for age, health, and other personal underwriting factors. This process keeps the premium expenditure level from renewal period to renewal period with the result that the death benefit amount decreases in a step-wise manner. A high CR produces "cheap insurance;" a low CR produces "expensive insurance. Thus, by varying the CR's value, the model produces differing schedules for optimal insurance coverage. The stochastic programming model sets parameters for life expectancies, pension benefits, living expenses, and investment returns under two scenarios: "normal" market and "crash" market. A Markov transition matrix sets a probability of .75 for a normal market. This results in a crash market occurring, on average, once during every four year period. The couple's goal is to maximize aggregate lifetime utility. The model tracks two performance measures—final surplus and expected final wealth—and four downside risk measures—average deviation from target consumption over time and across scenarios, probability of a final deficit, expected value of final deficit, and percentage of years in deficit. Utility is positive in final surplus and negative in lifetime consumption shortage. Utility value is a function of the couple's "risk preference" parameter 'α': Expected Utility = α[Final Surplus] – (1-α)[expected lifetime average	

		consumption shortage] Given the model's assumptions, Utility is decreasing as the husband's life expectancy decreases and/or as the wife's life expectancy increases. The couple's initial decision variable is the amount of insurance coverage they wish to maintain. Given this initial decision, the premium cost remains fixed for the remaining life of the husband. The model considers annual premium costs over the range of $0 to $40,000. After determining the optimal insurance coverage amount, the couple decides how to allocate remaining wealth. Performance is maximized by purchasing no insurance coverage while risk is minimized by implementing a less aggressive investment policy. Fixed-mix allocations range from 0% to 100% stocks; stocks and bond returns are assumed bi-variate log-normal with distribution parameters based on the current economic regime. In the base-case, risk and performance measures increase as the CR factor increases (i.e., insurance premiums are inexpensive). However, beyond a certain limit on the CR value, welfare does not increase with purchases of additional insurance. However, when the life expectancy of the husband decreases and that of the wife increases, results vary. Conservative investors will acquire additional insurance; but, balanced and aggressive investors will decrease coverage as the insurance cost increases: "Our explanation for these results is as follows. If the longevity risk could be eliminated with the life insurance at a low cost (high CR), the couple doesn't have an incentive to behave aggressively when their longevity risk increases. However, if the insurance is expensive, the couple should protect themselves from the downside risk by investing heavily on [sic] stocks. As their longevity risk diminishes, the couple doesn't need to worry about the downside risk any more. Therefore, they seek for higher surplus to increase their utility by heavily betting on stocks." The study examines eight possible cases of varying life expectancy parameters and CR adjustment factors. The authors conclude: "…there	

		is no generalizable pattern in the optimal stock weight and insurance investment." Finally, they explore the consequences of a cut in pension benefits. In this case, rather than purchasing additional life insurance, "…the couple's optimal strategy is to invest less on the life insurance…." The amount of the reduced benefit does not cover living expenses; and, a further reduction in income because of insurance premiums eliminates the couple's demand to hedge longevity risk.	
2013	"The True Impact of Immediate Annuities on Retirement Sustainability: A Total Wealth Perspective," Michael E. Kitces and Wade D. Pfau (July 15, 2013). Available at SSRN: http://ssrn.com/abstract=2296867	This article asserts that much of the value ascribed to Single Premium Immediate Annuity [SPIA] contracts because of mortality credits is, in fact, due to the rising "equity glidepath" that occurs whenever households utilize a "bucketed" retirement income strategy where fixed income assets are the source of initial retirement spending. The authors examine several portfolios—static stock/bond allocation, stock/SPIA allocation, and dynamic stock/bond allocations in an attempt to disentangle the impact of mortality credits and risky-asset returns. Their starting point is the assertion that retirement wealth should include the present value of expected SPIA payments. Previous research often "carves out" the cost of an SPIA so that the resulting portfolio consists only of a static allocation to stocks and bonds. Retaining the PV of SPIA payments is comparable to including the value of Social Security and Pension cash flows in a "total balance sheet perspective." The next step in the analysis is to consider systematic withdrawals (4 to 6 percent of initial portfolio value per year) from three portfolio allocations: (1) a 50-50 stock-to-bond allocation rebalanced annually, (2) a 50% SPIA allocation with 100% of *remaining* wealth invested in stocks, and (3) a hybrid model which has an initial 50-50 stock-to-bond allocation rebalanced under the assumption that bonds are "…allocated to be equivalent to the present value of the remaining SPIA payments, with the remainder allocated to stocks. This allows us to isolate the glidepath effect from the mortality credit effect of the stock/SPIA combination." Monte Carlo simulations generate stock/bond evolutions. Both fixed nominal and inflation-adjusted SPIAs are considered. For a 65-year-old male, the assumed payout for a nominal SPIA is 7.96% and for an inflation-adjusted SPIA the payment is 5.8%. No fees or	An interesting critique of the Kitces & Pfau paper is offered by James Shambo CPA in the May 13, 2014 issue of *Advisor Perspectives* found at http://www.advisorperspectives.com/articles/2014/05/13/revisiting-the-research-on-spias-in-retirement-portfolios. Shambo applauds the inclusion of the PV of annuity income on the investor's balance sheet as part of a total wealth perspective. However, he argues that the rising equity glidepath analysis leads to an incorrect conclusion because "…the actual financial portfolio used in their simulations starts with, stays and ends with 100% equities." The measured impacts are, in reality, mainly due to the initial withdrawal rate on the financial asset portfolio. The evidence that Shambo marshals for his argument lies in the demonstration that when high initial withdrawal rates remain even after an SPIA purchase, portfolio failure rates rise: "A total-wealth perspective…should not be the metric through which we evaluate future drawdowns. Cash flow is the correct way to measure drawdowns."

		investment costs enter the analysis. The compounding inflation rate is 2.91% which means that the nominal annuity provides a payout larger than the constant-dollar annuity for approximately 11 years. Although the SPIA contracts initially provide a high level of income, the present value of their remaining expected payment stream decreases their balance sheet worth. They note that the investor "…will be able to withdraw less from their financial portfolio to meet their overall spending goal in the early part of retirement, but these spending needs increase over the retirement period….Lower income needs from the stock portfolio allow it to grow more quickly than otherwise." It is this rising equity glidepath that helps extend portfolio sustainability—even in the face of an unfavorable sequence of returns early into retirement. The authors conclude: "…at least some of the improvement of going from the static stock/bond to the stock/SPIA is not actually a result of the SPIA at all, but instead simply a bucketing approach that leads to a liquidation strategy, where the fixed portion of the portfolio is disproportionately liquidated in the early years, such that the stock allocation from a total wealth perspective rises…." It is precisely the difference in investment outcomes between the stock/SPIA portfolio and the mimicking "glidepath" scenario that illustrates the "true benefit" of the SPIA itself. The observations on simulated results lead the authors to conclude that inflation-adjusted SPIAs are more valuable to investors because (1) the benefits attributed to nominal fixed SPIAs should be attributed to the equity glidepath; and (2) for truly longed-lived investors, the benefits of preserving purchasing power are substantial: "…for shorter time horizons, the contribution of mortality credits is actually *negative* (for both fixed and real SPIAs, though more so for fixed SPIAs) as the implied glidepath approach without a SPIA is actually superior to the stock/SPIA approach." Bottom line: "…the primary scenarios where SPIAs should be used are specifically those w[h]ere the intent is to hedge *significant* longevity beyond life expectancy….In the remaining scenarios for most retirees, though, the more effective way to improve retirement outcomes is	

		simply to implement a rising equity glidepath.”	
2013	“Strategies for mitigating the risk of outliving retirement wealth,” Vickie Bajtelsmit, LeAndra Ottem Foster, and Anna Rappaport,” Financial Services Review vol. 22 (2013), pp. 311 – 329.	Many studies purporting to present evidence regarding safe withdrawal rates may mislead retirees because incorporation of health costs, longevity risk, and long-term care expenses create significant tail risk in the distribution of retirement costs. Even at the 50th percentile of the distribution, health and long-term care expenses add an extra $260,000 of costs for the average married couple. The ’safe withdrawal literature’ generates retirement spending recommendations which flow from a consideration of portfolio sustainability under various withdrawal rules. By contrast, the authors assert that their risk model assumes “…that household spending is driven by spending needs rather than by a recommended drawdown strategy for assets.” The authors assert that a credible retirement income risk model should incorporate health-related expenses over the consumer’s life-cycle: “…extreme tail risks, such as early onset long-term care needs, investment declines (particularly in the early years of retirement), inflation risk, and unexpected health costs all contribute to the likelihood of retirement income inadequacy.” They establish a base case that considers the impact—both separately and in combination—of three strategies [(1) reduce discretionary spending; (2) delay retirement and social security claiming; and, (3) purchase long-term care insurance] on households with pre-retirement income of $60, $105 and $150 thousand. The case-study household consists of a male age 66 with a female spouse age 63. Given the number of variables in the risk model, it is not surprising to find that individual components have strikingly simplistic assumptions: • Investment portfolio evolves dynamically with the equity exposure adjusted yearly according to a ‘100-current male spouse age’ rule. Equity is evenly divided between small and large cap stocks; fixed income allocation reflects long-term corporate bond risks and returns. • Mortality data is from the Social Security Administration’s mortality table; • Inflation is modeled as a normally distributed variable with a	This article is a good example of incorporating variables beyond investment, inflation, and longevity into a retirement income risk model. Specifically, the authors point out that “the risk of outliving retirement wealth depends on spending and saving decisions, investment performance, qualification for defined benefit or other annuity income streams, health care costs, long-term care risk, and longevity.” The retirement income risk model incorporates the probability for incurring long-term care costs. The probability is calculated via the binomial distribution. The most effective control variable for increasing portfolio sustainability is to delay retirement. For most households, expenditures for Long Term Care Insurance hurt more than help.

		mean of 3.71% and standard deviation of 1.22% with an autocorrelation factor of 0.60. Health-related and long-term care costs assume the parameters reflecting CPI-Medical Care Costs from January 1947 through October 2011. • Investment return evolutions are modeled as a log-normal distribution. The assumed correlation between stocks and bonds is 0%--i.e., the model lacks a co-variance matrix. The authors state: "Given lower expectations on future equity returns and the current low interest rate environment, the results of this simulation can be seen as a lower bound on required retirement wealth." • Longevity risk, investment risk, inflation risk, and health risk are modeled as separate draws from their respective distributions. • Annual health expenditures are modeled as a log-normal distribution. • Long-term care cost is modeled in a two-step process. First, the probability of needing long-term care is modeled as a Bernoulli distribution with the probabilities based on age and gender. Second, the length of care is modeled as either three months or remaining life. The model assumes an annual cost of $80,000. The model reports the follow risk metrics: 1. Probability of portfolio depletion prior to death of the surviving spouse, 2. Average terminal wealth at the death of the surviving spouse, 3. Number of years income is insufficient to meet spending needs, and 4. Amount of wealth that would have been sufficient to meet retirement spending needs. Delaying retirement has the greatest benefit for lower pre-retirement income ($60,000) households simply because social security replaces a higher percentage of their pre-retirement income. However, delaying the date of retirement also generates substantial benefits for higher-income households: "…delayed retirement is clearly the most impactful	

		risk-mitigation strategy of those considered in this study." A reduction in discretionary expenditures by 15% has little impact on the lower income households but results in a significantly greater increase in portfolio sustainability for the two upper income level households. Although reduced discretionary spending results in small annual savings in household expenditures, these savings are easily overwhelmed by the extreme costs of shocks arising from possible adverse outcomes from investment, health, and long-term care events. This is especially true given the small reduction in dollar spending levels made by lower income households. The discussion about the effects of long-term care insurance presents interesting conclusions. Although model outputs suggest that LTC insurance produces some limited benefits for the lowest income households, "based solely on the probability of having wealth remaining at death and the years without wealth, it appears that LTC insurance does more harm than good." In terms of the amount of wealth required to have a 95% confidence level for lifetime portfolio sustainability, LTC benefits lower-income couples: "…the $60,000 couple needs half as much at retirement…." However, "at higher income/wealth levels, LTC insurance is not as beneficial because they can afford to pay directly for the LTC and therefore do not need to incur the extra expense." Generally, "relative to preretirement spending, the LTC insurance premiums result increased postretirement expenses, resulting in quicker depletion of retirement wealth." Furthermore, the authors point out that many LTC carriers have left the market because of unfavorable experience with this block of business: "…recent changes in the marketplace makes [sic] it less likely that the type of policy we have modeled here will be available at reasonable premium prices." Implementation of all three strategies improves outcomes for all households. Probability of portfolio sustainability changes from a base-case model output of 29%, 8% and 14% for the pre-retirement income households to 90%, 95%and 83% respectively. Most of the improvement in the sustainability risk metric is attributable to delayed retirement. "In general small decreases in discretionary expenditures do little to change	

		retirement outcomes because the large tail risks associated with health costs and investments are too great in magnitude to be offset by this strategy." With respect to LTC insurance, the authors conclude: "…higher income families have enough wealth such that LTC costs do not play a substantial role in determining adequacy."	
2013	"Measuring the Risk of Running Out of Money in Retirement," Grant Gardner and Sam Pittman Journal of Financial Planning vol. 26 no.12 (December, 2013), pp. 38 – 44.	The authors explore "sustainability risk" which is the risk to retired investors of "…running out of money before they die." Two sources of uncertainty that arise when trying to quantify sustainability risk are (1) uncertainty regarding investor longevity, and (2) uncertainty in investment returns. The authors point out that success rates are sensitive to the planning horizon. They test a base case for a hypothetical 65-year-old male retiree withdrawing an inflation-adjusted 5% per year from a portfolio with an initial value of $1 million. The allocation is a constant 40% stock / 60% bond; inflation is a constant 2 percent per year. The expected lifetime of the investor is 19 years; the 70th percentile of longevity is 25 years; and, the 95th percentile is 34 years. At the three percentiles of the longevity distribution the expected shortfall of the base case amounts to $10,000, $80,000 and $510,000 respectively where the shortfall is expressed relative to the investor's initial portfolio value. "Either horizon choice could be reasonably justified as "prudent," but each leads to very different spending decisions. The article next explores the impact of a change in asset allocation on the change in expected dollar shortfall. At the 50th percentile of the distribution—life expectancy—the lowest risk strategy is 100% fixed income. At the 70th percentile, the minimum risk allocation is 20% equity; and, at the 95th percentile, an allocation of 60% equity results in the lowest expected shortfall value. The model then utilizes the RP 2000 SOA mortality tables to probability adjust the length of the planning horizon. The adjustment substantially reduces the dollar value of the shortfall. Monte Carlo simulation indicates that the shortfall minimizing strategy is an allocation of 40% to equity and 60%to fixed income.	A good example of how sustainability risk metrics are very sensitive to assumptions regarding the longevity variable. "Safe" asset allocations are also highly dependent on the planning horizon(s) incorporated into the risk model. The loading for equity increases portfolio sustainability as the horizon increases. When longevity becomes a random variable in a simulation analysis, shortfall risk is minimized by a 40% weighting to equity.
2013	"The 6.0 Percent	This is a short article based on outputs from a simulation model	Portfolio allocations of 100% increased portfolio

	Rule," Gerald C. Wagner, Journal of Financial Planning vol. 26 no. 12 (December 2013), pp. 46 – 54.	incorporating retirement cash flows from both an investment portfolio and from a reverse mortgage. It provides a useful summary of withdrawal options and costs of a FHA home equity conversion mortgage [HECM]. Briefly, the risk model assumes a log-normal distribution for all stochastic variables. Model output suggests a fairly high success ratio for an investment portfolio allocated at least to a 60% weighting to equity operating under a 4% withdrawal rule for a 30-year horizon: "Portfolios that have equity exposure of 60 percent and higher reach close to or exceed the 90 percent success level if the initial withdrawal rate is 3.75 percent…." The author contends that integrating a withdrawal program from a HECM with portfolio withdrawals results in a 6 percent safe withdrawal rate: "…a 6.0 withdrawal rate can exceed 90 percent success with the use of a reverse mortgage as a portfolio supplement." The simulation model tests six ways of accessing funds from a reverse mortgage; and, it discusses the relative merits of each under differing client economic situations and objectives.	sustainability only marginally when compared to allocations of 70% equity. It reflects the trend to add additional variables to retirement income modeling. In this case, however, rather than adding additional variables that tend to increase cash-flow demands—e.g., long-term care costs—it adds a variable that reflects a source of retirement income—i.e., housing wealth.
2013	"Efficient Retirement Income Strategies and the Timing of Annuity Purchases," Sam Pittman Journal of Financial Planning vol. 26 no. 11 (November 2013), pp. 56 - 62.	The paper is an extension of research originally presented by Wade D. Pfau in a 2013 essay entitled "A Broader Framework for Determining an Efficient Frontier." The broader framework refers to optimizing a stock/bond/annuity asset mix given clients with a spectrum of utility for making bequests. Pfau's retirement model's outputs trace an efficient frontier where annuities crowd out bonds. The percentage of wealth allocated to the annuity is purely a function of the client's bequest motive—a client with no utility of bequest will, according to Pfau, achieve the most efficient allocation by holding a portfolio of 100% stocks. Pittman, to a great extent, replicates Pfau's simulation model—including the assumption of CRRA, and zero correlation among all variables of interest—in order to explore the consequences of delaying the option to annuitize. Not surprisingly, Pittman's model leads to the conclusion that a client lacking any motivation to make bequests will annuitize all wealth immediately—the Yaari solution.	Pittman lists some potential drawbacks of including an annuity in the portfolio allocation: (1) lack of liquidity, (2) low return if loads are excessive, and (3) difficulties with portfolio rebalancing. In Pittman's example, client wealth is at or above the threshold needed to purchase an immediate annuity capable of providing the targeted periodic income. Annuities are not risk free: "Unless the insurance company has the same financial strength as the U.S. government, the annuity payments have default risk." Pittman correctly asserts: "Making sure the pricing of an annuity—and the riskiness of annuity payments—are correctly modeled is extremely

		The author contends that for retirees exhibiting a bequest motive, the efficient frontier generated by Pfau's model can, in many circumstances, accommodate a delay in annuitization without sacrificing investor utility. The greater the bequest motive, the longer the utility of delaying the option to annuitize. It is interesting to note that Pittman's model triggers annuitization at the point where portfolio wealth is about to become less than the cost of an annuity designed to produce target income.	important when making comparisons to other product types, such as asset allocation portfolios that support retirement spending." [Note the confusion in vocabulary as Pittman jumps from "product types" to "asset allocation."] However, Pittman's model assumes that future annuity prices are priced the way they are currently—and that the pricing factors remain static. To compensate, Pittman uses a Treasury yield curve in the denominator of the actuarial pricing factor. However, this assumption is suspect because he also assumes a constant 2.45 percent nominal risk-free interest rate.
2013	"The Benefits of a Cash Reserve Strategy in Retirement Distribution Planning," Shaun Pfeiffer, John Salter and Harold Evensky, Journal of Financial Planning vol. 26 no. 9 (September, 2013), pp. 49 – 55.	The article attempts to rebut the 2012 argument in "Sustainable Withdrawal Rates: The Historical Evidence on Buffer Zone Strategies" by Walter Woerheide and David Nanigian. Woerheide and Nanigian argue that, under most all circumstances, the creation of a cash buffer zone results in lower portfolio success rates when compared to portfolios without such a zone. Indeed, under most variations, the success rates for portfolios without a cash money market component are substantially greater. Thus, the authors suggest that a static asset allocation strategy to stocks and bonds is superior to a strategy incorporating cash that acts as a source of withdrawals during bear markets. Pfeiffer et al. contrast a cash flow reserve distribution [CFR] strategy to a reverse dollar cost averaging [RDCA] strategy under four scenarios: (1) no taxes or transaction costs, (2) transaction costs only, (3) taxes and transaction costs on distributions taken from a tax-deferred account, and (4) taxes and transaction costs on distributions taken from a taxable account. The CFR is an example of a two-bucket strategy in which upcoming, near-term monthly distributions are held in cash—i.e. pre-funded. The strategy obviates the need to liquidate a portion of a fully invested portfolio to raise cash on a month-by-month basis. The preferencing criterion selected by the authors is portfolio sustainability as measured by improvement in a retirement plan's	The article suffers a 'scale' problem. The portfolio under examination is small relative to the transaction costs imposed on it. Most of the purported advantage of the CFR strategy exists because assumed monthly trading expenses of $60 ($720 per year) constitute, over time, a significant drag on the compounding of a portfolio holding assets valued at only $200,000. A footnote states: "…unreported results suggest a smaller CRR survival advantage…" for larger-sized portfolios. The authors provide no analysis of portfolios in excess of $500,000. One suspects that their central thesis reduces merely to the statement that relatively modest costs (a flat $30 per trade) have a measurably detrimental impact if a retirement income portfolio is sufficiently small.

		survival rate. The base case presents results for a two-security portfolio allocated 60% to a diversified stock fund and 40% to a diversified bond fund operating over a maximum 30-year planning horizon. The investor rebalances the portfolio when the weight of either asset class drifts beyond 5 percent from its targeted asset allocation. The authors compare sustainability under constant-dollar withdrawal rates of 3-, 4-, and 5-percent. They conclude: "In a world of no taxes, no transaction costs, and no behavioral concerns about market volatility, then a RDCA strategy leads to an approximate 1 percentage point survival rate advantage over the CFR strategy for a 4 percent real withdrawal rate at year 30 in retirement. In the three remaining and more realistic environments . . . the results are flipped, and the CFR strategy shows between a 4.4 and 6.0 percentage point survival advantage…." Their retirement income risk model is a Monte Carlo simulation engine. The distribution parameters assume a nominal pre-tax return of 8.75 for stocks, 4.75 for bonds, and 3.50 percent for cash with inflation projected at a constant 3 percent, and stock/bond correlation a constant 30 percent. Standard deviations are 21-, 6.5-, and 2-percent respectively. The author's case example evaluates a portfolio with an initial value of $200,000 with a 60-40 allocation for the RDCA strategy with investment positions liquidated monthly to raise cash required under a 4% withdrawal rule; and, a CFR strategy initially allocated approximately 58% to stocks, 38% to bonds and 4% to cash. The authors create a set of rules under which the cash reserve—i.e., pre-funded yearly distribution—is refilled as periodic distributions are extracted from the portfolio. For example: "If both asset classes' prior year returns are negative, and there is more than two months' of withdrawal needs, and there is a need to rebalance, then the portfolio is rebalanced but no cash refill occurs due to the potential of selling a depreciated asset." The authors conclude that the primary reason for the enhanced survival probabilities for portfolios following a CFR strategy operating with tax and transaction costs is traceable to a decrease in transaction costs. Pre-funding several months' withdrawals eliminates the need to incur	

		month-by-month trading costs. Indeed, in the absence of these costs, the advantages of the CFR become almost non-existent.	
2014	"Determining the Optimal Fixed Annuity for Retirees: Immediate versus Deferred," David Blanchett. Journal of Financial Planning Vol. 27, No. 8, pp. 36 - 44.	The study creates a simulation model of a portfolio consisting of U.S. stocks, bonds and cash in order to develop a benchmark against which the investor can compare Single Premium Immediate Annuity [SPIA] contracts offering a variety of provisions with Deferred Income Annuity [DIA] contracts also offering a spectrum of contract features. The bond component is a single one-year autoregressive model with the initial interest rate set to 2.5% (nominal yield to maturity ranging from a low of 1% to a maximum of 10%) and with a white noise error term. The variance of the error term depends on future yields to maturity which are grouped in three categories—Low (Variance = 0.300%); Medium (Variance = 0.960%; and High (Variance = 2.000%). The return to cash (lower annual bound = 0%; upper annual bound = 10%) is a function of the realized bond yield in any year. Stock values are also a function of the Low, Medium and High interest rate and inflation regimes with an annual total return range of -15% to +40%. Thus, the financial (stock/bond/cash) simulations reflect the model's outputs at (1) various asset allocations (10%, 40% and 70% equity, with a fixed 10% weight to cash); (2) various assumed inflation rates (Low, Medium and High) and various nominal return assumptions (Low, Medium and High). Blanchett tests each scenario combination for its ability to sustain a fixed withdrawal based on 3%, 4%, or 5% of the portfolio's initial value. A 40 basis point investment fee is applied in each year of each simulation. Blanchett observes that the probability of shortfall—defined as the inability of the portfolio to sustain target periodic income over a 30-year planning horizon—is larger than generally noted in the risk-of-shortfall literature. Values for the likelihood of success [1 – probability of shortfall] range from 0.0% for the 10% equity portfolio seeking to sustain a 5% initial withdrawal rate in a high inflation environment, to 98.8% for the 40% equity portfolio seeking to sustain a 3% initial withdrawal rate in a low inflation environment. Comparing results to the often-cited 4% withdrawal rate rule, Blanchett	This article follows a trend in which researchers create large grids of computer-generated scenarios—in this case 81 scenarios for a stock/bond financial asset portfolio and 6,561 for annuity options in which the annuity options are ranked across a grid of 2,217 difference assumptions and preferences—in an attempt to provide investors and advisors with a 'contour map' to illustrate the financial results of various planning strategies or products within the bounds of the 'n-dimensional' grid. The analysis is product, company and date specific. The conclusions, therefore, derive from a snapshot of data; generalization across time is therefore problematic. Blanchett acknowledges that the U.S. annuity market—especially for DIAs—is evolving, and that future developments may lead to different evaluative rankings.

		notes: "…a 4 percent initial withdrawal rate has a 67.4 percent probability of success over a 30-year period based on moderate return and inflation expectations for a 40 percent equity portfolio." Against this backdrop, the article compares "…the relative efficiency of SPIAs and DIAs for a variety of retirement scenarios." The annuity contracts are ranked (1 = best / 8 = worst) using a "preference model based on utility." Rankings are weighted across all scenarios giving a weighted average rank—each scenario is assigned equal weight. Furthermore, the article presents the ranking of each type of annuity contract under the various scenarios representing fixed assumptions or static investor preferences—a total of 2,217 scenarios. The assumptions and preferences employed in the analysis are as follows: 1. Initial withdrawal rate 2. Equity allocation 3. Percentage of target retirement income covered by Social Security 4. Nominal returns 5. Inflation 6. Life expectancy 7. Shortfall risk aversion 8. Bequest preference Each of the above assumptions or investor preferences was tested at Low, Medium and High Values. Thus, for example, the study tests the utility value of a DIA providing a high income stream relative to initial portfolio value for a consumer expecting a long life and preferring a high exposure to equity, within a moderate inflation environment, with a moderate shortfall risk aversion and bequest preference against a comparable SPIA contract operating under similar circumstances. Blanchett notes that his assumptions/preferences list is not exhaustive.	

For example, a DIA puts an upper bound on the need for active financial asset portfolio management thus simplifying the late-in-life income generating task. This is a type of hedge against cognitive impairment risk as well as longevity risk.

The annuities under evaluation are specific products offered by insurance carriers at a precise moment in time—the most favorable quote on January 26, 2014 for joint and 100% survivor annuities for a retired couple each age 65. The list of contract types is as follows:

1. Nominal SPIA with life only payment feature
2. Nominal SPIA with 20-year term certain payment guarantee
3. Real SPIA with life only payment feature
4. Real SPIA with 20-year term certain payment guarantee
5. Nominal DIA with life only payment feature (payout starts at age 85)
6. Nominal DIA with cash refund feature (payout starts at age 85)
7. Real DIA with life only payment feature (payout starts at age 85)
8. Real DIA with cash refund feature (payout starts at age 85).

The model output assigned the best rankings across all the Low/Medium/High scenarios to the nominal SPIA with the 20-year refund feature and the nominal SPIA with the life only payments. The real DIA contracts received the worst rankings under the aggregated scenario rank compilation approach. However, Blanchett also considers specific preferences or assumptions—e.g., Low, Medium or High investment returns over the retirement period—and ranks contracts holding other scenarios constant—a type of partial first derivative model. His results indicate that "…an annuity may be slightly less efficient for one scenario, yet materially more efficient in another." Blanchett builds a multivariate regression equation to determine the relative "importance" of each of the eight assumptions/preferences on the relative "attractiveness" of annuity contract. For example, the regression model's outputs suggest that investors with a high percentage

		of income coming from Social Security benefits would be likely to find DIAs attractive. The study concludes: "…the relative attractiveness of different annuity types considered varies significantly by scenario and assumptions…SPIAs—especially SPIAs with 20-year period certain payments—are the most efficient, on average, across all the scenarios considered. Real SPIAs were the most volatile annuity type from an efficiency perspective, where these products tended to be either the absolute best or absolute worst annuity type. Real DIAs were also the least efficient of the eight annuity types considered. Real DIAs were not optimal in any of the 6,561 scenarios considered…."	
2014	Falling Short: The Coming Retirement Crisis and What to Do About It Charles D. Ellis, Alicia H. Munnell and Andrew D. Eschtruth, Oxford University Press (2014).	This short book is a good place at to begin a study of the history of retirement in the United States ("well into the nineteenth century, about half of all 80-year-old men in America still worked."). It provides a valuable long term perspective on our public and private retirement support systems, as well as helpful insights into how individual investors can enhance their personal retirement financial security in the face of increased longevity expectation and relatively meager pension wealth—Work longer, boost savings (including reforms of the Social Security system and 401(k) provisions), and utilize home equity. The authors term the 1980s and 1990s the "golden age of retirement." Full Social Security benefits were available at age 65, Medicare covered most healthcare costs, Social Security replaced approximately 40% of pre-retirement income for the average claimant, and a larger percentage of workers were covered by Defined Benefit Pensions guaranteeing lifetime income. Throughout the book, the authors provide critical insights that encapsulate the essence of the modern day retirement income challenge. For example: • "…a couple retiring today faces a 50 percent chance of at least one member living to 92." • In 2012, Medicate annual out-of-pocket expenses averaged $7,500 for a couple. In addition, retirees are responsible for all	This is a non-technical book written for a primary intended audience of government policy makers and general investors. It contains, however, a good bibliography and is well documented via a footnote apparatus that calls attention to many important academic studies and government public policy papers. The book defines retirement financial risk as follows: "Retirees face the risk of either spending too quickly and outliving their resources or spending too conservatively and depriving themselves of necessities or modest pleasures." The authors provide short discussions of controversial academic topics like definitions of income "optimality" in terms of replacement ratios, level income streams, life-cycle optimization models, and so forth. Likewise, they incorporate and evaluate the conclusions of some studies that indicate that current and prospective retirees face generally favorable financial prospects. The strategy of recognizing and discussing competing

		expenses not covered by Medicare, such as dental care, eyeglasses, and hearing aids….Estimates are that the average coupe at age 65 faces about $200,000 in out-of-pocket and noncovered medical expenses during their retirement years.” • “…in 2013, the median household approaching retirement with a 401(k) had a total of only $111,000in 401(k) and IRA balances.” • “…53 percent of working American households, roughly 35 million households, will not have enough retirement income to maintain their preretirement standard of living if they retire at age 65—which is later than the current average retirement age—and annuitize all their financial assets, including a reverse mortgage on their homes.” The book strongly recommends “buying” an annuity from Social Security over the purchase of a commercial annuity contract from an insurance carrier. The investor implements this strategy whenever he or she delays claiming the social security benefit by utilizing the financial assets on their personal balance sheet. The authors point out that (1) unlike a commercial annuity, the Social Security benefit calculation is actuarially fair, (2) the benefit is calculated based on general population mortality, and, therefore, need not reflect large adverse selection risk in the annuity's pricing, (3) the benefit is inflation adjusted, and (4) Social Security benefits are priced advantageously in low interest rate environments. All this leads to the recommendation: “…use a portion of our 401(k)/IRA assets to cover living expenses for a few years and delay claiming Social Security.” They calculate that Social Security benefits claimed at age 70 are 76% higher than benefits claimed at age 62.	perspectives keeps the work from devolving into a polemic about either public policy requirements or individual investment imperatives.
2014	“Retirement income among wealthier retirees,” Anna Madamba, Stephen P. Utkus, John Ameriks. Vanguard Research (May,	Based on results from a survey of wealthy, retired households, the authors utilize cluster analysis to identify eight types or categories of retired investors. An investor is assigned a type based on the predominant source of their retirement income. Categories and population percentages are:	Survey data presents an interesting decomposition of demographic data within and among the various household types. For example “annuity investors” tend to hold variable annuity contracts, live in households in which one or more members are in poor health, less motivated by leaving bequests, had lower-than-average income, and were more

	2014), pp. 1 -16.	1. Social Security Recipients (26%) 2. Pensioners (24%) 3. Retirement Investors [Self-funded tax-deferred accounts] (18%) 4. Taxable Investors (17%) 5. Liquidity Investors [bank accounts] (4%) 6. Annuity Investors (5%) 7. Real Estate Investors (3%) 8. Business Investors (2%). Median household wealth for households with at least $100,000 in non-housing wealth is $395,000. Adding the present value of pension and Social Security guarantees pushes non-housing wealth to $1.1 million. The sample population was limited to households with at least one retired member ranging in age 60 to 79. They conclude: "Households are clearly different, even at a high level, in terms of risk characteristics, the nature of income guarantees they own, tax status of accounts, liquidity preferences, and the desire for help from an advisor."	likely to have a financial advisor. The survey data is useful for a number of reasons including its clear demonstration that the retired wealthy investor population is not homogeneous.
2014	"Differential Mortality and Retirement Benefits in the Health and Retirement Study, Barry P. Bosworth and Kathleen Burke, Economic Studies at Brookings (PRIL 2014), PP. 1 – 22.	This study reviews a multitude of earlier papers that, for the most part, conclude that life expectancy is rising more rapidly for individuals in the higher levels of wealth and education. It considers data from the Health and Retirement Study with a primary focus on the progressivity of benefits (Disability, Retirement, and Survivor) from the Social Security System. An early seminal work in the area of mortality differentials by socioeconomic status is the 1973 study by E.M. Kitagawa and P.M. Hauser: *Differential Mortality in the United States: A Study in Socioeconomic Epidemiology*. Follow on studies by various authors general text for disparities in mortality either using income or educational attainment as the primary variable of interest. Many studies conclude that there is an increasing differential in mortality between the highly educated, high income group and other population groups: "Our analysis of the mortality experience of participants in the HRS shows a strong pattern of increasing differential morality in which life expectancy	The authors conclude that the Social Security benefit system remains progressive despite the longer expected claiming by the top socio-economic bracket. A primary reason for this is that the disability and survivorship benefits are more likely to be claimed by members of the lower socio-economic population groups.

		is rising for those at the top of the distribution of individuals ranked by alternative measures of socio-economic status, but it is stagnate or declining for those at the bottom. The study presents several data tables of interest. For example, for men age 75 and over, the mortality rate of those with a college degree or above is 0.74 of the category's average rate; households with career earnings in the top quintile exhibit a mortality rate of 0.73 of the category's average rate.	
2014	"Retirement Planning by Targeting Safe Withdrawal Rates," David M. Zolt Journal of Financial Planning vol. 27 no. 10 (October 2014), pp. 20 - 23.	The article seeks to provide "lookup tables" of "safe withdrawal rates" for various planning horizons over a spectrum of asset allocations. Although the tables are products of Monte Carlo simulations that assume a normal distribution of financial asset class returns, the tables are recommended to advisors because they obviate the need for "cumbersome Monte Carlo simulations." The output from Zolt's model confirms the 95-percent success rate of the 4% withdrawal rule for portfolios holding at least 50% equity. The tables , however, provide information regarding safe withdrawal rates for older age clients whose planning horizons are expected to be less than 30 years. Finally, the tables provide success ratios for both the traditional 4% rule, and two other modifications: (1) a no-inflation increase rule where "target percentage tests" are failed because of lackluster portfolio performance; and (2) a withdrawal rate with a catch-up (maximum 3% annual adjustment) when portfolio performance exceeds expectations. Using the latter withdrawal rule increases, according to the author, the safe withdrawal rate to 5 percent—at a 95% confidence level—over a 30-year horizon.	This article seeks to calculate and present safe withdrawal rules for a spectrum of retirement ages. Note: tables of sustainable withdrawal rates over various planning horizons are found in previous research articles. See, for example, "Revisiting the '4% spending rule'," Maria A. Bruno, Colleen M. Jaconetti and Yan Zilbering Vanguard Research (August 2012).
2014	"Estimating the True Cost of Retirement," David Blanchett, Presented at the Society of Actuaries 'Living to 100 Symposium' (January	The author examines empirical data on spending patterns during retirement. Generally, retired investors do not exhibit a pattern of spending that matches a constant amount periodically adjusted for inflation—e.g., the 4% rule. Rather, the cost curve followed by most retirees is 'smile' shaped with expenditures gradually decreasing with age until they move upwards towards the end of life. For example, the author cites one study that finds that "…consumption-expenditures	A helpful look at empirical studies showing expenditures over time for households with retired investors. Estimating expenditure curves for households at various income levels, increases the accuracy of retirement cost projections. When using traditional Monte Carlo analysis, the sustainability rates are greater with realistic spending patterns

	8 – 10, 2014).	decrease by about 2.5 percent when individuals retire, expenditures continue to decline at about a rate of 1 percent per year after that." The presentation analyzes expenditure curves as a function of household income; and explores how household expenditures change for various ages. It notes that the commonly used Consumer Price Index for Urban Consumers [CPI-U] differs modestly from the newer Consumer Price Index for the Elderly [CPI-E]: "from December 1982 to December 2012, the average annual change in the CPI-E has been 3.07 percent versus 2.92 percent for CPI-U…." Medical expenses are, not surprisingly, an important factor for retired households: "The median percentage of medical expenses increases from approximately 5 percent of total expenditures at age 60 to 15 percent by age 80." Despite the increasing percentage of the budget devoted to health costs as the investor ages, "…the real change in annual spending through retirement is clearly negative." Some of the decrease may be attributable to forced spending reductions by investors who lack sufficient resources to sustain their target spending level. This conjecture leads the author to segregate retired households by income levels in order to ascertain whether the spending curves differ significantly by wealth. The data suggest that low spending / low net worth households tend to have increases in spending over time; high spending / low net worth households, however, tend to exhibit considerable declines in consumption over time. This data forms the basis for creating various spending curves—via a regression equation--to project changes in retirement expenditures throughout retirement. The steps in the analysis are as follows: 1. For the aggregate group of retired investors, estimate via a Monte Carlo simulation of a simplified investment portfolio, the cost of sustaining retirement income over a 30-year period given a constant real withdrawal. The constant-dollar spending rule considers initial withdrawal rates from 2% through 8%. 2. For each sub-group from the retired investor population perform a Monte Carlo simulation using the projected real spending	than they are under the 4% spending rule. This suggests that the true cost of retirement may be significantly less than most retirement income models suggest.

		curve characteristic of that group. 3. Rerun the above analysis incorporating the force of mortality. The use of projected spending curves increases portfolio sustainability rates and lowers the estimated cost of retirement. For example, under the 4% spending rule for a fixed 30-year horizon, the portfolio sustainability rate is 73.3 percent. Mortality adjusting the planning horizon increases the rate to 81.5 percent assuming a joint age 65 retired couple. Using the projected spending curve for the $50,000 household, increases the rate to 89.9 percent. [Using the $100 spending curve increases the mortality-adjusted portfolio survival rate to 93.2% assuming an initial withdrawal amount equal to 4% of portfolio value]. The author asserts: "…modeling the cost over the expected lifetime of the household, along with incorporating the actual spending curve, results in a required account balance at retirement that can be 25 percent less that the amount required using traditional models."	
2014	"The Perfect Withdrawal Amount: A Methodology for Creating Retirement Account Distribution Strategies," E. Dante Suarez, Antonio Suarez, and Daniel T. Walz SSRN, November 11, 2014.	The authors characterize retirement income planning in the following terms: "…retirees generally wish to use their retirement funds to support a standard of living that is as high as possible, but without depleting their account so quickly that the years still ahead become difficult to finance—the so called failure risk. On the other hand, withdrawing 'too little' money might simply translate into excessively high balances in their accounts at the end of the lifespan horizon—the so-called surplus risk—and the 'golden years' lifestyle would have been restricted unnecessarily." The research goal is to develop formulas that can be applied in each period as a function of age, assets, desire for safety, and spending flexibility. In a brief literature review section, the article notes that early papers on the topic of withdrawals from retirement accounts focuses on a constant amount (perhaps adjusted for inflation) that is set at retirement outset. A series of later-generation research explored "adaptive" rules which were pre-set and which the investor applied throughout retirement. Later studies advocated reassessing the viability of withdrawals as a function of current age and stochastic lifespan so that midcourse	The sum-product formula representation of sequence risk that the authors introduce early in the article is somewhat confusing. It is not until the middle of the article that the reader learns that the sum of the sequences of return products is, in fact, the series of annually recalculated—i.e., simulated return sequences—with the number of terms ['n'] reduced by one with the passage of each year. Thus, it is the product of return sequences measured over n years + n-1 years + n-2 years, and so forth. The authors acknowledge that the article's primary focus is to elaborate the structure of a general solution to retirement income planning as opposed to providing outputs to justify financial planning recommendations: "…we see an opportunity to improve the computation of the PWA probability distributions by using more advanced methods to

		corrections are not derived from a mechanical application of pre-set rules. Against this strand of investigation, the authors contrast papers that test withdrawal amounts in terms of investor utility with studies incorporating a subjective discount rate to reflect 'impatience.' The study begins with the presentation of the perfect withdrawal amount [PWA] under conditions of perfect certainty with respect to future returns and investor lifespan. Essentially, the solution to the PWA is comparable to finding the fixed periodic payment required to amortize a variable-rate loan in exactly 'n' periods. In this case, however, the period is unknown, the target ending value may not equal $0 and amortization applies to a portfolio of assets rather than to a liability. Also, the applicable interest rate is not fixed as in a fixed-rate loan, but reflects the variable sequence of period-to-period returns from the investor's portfolio. The basic mathematical structure of the equation employed to solve for PWA is expressed as follows: Withdrawal amount = (Cumulative Portfolio Return for 'n' years – Target Bequest) / (The sum-product of return sequence) from n = 1 to n = the end of the planning period. Given that critical inputs for determining the PWA are not known in advance, "…the retirement withdrawal question is, at its core, a matter of 'guessing' what the PWA will turn out to be (eventually) for each retiree's portfolio and objectives." The sequence risk is, of course, a function of when large or small returns occur in retirement. Two retirement return vectors can exhibit the same cumulative investment returns but generate different amounts of wealth depending on the order in which the returns appear in the sequence. The authors term the return sequence sum-product: S_n. This simplifies the PWA equation to: $$W = (R_n K_S - K_E)S_n$$ Where K_S and K_E represent beginning and ending portfolio value. The	construct the probability profiles for the financial assets' returns. The bootstrapping procedure used here is very practical, but we are aware of its numerous drawbacks." Although the authors claim that many of the insights and procedures outlined in the article are unique, a similar approach appears in greater detail in Gordon Pye's 2012 book <u>The Retrenchment Rule: When It's Too Late to Save More For Retirement.</u> They appear to be unaware of Pye's work.

		authors note that the "…sequencing factor formula is the same as the sinking fund factor under variable rates of return, but the interpretation and application is thoroughly different in each case." The authors transform the above equation to express the PWA as a *rate* rather than an *amount*. This rate takes into account the fraction of the initial portfolio value that the investor wishes to leave as a bequest. The model's final expression becomes: $$w/K_S = R_nS_n - S_n(K_E/K_S)$$ This model is applied by creating "…an actual probability distribution for PWAs, assuming a 100% equity portfolio….This allocation was chosen because it maximizes the variance of the returns on the account's assets, and thus produces the PWA distribution with the largest dispersion possible." The probability distribution reflects 20,000 Monte Carlo simulations in which aggregate returns are determined for each vector and sequencing of monthly returns for each vector is determined. Thus, for each aggregate return vector [R_n], the sequencing [S_n] is simultaneously determined to give 20,000 (R_n,S_n) pairs over a 30-year planning horizon 'n'. To evaluate how any given portfolio performs, the target bequest, initial portfolio value, and periodic withdrawal amount is applied to each return-sequence pair to arrive at a final success/failure profile. The construction of a probability distribution allows the advisor to demonstrate the likelihood of financial success when selecting target bequests and/or target withdrawal amounts: "…the statistically appropriate thing to do is to calculate the probability distribution of PWAs for the current situation and choose from there—every year." Such a calculation provides insight into the probability that a retiree will have to lower future withdrawals—i.e., a confidence level for maintaining a withdrawal amount—as opposed to measuring the likelihood that a portfolio will become depleted. Each year the advisor uses the model to recalculate the PWA based on	

		the updated probability distribution, the clients current age, and the new portfolio balance. New return sequences are simulated and new PWAs are calculated under application of the same procedure. The plethora of withdrawal rate rules is unnecessary because the PWA calculation does not trigger "…different schemes if certain conditions are met." ""…the simplicity of our approach…cuts through the Gordian knot of some of the adaptive rules found in the previous literature…." The authors provide several short case studies to illustrate how the PWA model can adapt to specific investor needs and preferences—e.g., income stability, safety, and withdrawal amount).	
2014	"From Savings to Income: Retirement Drawdown Strategies," Luke Delorme American Institute for Economic Research (2014)	This research paper examines retirement distribution strategies for a portfolio allocated 50% to stocks [S&P 500 Index] and 50% to bonds [U.S. Long-Term Corporate Bond Index], and rebalanced annually. The author makes the interesting point that not only is a percentage of shortfall risk metric inappropriate as a measure of retirement risk unless the magnitude and length of shortfall are also considered; but also that such a measure is deficient because it does not take into account excess amounts—i.e., terminal wealth—that could have been used to boost lifetime spending. Thus, in terms of generating investor utility, no positive weighting is put on gifts or bequests. Data is broken into 553 historical 40-year periods from January 1928 through December 2013. This is a rolling period methodology which, the author points out, has the drawbacks of (1) over-representing months in the middle of the series relative to months at the beginning and end; and (2) producing highly correlated results due to lack of independent sampling. Nevertheless, he contends that this method produces results that "…are similar to the Monte Carlo simulation, which randomizes return streams." An appendix at the end of the study presents results from a Monte Carlo analysis which, in the author's opinion, generally confirm the rolling-period historical results. The study reports on eight retirement spending strategies each of which employs one of twelve initial annual percentage withdrawals ranging from 2 to 7.5 percent. For each strategy the author calculates a utility score which acts as a preferencing metric. Each combination of the	Study points out that although there is "…ample information about how and where to invest assets during accumulation… information about post-retirement spending strategies is less plentiful." Although the author uses a utility score preferencing criterion, his calculation of utility, does not aggregate a utility function over the range of results. Rather, utility is an equally weighted average of two points: Utility = [(Average Spending %) + (Minimum Spending %0] where 'spending %' is the level of spending as a percentage of original portfolio value. The author makes the somewhat dubious assertion that this type of utility score calculation is valid because it considers both the average spending and the downside deviation from the average. The article contains the following statements: • "The model shown here does not account for specific levels of risk aversion." • "…the utility function used here does not allow for preferences in near-term versus long-term spending." • "…the preference for near-term spending

		drawdown and initial percentage withdrawal strategies produces a different retirement income pattern and generates a different utility score. The study presents best case, worst case and average results. The utility scores reflect tradeoffs faced by retirees. For example, "…an increasing drawdown percentage starting at 4.5 percent offers the highest average utility. However, under the worst-case scenario, this strategy offers significantly lower utility than the same strategy that starts with a 3 percent drawdown." The withdrawal strategies under examination are: 1. Constant dollar—e.g., the 4% rule 2. Constant Percentage—e.g., a unitrust distribution formula 3. Smoothed Percentage—a three-year average rule 4. Constant-Percentage with ceiling 5. Constant-Percentage with floor 6. Inflation-adjusted Percentage 7. Increasing Percentage 8. Required Minimum Distribution per the IRS requirements from age 70+. The results fail to reveal a clear winning strategy. Drawdown strategies that are slightly higher than 4% of initial value "tend to achieve the highest utility scores." The study concludes: "Initial drawdowns below 3 percent should be considered only for the most risk-averse retirees, or when return expectations are well below the historical average. Initial drawdowns above 5 percent should be considered only if a flexible strategy that responds to changing market returns is chosen and risk tolerance is high." The 4 percent withdrawal rule is more likely than other strategies to result in portfolio depletion. Generally, an increasing percentage strategy produces better results. In many historical periods, the constant percentage with floor strategy fares well.	has little impact in comparable studies and adds unnecessary complexity."
2014	"'Short' Falls: Who's Most Likely to Come up Short in Retirement, and	The author utilizes the EBRI [Employee Benefit Research Institute] Retirement Security Projection Model to estimate (1) the percentage of retirees likely to run short of funds during retirement; and (2) the time that the shortfall takes place. Rather than simulating a representative	The model follows a shortfall risk measurement approach with shortfall calculated on an asset/liability basis. Specifically, for each population quartile of retirees—where the quartiles

	When?" Jack VanDerhei Employee Benefit Research Institute Notes Vol. 35, No. 6 (June, 2014), pp. 2 – 18.	individual's ability to utilize personal resources for production of a target retirement income, the model operates over wide populations of retirees in specific states—e.g., Oregon or Kansas—or within the U.S. generally. The model was built to quantify the likelihood, timing, and magnitude of poverty for the population of retirees so that state and federal agencies could adequately plan public policy programs. Specifically, the analysis divides the population into four quartiles based on pre-retirement income. The model avoids the 'replacement rate' target approach to determining retirement income adequacy. Generally, replacement rate calculations are based on the proportion of pre-retirement income that can be converted into lifetime income via annuitization of financial/pension wealth. However, the author points out that few actual retirees elect an annuity option; and, therefore, a replacement rate (ratio) analysis is not realistic. A life-cycle modelling approach is also rejected. Although the life-cycle model assumes that investors derive maximum utility by smoothing consumption over the planning horizon, such a budget strategy would, for investors at the lowest quartile of pre-retirement income, fail to produce even a subsistence income stream. The study estimates the percentage of the general population that will have resources sufficient to meet "average expenses" and, in an extension of the model, to also meet uninsured health care costs. The model simulates the financial resources that will be available, over time, from Social Security, Defined Contribution and Defined Benefit Pensions, IRAs, Savings, and Net housing equity. A household runs short of money in any trial in which aggregate resources are insufficient to meet average retirement expenditures. The model divides expenditures into deterministic costs (food, housing, transportation, etc.) calculated from the Consumer Expenditure Survey, and into stochastic costs from uninsured health care expenditures. The model considers three expense	are assigned by the level of pre-retirement income—the average financial resources are matched to the average expected retirement costs for basic needs and to the stochastic costs for health care. The study concludes that shortfall risk is age and income dependent—lower pre-retirement income foreshadows higher shortfall likelihood—i.e., higher probability of needing government assistance (welfare) in old age.

		levels: (1) 100% of average deterministic expenses for each population quartile, (2) 90% of average deterministic expenses, and (3) 80% of average deterministic expenses. Model outputs vary significantly across the quartiles ranging from a success rate of only 16.8 percent for the lowest income quartile to 86.4 percent for the highest income quartile assuming retirement at age 65 and a planning horizon of 35 years—i.e., to age 100. "…[The] stark conclusion: The lowest preretirement income quartile is the cohort where the vast majority of the shortfall occurs, and the soonest….Indeed, as the results across multiple scenarios and assumptions show, the lowest-income quartile is the most vulnerable, while longevity and long-term care are the biggest risk factors across the entire income spectrum."	
2014	"The Use of Annuities in an Optimal Retirement Portfolio," Don Ezra Risks & Rewards: The Investment Section of the Society of Actuaries Issue 64 (August 2014), pp. 1 – 8.	The author observes that there comes a time when longevity risk dominates equity risk; and, therefore, the most risky course of action is not to secure longevity protection. However, investors also seek growth; and the best method to hedge longevity risk is through acquisition of a contingent annuity that begins to provide an income at the investor's life expectancy. Prior to that time, equity is the preferred method to insure growth. However, Ezra also points out the need for safety. He suggests that investors consider adding a fixed income position to the portfolio in the form of maintaining "five years of spending guaranteed via government bonds and TIPS…." This gives the investor a five-year window to avoid drawing down on the equity portion of the portfolio. He recognizes that this strategy oversimplifies the retirement income problem. He parallels Milevsky's observation that spending fixed income during bear markets creates a portfolio that increases volatility at an increasing rate. Even over the long term, equities remain risky investments; however, the fixed income component means that "we still have time to prepare." The strategy under consideration provides, according to Ezra, (1)	Although the author asserts that purchasing an immediate annuity upon retirement is often not the optimal retirement solution; nevertheless, the measurement of the present value of lifetime retirement income is best made by reference to this instrument.

		Growth, (2) Safety of Income, and (3) Longevity Protection.	
2014	"Age-dependent investing: Optimal funding and investment strategies in defined contribution pension plans when members are rational life cycle financial planners," David Blake, Douglas Wright and Yumeng Zhang, Journal of Economic Dynamics & Control Vol. 38 (2014), pp. 105 – 124.	The authors use the term 'deterministic lifestyling' to characterize pre-retirement 'glide path' investing. They argue that a strategy of 'stochastic lifestyling' is optimal. In this strategy, the investor begins with a high equity allocation followed by a gradual shift towards bonds as retirement age approaches. However, the shift towards bonds depends on the realized outcome of specific state variables including the return on equities and the evolution of the labor income growth process: "…the switch away from equities is stochastic rather than predetermined, and is dependent on past investment and salary growth experience." At retirement, the investor converts all bonds to annuities and considers annuitizing some or all of the equity position. Absent any bequest motive, the investment decision must account for three investor preferences: 1. Smoothing consumption across different possible states of nature within a given time period; 2. Smoothing consumptions across different time periods; and, 3. The extent to which current consumption is valued relative to future consumption. Assuming that the investor is a rational, life-cycle financial planner, "The plan member…needs to form a view on both the trade-off between consumption in different states of nature in the same time period and the trade-off between consumption and consumption variability in different time periods. Attitudes to these trade-offs will influence the optimal funding and investment strategies of the pension plan." The authors' post-retirement model assumes constant relative risk aversion, Epstein and Zin preferences, a constant subjective discount factor, a bond fund with a constant annual real return (2%), and a risky asset (equity) fund with a real return distribution expressed as:	Given that the model assumes that utility is derived exclusively from consumption (no bequests), its focus is on investor preferences with respect to the timing, level, and variability of consumption opportunities. The article provides (1) a clear explanation of three types of assumptions often found in life-cycle models and (2) a clear roadmap for how changes in the value of these assumptions impact model outputs—i.e., the role of each assumption in model sensitivity testing. Risk aversion: reflects the investor's desire to stabilize income across different states of nature (e.g., prosperity, recession, depression) at a specific period in time. The higher the investor's risk aversion coefficient, the more reluctant to invest in a strategy that would force a decreased standard of living in the wake of a decrease in equity value. EIS (elasticity of intertemporal substitution): reflects the investor's desire to smooth consumption over time. A low EIS value indicates that the investor has little tolerance for period-to-period consumption volatility. The investor prefers to maintain consumption in the current period relative to the level of consumption enjoyed in the previous period(s). The sign and magnitude of EIS reflects the relationship between a 'substitution effect' and an 'income effect.' For an increase in the risk free rate [a state variable] the substitution effect is always negative—i.e., motivates a decrease in current consumption—because the increase in interest rates makes future consumption relatively

		$$R_x = r + (\mu - ½\sigma^2) + \sigma Z_{1,x}$$ Where μ is the annual equity risk premium (4%), σ is standard deviation (20% per year) and Z is a series of IID standard normal random variables. Annuity pricing factors also reflect the simplifying assumption of a constant real rate; and, the annuity pricing formula's cost reflects the annuitant's age rather than a series of stochastic interest rate changes. Given the authors' assumption that a long-term investor focuses not on a wealth level but on the consumption stream that can be financed from a given level of wealth, investment decisions are influenced by preferences between current and future consumption. In 1989, Epstein and Zin presented the following utility function that incorporates consumption [C], the coefficient of relative risk aversion [γ], the coefficient of EIS [φ], and the personal discount factor reflecting consumption time preference, or impatience [β]: $$U_t = \{(1-\beta)(C_t)^{(1 - (1/\varphi))} + \beta(E_t[(U_{t+1})^{1-\gamma}])^{(1 - (1/\varphi)/(1-\gamma))}\}^{1/(1 - 1/\varphi))}$$ For an investor with Epstein-Zin utility, the general solution to the optimal asset allocation to the risky asset [Θ_t] is based on a weighted average of the short-term demand to hold the risk asset—i.e., the Merton Optimum—and, an intertemporal hedging demand with depends on the covariance between the risky asset return and the investor's utility per unit of wealth over time: $$\theta_t = \frac{\mu_t}{\gamma\sigma_t^2} + \left(1 - \frac{1}{\gamma}\right)\frac{cov_t(R_{t+1} - \left(\frac{U_{t+1}}{F_{t+1}}\right))}{\sigma_t^2}$$ Optimal allocation to the risky asset is constant over time if the investment opportunity set also remains constant. At retirement age 65, the investor can choose to annuitize all wealth or	cheaper. Current consumption decreases and the current rate of savings increases. However, an increase in the risk-free rate can result in a positive income effect if it motivates an increase in current consumption because of the higher rate of earnings on existing savings. Depending on which effect dominates, the investor will increase or decrease current consumption. If neither effect dominates, the EIS will be zero and consumption will not change despite the volatility in the risk free rate. <u>Subjective or Personal Discount Factor</u>: reflects the investor's preference for current consumption relative to future consumption. The lower the personal discount factor, the higher the personal discount rate—i.e., the more the investor values current over future consumption. Note: EIS is often calculated as the reciprocal of the coefficient of Constant Relative Risk Aversion despite the fact that the empirical support for such a value is mixed. That is to say, a high coefficient value for CRRA tends to be associated with a low value for EIS. High risk aversion portends unwillingness to tolerate consumption variability.

		hold a proportion of wealth [ϴ] in the risky asset portfolio. Subject to some reasonable constraints such as no short selling, the optimization problem is to maximize Utility with consumption and allocation to the risky asset as the control variables. An analytical solution does not exist, and the authors generate 10,000 simulations of the wealth level supporting retirement consumption. At retirement, all bonds are exchanged for annuities; and, by age 76, the investor annuitizes all wealth with the pace of phased annuitization dependent on the coefficient of relative risk aversion. Lower risk aversion generally leads to increased consumption during retirement, however, the increased consumption level and the delay in converting risky assets to annuities means that the investor is exposed to higher consumption variability. After retirement, a high coefficient of EIS suggests that the investor does not consume all current available income; but, rather, reinvests surplus income to buy additional future annuity income. However, investors who cannot tolerate consumption volatility across states of nature in a given time period are also unlikely to tolerate consumption volatility over time—i.e., they are likely to exhibit a low EIS. Finally, the personal discount factor does not influence the optimal—i.e., utility maximizing—investment strategy for retirees under the authors' model: "The size of the personal discount factor has no effect on the optimal asset allocation after retirement. The size of the EIS has a marginal impact on the optimal asset allocation both before and after retirement."	
2014	Evangelos Karagiannis, "Stochastic Investment Horizons in the Asset Allocation Decision and Liability-Driven Investing," CFA Institute (2014), pp. 1	The author considers two base cases: (1) an investor with a fixed investment horizon; and, (2) an investor committed to holding a portfolio for whatever period of time is required to achieve a pre-determined rate of return. In the first case, the horizon is fixed and the rate of return on terminal wealth is a random variable; in the second case, the target return is fixed and the time required to achieve it is the random variable. In the first case, assuming continuous compounding of annualized returns, an expression for the dynamics of the price process is:	This short article is a primer on the continuous time finance approach to asset price dynamics. Karagiannis illustrates, using a simplified model, how the stochastic investment horizon (time to attain target wealth) approach is complementary to traditional mean-variance analysis. The article offers insight into the promises and pitfalls of attempting to remedy a shortfall [funding ratio < 1] by increasing investment risk. However,

	– 4. www.cfainstitute.org	$$dY = (r+\delta)Ydt + \sigma YdZ$$ where the instantaneous change in Price ['Y'] follows a random walk process with drift 'μ' (sum of the risk free rate 'r' plus the risk premium 'δ') and constant volatility 'σ.' In the price change process, 'Z' represents standard Brownian motion. Utility ('u') of the excess return ('δ') is expressed as: $$U = E(\text{excess rate of return}) - \tfrac{1}{2}a\text{Var}(\text{excess rate of return})$$ Where 'a' is the investor's risk aversion. Given this process, the optimal portion of wealth ('y') invested in the risky asset is the Merton optimum: $$y = \frac{E(\text{excess rate of return})}{a\text{Var}(\text{excess rate of return})}$$ If 'A' represents the risk aversion coefficient of the representative (average) investor—"societal risk aversion"—then, in a market set by fixed horizon investors, the market risk premium equation becomes: $$\delta - \tfrac{1}{2}\sigma^2 = \tfrac{1}{2}A\sigma^2$$ For a fixed-horizon investor, the risk premium [$\varepsilon = \delta - \tfrac{1}{2}\sigma^2$] is a linear function of 'A' and 'σ^2'; and the fixed horizon investor allocates the following proportion of wealth to the risky asset: $$y_{FH} = \frac{\delta - \tfrac{1}{2}\sigma 2}{a\sigma^2} \text{ or, } \frac{1}{2}\frac{A}{a}$$ The commitment to risky assets is a function of the ratio of the individual	the analytics provided by the author are appropriate only in the face of a host of simplifying assumptions. Karagiannis warns that closed-form solutions are not available when: • the interest rate varies throughout the planning horizon; • both liabilities and assets are stochastic and correlated; and, • assets are not log-normally distributed. However, analytical calculation of time to "hitting time probabilities" offers valuable perspective on the risks of ALM investing.

		investor's risk aversion level ['a'] to the risk aversion coefficient for the weighted average investor ['A']. By contrast, a stochastic horizon investor sets a target rate of return 'R' for his risky asset portfolio. In this case, the time 't' it will take to achieve 'R' is the unknown. The decision process is based on the distribution of 't'; and the expected excess return on the risky asset portfolio is R – rt, where r is the risk free rate compounded over time. Karagiannis notes that uncertainty (t) is transferred to the term with the risk-free rate 'r'. This means that the variable is not the risky asset portfolio's return 'R' because the investor knows that given a sufficiently long planning horizon, the drift and diffusion terms will push the portfolio to 'R'. Rather, uncertainty over the stochastic planning horizon is the risk free rate available to the investor during the holding period. The risk premium equation becomes: $$\varepsilon(r+\varepsilon)^2 = ½Ar^2\sigma^2$$ which means that the risk premium is no longer a linear function of 'A' and σ^2. The risk premium [$\varepsilon(r+\varepsilon)^2$] increases with 'A,' r,' and 'σ^2'. When the available risk-free rate is high, it requires a higher risk premium to induce a target-return investor to hold risky assets. In a low interest rate environment, the target-return investor will allocate more to the risky asset. The allocation weighting, detailed in the article, is a function of "hitting time"—the uncertain time it will take for the portfolio to achieve the target return 'R'. Time is the stochastic variable and allocation is a function of the risk-free rate. The article next applies the above price-change and asset allocation principles to ALM management for investors with stochastic planning horizons—i.e., target return investors. Using a simple fact pattern of a liability 'L' due at time 'T' with a constant discounting factor of 'r', the example considers a fund with a current value less than the present value of L—e.g., an underfunded pension plan. If 'r' is the risk-free rate, then the only way to meet the liability is to hold a risky asset portfolio	

		with a drift of r+δ and volatility σ. In the future, if the current value of assets [A] equals the present value of liabilities [L], the portfolio can be immunized with no further risk to its funded status. Karagiannis generates the deterministic path for a liability due in 30 years with r = 3%, and simulated paths for risky portfolios with μ = 6% and σ = 30%. In some cases, the hitting time occurs prior to 30 years (success); in other cases, the path of the risky asset always remains below the liability value (failure). An analytic expression for the probability of achieving success—i.e. funding an underfunded investment program by holding a risky asset portfolio (making the simplifying assumption that the portfolio follows a Brownian Motion: $X_1 = X_0 + \mu t + \sigma W_t$) is available--where W is standard Brownian motion and X_t is the log of the asset portfolio value: $X_0 = -\ln(Le^{-rt} / A)$ where L/A is the funding ratio. Applying the log function to the exponential liability growth converts the problem to a linear framework. The investor can calculate, given the expected equity risk premium, both the minimum and maximum hitting time ("the first passage time") for the funding ratio to reach a value of 1. The calculation utilizes the following formula for the probability of the first "hitting time" ('τ'): $$P[\tau \in (t, t+dt)] = \frac{\lvert X_0 \rvert}{\sigma\sqrt{2\pi t^3}} e^{-(\lvert X_0 \rvert - \delta t)^2 / 2\sigma^2 t} dt.$$ The expected first passage time is $\lvert X_0 \rvert / \delta$; and the variance is $\lvert X_0 \rvert \sigma^2 / \delta^3$. τ_{MAX} equals $\frac{X_0^2}{3\sigma^2}$.	
2014	Andreas Hubener, Raimond Maurer and Ralph Rogalla,	This study has a two-fold objective: (1) determine a retired couple's demand to hold annuities, life insurance, bonds, and stocks over an uncertain lifetime; and (2) determine if the demand to hold life	This essay is one of several in this bibliography dealing with the demand to hold life insurance during retirement. See, also, Woo Chang Kim, John

	"Optimal Portfolio Choice with Annuities and Life Insurance for Retired Couples," Review of Finance, Vol. 18 (2014),pp. 147 – 188.	insurance is motivated primarily by "bequest" or "provision" (i.e., income replacement) goals. The authors present "…a realistically calibrated dynamic two-person lifecycle consumption and portfolio choice model…." The model assumes that, other than social security retirement benefits, "…a large fraction of retirement wealth is pre-annuitized for only one spouse…." They model the consequences of the early death of one spouse when such an event results in a drop in income to the survivor. Stocks, bonds, annuities, and term insurance contracts are evaluated in terms of their ability to hedge the risk of an unacceptable drop in consumption opportunities available to the surviving spouse. Although previous studies by Maurer *et al*. suggest that a gradual purchase of individual annuities is "…welfare enhancing, in that they offer consumers an effective hedge against individual longevity risk as well as the opportunity to trade liquidity for the survival-contingent extra return known as the 'survival credit'...," nevertheless, the current paper indicates, "…for retired couples with moderate financial wealth, purchasing additional annuities in the private market only provides marginal welfare gains. This might explain why so few households participate in the private annuity market and therefore—at least in part—the annuity puzzle." The optimal control model incorporates four "family states:" both spouses alive, husband alive/wife dead, wife alive/husband dead, and both deceased. Given mortality data from actuarial tables, the authors develop a Markov, time-dependent transition matrix based on individual (and independent) one-year survival probabilities. Bonds are risk-free instruments paying a constant rate of return (2%); stocks returns are serially independent and are log-normally distributed (equity risk premium of 4%, stock volatility of 20%). In each period, a 1-year term insurance contract can be purchased on the life of each spouse. The premium for coverage is the actuarial fair cost (i.e., present value of expected payout) increased by a loading factor. [Note: where the present value is determined by a risk-free bond's constant discount rate, the unconditional expected one-period return of	M. Mulvey, Koray D. Simsek and Min Jeong Kim, "Longevity Risk Management for Individual Investors," Stochastic Programming Applications in Finance, Energy, Planning and Logistics (Chapter 2) http://www.worldscientific.com/doi/pdf/10.1142/9789814407519_0002 (January, 2013), pp. 9 – 41; and, Patrick Collins & Huy Lam," Asset Allocation, Human Capital, and the Demand to Hold Life Insurance in Retirement," Financial Services Review (Winter, 2011), pp. 303 – 325.

		a term insurance contract equals (R_f / 1+load factor) which is less than the risk-free rate. The all-or-nothing payout from the term insurance contract generates an extremely high value for the standard deviation statistic. The payout, however, is orthogonal to a life-only income stream.] The mortality rates reflect the US 2001 Commissioners Standard ordinary Mortality Table which is the legally required minimum standard for calculating the actuarial reserves for life insurance contracts. Actual mortality experience across the insurance industry is more favorable during the estimation period. Annuities (single or joint life) are purchased at the actuarial rate reflected in the Society of Actuaries Annuity 2000 Mortality Table. The authors point out that annuity loading factors change as the age at purchase increases: "implied loadings vary between about 11.5% for joint annuities at the age of 65 years and about 31.5% for single annuities for males aged 85 years. These loadings serve insurance companies as safety margins, for example, to manage systematic longevity risk, and they also help to address the problem of adverse selection in the private annuity market." The model reflects a pecking order for financial decision making within each period: 1. Consumption expenditures, 2. Life Insurance / Annuity premium expenditures, 3. Investment Allocations Borrowing and short positions in securities are disallowed; insurance/annuity products cannot be sold on a secondary market. The model evaluates a range of private joint annuity contract survivorship benefits from 50% to 100%. Utility, assuming a constant relative risk aversion utility function, is derived from consumption and, possibly, from bequest. Consumption is scaled by a family factor to reflect the fact that a multiple person family (living spouse and children) may realize cost savings relative to a family where each member lives separately and independently. The scaling	

		factor "…can be interpreted as the effective family size." The base-case model assumes that consumption is equally shared between the spouses. Maximum expected lifetime utility is determined recursively through the solution of the Bellman [HBJ] equation where the controls are: Consumption, percent allocated to stocks, annuity purchases, and life insurance purchases. Expectations are conditioned on the family state matrix: both alive, husband only alive, wife only alive, neither alive. Utility in the 'neither alive' state is a function of the couple's bequest preferences. In the base case, the husband's initial age is set to 65, and that of the wife to 62. The coefficient of risk aversion is 5; the time preference factor is .96; and, the family consumption scaling factor is 1.3. The article summarizes the results from 10,000 simulated lifecyles based "…on previously derived optimal controls." The authors evaluate three initial wealth endowments: (1) stocks and bonds only; (2) stocks, bonds, and a joint and 2/3 survivor benefit annuity; and (3) a highly asymmetrically distributed endowment in which income is reduced by 17% on the death of the wife; or, by 67% on the death of the husband. The last case is comparable to the husband receiving a defined benefit plan payout benefit without a survivor benefit. The three initial wealth endowments are studied for couples wishing to optimize utility in both markets that do not allow them access to private annuity contracts, and in markets that permit purchase of additional private annuities. In the market lacking annuity contracts, the solution to the couple's asset allocation problem (maximizing aggregate lifetime utility assuming constant relative risk aversion) is the Merton optimum. The relative amount invested in stocks remains constant; and, even in the face of a bequest motive, the couple does not purchase life insurance: "the death of one spouse is not a negative shock, economically." [Note: family scaling factor is 1.30.] "The surviving spouse must consume less than the couple to draw the same utility, while the financial wealth is unaffected by the death of the spouse. Consequently, there is no risk for the couple that must be hedged through life insurance. Consumption is high early	

		in retirement since they anticipate that one spouse may die and that subsequently only the survivor will need to finance consumption. As the couple ages, however, average consumption declines quickly." With symmetrically distributed preexisting retirement income, consumption decreases less rapidly "…since the couple receives a constant lifelong income even at advanced ages…Between ages 65 and 80 years, there is small and diminishing demand for life insurance on the husband." Given the absence of annuities in this market, the couple allocates 71.6% of financial wealth to stocks. "The share of total wealth (i.e., the sum of stocks, bonds, and the present value of retirement income) invested in stocks, however, is projected to remain fairly constant over time, between 22% and 24%, which is nearly the same value as in the first case…." Finally, in the third case—asymmetrical retirement income distribution—the couple must cut consumption significantly in order to pay for life insurance on the husband: "If the husband dies first, the widow's income drops to only 33% of its previous level, well short of what is required to finance a utility-equivalent consumption stream." The fraction of financial wealth allocated to stocks is comparable to the symmetrical case. The above analysis is repeated for a market that allows access to private annuities. The case with symmetrically distributed income generates a pattern of results for wealth, life insurance, consumption, income, and asset allocation that "…are essentially identical to the case without access to the private annuity market, since the couple has effectively no demand for private annuities….the couple postpones purchasing these annuities in an effort to hedge against the risk of early death of one of the spouses….Only at the age of 85 years, the last year in which annuities can be purchased in the private market, does the couple substantially increase its annuity holdings." When there is asymmetrical preexisting retirement income, the couple "…will immediately begin purchasing single annuities for the wife to reduce the degree of asymmetry in the distribution of lifelong income. From the age of 65 to 85 years, the wife's annuity income is build up only gradually, as there is some risk that she might die early." The couple will	

		hold large amounts of insurance on the life of the husband. The allocation to actuarial instruments decreases the couple's stock allocation. Further analysis (comparative statics) indicates that the demand for life insurance during retirement "…is almost exclusively driven by the provision motive." The factors most influencing the demand for insurance are the family consumption scaling factor (widow can or cannot decrease expenses and consumption post husband's death); and, age difference between spouses. The risk aversion level factor has an uncertain sign—both high and low risk aversion produce a lower demand for life insurance compared to an average risk aversion level. Interestingly, a comparison of model output corresponds closely with empirical analysis using data from the RAND HRS Data file on U.S. retired investor households. The authors conduct both a probit analysis and an OLS regression estimation regarding life insurance ownership: "…econometric analysis of the key factors driving the demand for term life insurance of retired couples reveals a statistically significant positive relationship between the asymmetry of the couple's pension income and the probability of holding life insurance policies, as well as the corresponding face values." Finally, when reformulating the couple's decision regarding annuity purchases such that the decision is an all-or-nothing decision that can be made only at the start of retirement, analysis of the HRS data set indicates a positive correlation between wealth level and demand to hold annuities. The population in lower wealth deciles has the demand for annuities satisfied by Social Security benefits. Although econometric analysis indicates "Those in the two highest deciles should optimally invest more than half of their liquid wealth in private annuities…," nevertheless, following such a strategy only increases aggregate utility slightly. "This may explain why even wealthier households refrain from purchasing private annuities and, hence why empirically observable annuity demands are negligible throughout the wealth distribution."	
2014	"Better Financial	The paper explores "…the economic potential of longevity annuities to	The authors note that the pricing for long-term care

	Security in Retirement? Realizing the Promise of Longevity Annuities," Katharine G. Abraham and Benjamin H. Harris, Economic Studies at Brookings (November 2014), pp. 1 – 20.	play a useful role in a retirement landscape increasingly dominated by defined contribution accounts." They distinguish annuities from other financial products, and define a longevity annuity as a product that "…provides a fixed stream of payments that begins with a substantial delay from the time the contract is purchased…." The article presents a partial list of risks confronting retirees: 1. Longevity risk ("…the risk of living longer than expected and exhausting his or her assets….") 2. Health Care Cost risk 3. Financial risk ("…risk of low financial returns…") 4. Housing Market risk 5. Inflation risk. The authors contend: "Uncertainty about how long a person will live is perhaps the most daunting of these risks." The article quantifies longevity risk with general population data from the Social Security Administration's 2010 period life tables. In 2010, for men aged 60, 30 percent can be expected to live to age 86 and close to 10 percent to age 92; among 60-year-old women, 30 percent can be expected to live to age 89 and close to 10 percent to age 95. Individuals with higher earnings have longer life expectancies. Turning their attention to longevity annuities, the study asserts that several factors give them a pricing advantage relative to single-premium immediate annuities: (1) the deferral of the payout start date provides a period of time for the insurance carrier to generate investment earnings on the premium; (2) the expected length of the benefit period is shorter; and (3) a large percentage of annuitant contract holders will forfeit benefits to the advanced-age survivors. They provide a table of costs for a 60-year-old male and female as a function of the length of payment delay given an initial premium of	insurance in the private marketplace "…reflects severe adverse selection…." They are unaware of any longevity annuity contract that promises a constant dollar payout. They describe the current marketplace for longevity annuities as "thin." Although "…deferred income annuities reached $2 billion sales in 2013," most of the deferred payment contracts were issued to buyers under age 59, and most contracts had deferral periods of five years or less. They assert: "…few annuitants are using deferred-income annuities to protect against longevity risk." The authors are more sanguine about counterparty risk ("Consumer fears about life insurance insolvencies in the wake of the financial crisis are not justified by actual experience.") than some other commentators. See, for example, "Products and Strategies for Lifelong Retirement Distributions," Mark J. Warshawsky (2011) available at http://www.dol.gov/ebsa/pdf/TowersWatson091510.pdf; or, Life Annuitie<u>s: An Optimal Product for Retirement Income</u>. Moshe A. Milevsky The Research Foundation of the CFA Institute (2013): "More than 165,000 policyholders had purchased high-yield annuities from Baldwin-United, and the money was frozen for more than three years while regulators and the courts picked up the pieces." "Another saga that has been ongoing for 20 years…is Executive Life Insurance Company of New York." As the insurance industry brings additional annuity

		$100,000 for a non-qualified annuity. For example, a male investor could purchase, in 2014, a single premium immediate annuity paying a monthly lifetime benefit of $534.50. An election to delay payments to age 80 secures a $2,538.70 monthly lifetime benefit; an election to defer payments to age 85 results in a $4,501.86 monthly lifetime benefit. The article also provides a good review of state guarantee associations and their promise to pay promised benefits in the event of an insurer default ("...holders of annuity policies written by companies that failed received 94 percent on the value of their claims…."). From the insurance company perspective, annuities pose a risk because, in the authors' opinion, "…there exists no practical mechanism for hedging effectively against aggregate mortality risk." They do not agree with some commentators who suggest that a combination portfolio of life insurance policies and annuity contracts can offset systematic mortality risk for carriers because "…gains in aggregate mortality are likely to be realized mainly by those at the oldest ages, while life insurance risk is spread throughout the adult age distribution." Note: compare to argument made by Norma Nielson (2012) "Annuities and Your Nest Egg: Reforms to Promote Optimal Annuitization of Retirement Capital" (2012): "Most annuities are sold by firms with large blocks of life insurance. Some of these companies may be able to offer better than expected annuity returns because 'The sale of an annuity reduces their overall risk faced by the firm and produces a corresponding reduction in the needed (risk-adjusted) rate of return.'"	products to market, the safety and security of contractual obligations may well assume a greater role in discussions of how best to mitigate longevity risk. Although far afield from the central focus of this bibliography, researchers might benefit from reviewing the work of Joseph M. Belth, former professor of insurance at Indiana University (…any business built on the nondisclosure of information vital to its customers will not survive—and will not deserve to survive—over the long term.") [The Insurance Forum: A Memoir (Ellettsville, Indiana (2015), especially pp. 187 – 196. The Belth memoir chronicles a history of the insurance industry's attempt to quash consumer protections, and to circumvent regulatory requirements. Interestingly, the Abraham/Harris article points out similar concerns: "…insurers effectively reduce their capital reserves by shifting liabilities to offshore entities that are not subject to domestic regulations….New York's Superintendent of Financial Services released a report in 2013 highlighting this threat, comparing the practice of captive reinsurance to the financial products such as credit-default swaps on subprime mortgages that precipitated the financial crisis." Without question, the investigative journalism / 'muckraking' press have much to consider in both the insurance and securities industries.
2014	"Minimizing the Probability of Ruin in Retirement,"	The author makes the observation that previous research often begins with "an intuition-based heuristic" that is tested by historical backtesting or simulation methods such as bootstrapping or Monte Carlo. The paper	This essay is something of a short-course in dynamic programming. In addition to various proofs and derivations, the author presents extensive

	Christopher J. Rook (August 7, 2014). http://papers.ssrn.com/sol3/papers.cfm?abstract_id=2420747	reviews some representative studies beginning with Bengen's 1994 work. Many of the referenced studies test for the optimal static asset allocation—i.e., the allocation which, if maintained throughout the planning horizon (fixed or stochastic), minimizes the likelihood that a portfolio will run out of money. More recent studies employ models exhibiting dynamic or adaptive asset allocation. The author wishes to provide a "…roadmap to guide the retiree/advisor how to maintain optimality over time." Specifically, "At each point in time, as the retiree ages, there is a minimum probability of ruin based on an optimal asset allocation, and it is the goal of this research to find those values." The analysis studies the probability of Ruin at time 't.' If the event 'Ruin' fails to occur at any time interval, then its complement [R^c] must occur. This means that the probability of ruin at a specific period is conditioned on the probability of not being ruined in all previous periods. As an example, the expression for the probability of ruin in time period 3 is: $P(Ruin(3)\ \vert Ruin^c(1) \cap Ruin^c(2)) = \frac{P(Ruin^C(1) \cap Ruin^C(2) \cap Ruin\ (3))}{P\left(Ruin^C(1)\right) * P(Ruin^C(2)\ \mathrm{I}\ P(Ruin^C(1))}$ Where 'I' is the symbol for "given that." Assume that the account value [\$A] at t=0 is the account value at the start of the period multiplied by two factors: (1) the initial period's total return [$1+R_{(1,a)}$] with 'a' the percentage allocation to stocks, and (2) the period's investment expenses ($1-E_R$). Assume, further, that the period's required withdrawal amount is the account value [\$A] multiplied by the inflation-adjusted withdrawal rate [$(W_R)+(1+I_1)$], then ruin occurs if and only if: $\$A*(1+R_{(1,a)})*(1-E_R) \leq \$A*(W_R)*(1+I_1)$. Where ($1+R_{(1,a)}$) can be decomposed into a nominal return piece and an inflation-adjustment piece [$(1+I_1)$]. This permits cancellation of the inflation elements and the account value elements on both sides of the equation with the resulting simplification for ruin as: Ruin occurs if and only if the inflation and expense adjusted return ≤ the required withdrawal from the portfolio. If return is the random variable 'X,' and	appendices discussing the method of backward induction and program implementation in C++. It partially circumvents the 'model risk' problem associated with making distributional assumptions for the financial asset return generating process. A critical element for assessing the probability for ruin minimization is the inspection of the left tail of a distribution—not the entire PDF. Of course, given the goal of inflation-adjusted withdrawals, results depend on the choice of the distributional assumption made for the inflation process. Perhaps its most valuable contribution is to provide a critical prospective on several much-discussed investment strategies used to mitigate the sequence of returns risk. Bucketing, an increasing equity glide path, and other conservative investment methods at the start of retirement are frequently recommended. While Rook does not directly address these retirement income investment strategies, he outlines, in a well-conceived mathematical context, the interrelation between decision making that seems optimal in the short term and decision making that, in actuality, produces the most favorable long-term outcome. "Selecting a low volatility portfolio…early in retirement to avoid ruin at the next time point now comes with a price because that portfolio results in a higher value for the next ruin factor….A larger ruin factor at the next time point increases the probability of ruin after the next time point…." A dynamic programming approach helps to identify a balance point between the competing factors.

		the required withdrawal is the constant 'x,' then $P(X \leq x)$ can be measured by the cumulative density function of the random variable 'X.' The author defines the term "ruin factor" at time 't' [RF(t)] recursively as the Ruin Factor(t-1) / [(1+$R_{(1,a)}$)*(1-E_R) - RF(t-1)], for all periods 't' through T_D = time of withdrawal just prior to the investor's death. Stated otherwise, ruin will occur if and only if the net constant dollar return is less than the Ruin Factor: Ruin is then experienced at time t if the random variable realization of inflation-adjusted net return is less than or equal to the Ruin Factor at t-1. Assuming that the CDF of the return can be estimated, the investor can calculate the Ruin Factor and the Probability of Ruin at various times and for various asset allocations. "At time t=0, the probability of ruin can be viewed as a *prior* probability that should be updated once returns are observed in what can loosely be called the *posterior* probability of ruin. It is appropriate for the retiree/advisor to re-evaluate ruin probabilities at scheduled intervals." Specifically, given the sequence of returns during retirement, the investor evaluates the probability of ruin and adjusts the portfolio allocation accordingly. The ruin factor "…helps to reveal the precise asset allocation that minimizes the probability of ruin for the remainder of retirement….an inflation/expense-adjusted market return that exceeds the prior year's ruin factor (plus 1) will improve the retiree's position with respect to ruin." Future Ruin Factors are random variables because their values are a function of investment returns and inflation realizations. Two aspects of the ruin factor are noteworthy: 1. The reciprocal of the ruin factor equals the number of constant-dollar withdrawals remaining at time t. The uncertainty of future Ruin Factor value increases as the weighting of stocks in the portfolio increases; and, 2. To improve the chance of avoiding future portfolio depletion, the retired investor must earn a net, after-inflation return in excess of the current year's ruin factor. The goal is to use the ruin factor as a guide to the asset allocation	

		decision that, at every time t, minimizes the probability of ruin: "The retiree would benefit from a roadmap that informs them how their P(Ruin) is changing over time, and the actions needed to maintain optimality." A key to minimizing the likelihood of portfolio depletion is to recognize that the optimal asset allocation at any point in time can be determined by inspecting the (left) tail behavior of the Probability Density Functions for various asset allocations. This means that return distributions do not necessarily have to conform to normal or log normal distributions characterized only by their first two moments. Although the author presents two solutions, he rejects the first; and explains why it does not minimize the probability of ruin over the entire planning horizon. The first solution is to select the asset allocation exhibiting a PDF with the smallest tail area to the left of the value of the Ruin Function at t-1. Selecting the allocation using this preferencing criterion "…allows the retiree to act optimally with respect to the next time point, having an asset allocation that minimizes P(Ruin(t))." Such a portfolio selection criterion, however is flawed because "…it acts optimally on a local basis and ignores the long-term impact of each short-term decision." During the initial retirement periods "…it would place the retiree into a low volatility portfolio attempting to minimize P(Ruin) before the next withdrawal. The inflation/-expense-adjusted returns with such a portfolio will struggle to outperform the ruin factor, causing it to increase over time. Eventually, the ruin factor may spike and the strategy would respond by shifting the retiree into stocks reflecting desperation." The study emphasizes that "Minimizing P(Ruin) by using sequential optimization fails because the [allocation] at time t=0 has an impact on the probability of ruin at later time points." Stated in mathematical terms: '…a collection of local optimums does not necessarily aggregate to a global optimum." Having rejected the sequential period-by-period optimization approach, the study describes the merits of using backward induction. The article provides an outline of this methodology starting with the final period—the last withdrawal prior to the investor's death [T_D] for which, given the	

		restricted sample space [S = $Ruin^C$ ($\leq T_D$), there is no need to compute the Ruin Factor [RF] because this is the last withdrawal. By the laws of probability, a boundary condition is established at TD where P(Ruin) = 0; and, for all prior periods, the state of $Ruin^C$ has occurred. Moving backwards one period of time, the retiree faces a restricted sample space which covers two periods and which requires the investor to examine the tail probabilities of allocations to select the allocation that minimizes the probability of ruin over the two-period interval. As the retiree works towards t = 0, the sample space becomes: S={Ruin(t+1) ∪ Ruin(t+2) ∪ . . ., ∪ Ruin(T_D), $Ruin^C$ ($\leq T_D$); and the retiree selects the asset allocation that minimizes the probability of ruin. The choice of this allocation "…at time t becomes a balancing act between minimizing P(Ruin) at time t+1, and after time t+1." The asset allocation [a] and Ruin Factor [RF(t)] are discretized to approximate a numerical solution which is not based on simulation. Each RF is discretized with precision P_R along the ruin factor dimension. This allows for a comparable discretizing of the probability distribution of the asset allocation return realized in each P_R interval with boundary conditions set by virtue of a finite planning horizon. Thus the optimal Value Function "…V(t, RF(t)) is a continuous function along 4 dimensions (t, RF(t), a, V)…." The article maps results for various expense factors and withdrawal rates. The Value Function that is maximized is the minimization of the probability of portfolio depletion prior to the end of the planning horizon [P(Ruin)]. For favorable evolutions of the risky asset portfolio [Return > Ruin Factor] the optimal asset allocation shifts from stocks towards bonds in a type of equity glide-path. However, in unfavorable markets "…the optimal equity allocation remains roughly constant or increases…." Comparing the optimal dynamic allocation strategy with a stay-the-course 50-50 stock to bond strategy indicates that the static allocation results in at least a doubling of the probability of ruin when the planning horizon is fixed.	
2014	"Floor and Upside Investing in	Irlam employs stochastic dynamic programming [SDP] to map asset allocation / periodic consumption as functions of investor age and	A commonly recommended approach to designing a retirement income portfolio is the "floor + upside"

	Retirement without Annuities, Gordon Irlam (October 2, 2014), SSRN: http://ssrn.com/abstract=2504746	portfolio size. Given the SDP-generated map, simulations generate multiple paths. The investor can determine the optimal allocations and consumption amounts along these paths. The utility function divides into several parts: one level of risk aversion applies to non-discretionary expenses and the other to discretionary expenses. The author characterizes the multi-part utility function in terms of separate floor and upside utility. Such a characterization rules out use of a homogeneous CRRA function across the investment portfolio's wealth domain. Rather, for consumption utility, Irlam divides the function into three regions (floor, transitional, and extra spending), and assigns a different coefficient of relative risk aversion [γ] to each region. For example, given the standard expression for the CRRA utility function: $$U = (1/(1-\gamma))*C^{1-\gamma}$$ where C = annual consumption, for an investor neutral to (1) a 50-50 chance of 100% consumption or 200% consumption, or (2) a guaranteed consumption of $((1+2^{1-\gamma}) / 2)^{1/(1-\gamma)}$, given a γ value equal to 4, a dollar at a 50% consumption level is worth 16 incremental dollars at a 100% consumption level [$2^4 = 16$]. The author notes: "The choice of γ is an important one. A low γ will favor stocks, and a high γ will favor bonds." The γ value is higher for the non-discretionary ('floor') expenses than for the region associated discretionary expenses. Ingram assigns a γ value of 4 to the floor region, and a γ value of 1 to the extra spending region. He joins the regions ('transitional' region) with a cubic polynomial curve: "A factor of 20 is defined as the ratio between the slopes of the CRRA utility functions and the start of the transition region." This region occupies the interval \$30,000 to \$40,000 in the sample wealth domain. The base case scenario contemplates a \$30,000 required income for a male age 65 entitled to \$15,000 annual inflation-adjusted Social Security benefit. Desired additional spending amounts to \$10,000 per year (consumption range = \$30 to \$40 thousand) from a portfolio with an initial value of \$750,000. The stock portion earns a 5.2% geometric real return with a	investment strategy. In this case, the investor purchases investments that offer a high degree of safety with respect to their ability to provide periodic cash for critical expenses. Once threshold expenses are secured, the investor purchases growth-oriented investments in the hopes of achieving a comfortable standard of living in retirement. Although this strategy is often compared to Tobin's two-fund solution, it also shares many characteristics of A. Maslow's psychological self-actualization pyramid. In the 1970s, industrial psychologists, often at the behest of corporate marketing and sales departments, generated sales/product pyramids purporting to conform to Maslow's insights and recommendations. Basic needs were critical—a hungry person could not fully self-actualize; and, such needs should be filled with "guaranteed" products. Risky-asset portfolios were something that the cautious investor implemented only when the more "important' life goals were fully funded by "safe" assets. Stocks—i.e., higher expected return assets—are acquired with residual funds and, with luck, allow investors to purchase luxury goods. The pseudo-psych approach to financial planning became especially popular with certain marketing segments within the insurance industry. In some respects, it was an attempt to counter the more integrated investment wealth-creation approach represented by Markowitz's efficient frontier—another 'geometric prop' quickly co-opted by sales organizations promising to "optimize" everything under the sun. Interestingly, 'pyramid' planning co-

		19.4% standard deviation; the bond portion earns a 2.7% real return with a 9.7% standard deviation. Portfolio management expenses are 10 basis points per year. The applicable annual discount rate for future consumption is a constant 3%. The base case does not consider taxes or a bequest objective. Asset allocation as a function of age and portfolio size indicates the optimality of maintaining fixed weighting under a CRRA utility function. Stock/bond allocation remains balanced in the 60-40 region for portfolio values in a band between $600 and $400 thousand dollars. This is the wealth region required to support the floor target. However, for portfolio values beneath this wealth level, the percentage allocated to stock increases to a predominately stock allocation. Likewise, given the low γ value for wealth regions supporting the excess spending consumption level, the allocation tilts exclusively to stocks: "The lower degree of risk aversion at high consumption levels means you can afford to take a higher degree of risk on equities." "For small portfolio sizes there are no bond holdings, then as portfolio size increases bonds ramp up, reach a peak, and finally decline. The bond hill functions as a buffer that attempts to stop consumption dipping into the required space. Beyond the crest of the till it is OK to play a very stock heavy game because the bond hill can be fallen back on should thing [sic] turn out poorly….The decline in absolute bond holdings beyond the crest of the bond hill represents something of a challenge to the use of SPIAs to provide a floor in the floor and upside approach." Although simulation outputs indicates a high degree of consumption variability, most all of the variability occurs in a region of excess consumption: "The simulation runs achieved a discounted certainty equivalence for floor consumption in retirement of $29,945, indicated that the goal of having sufficient assets to fund required spending was nearly entirely satisfied. Another metric is that the expected proportion of the discounted time spent consuming above the floor level is 99.6%...."	exists comfortably with the behavioral finance insights regarding the tendency of "normal"—as opposed to "rational"—investors to exhibit asset and goal segregation. The "rationalist school's" counterargument is that portfolio construction through piecemeal funding of segregated goals almost certainly creates a "pay-day" structure that, when considered as a whole, rarely, if ever, reflects the investors risk/return preferences and constraints (risk tolerance) with respect to total financial wealth. Just to make things interesting, in the practitioner world, organizations like the *Retirement Industry Investment Association* anchor planning recommendations to the reality of the "household balance sheet." Rather than funding identifiable 'monads' (to use the Cartesian expression) within a pyramid-like financial structure, the approach is a form of asset/liability match. For personal wealth management, this is an approach similar to Jarrod Wilcox's concepts of (1) the "time series of implied balance sheets," and (2) "management of discretionary wealth" as expressed in books like <u>Investing by the Numbers</u>, (Frank J. Fabozzi Associates, New Hope Pennsylvania). 1999. Note that Wilcox advocates use of the "Kelly formula" for optimal growth. See also, the discussion of multi attribute decision making in "Decision analysis using targets instead of utility functions," Robert Bordley & Marco LiCalzi Decisions in Economics and Finance (May 2000), where pyramids are "converted" into trees.

2014	"Market Risk, Mortality Risk, and Sustainable Retirement Asset Allocation: A Downside Risk Perspective," W.V. Harlow and Keith C. Brown Working Paper Draft December 29, 2014 http://faculty.mccombs.utexas.edu/keith.brown/Research/retireallocate-wp.pdf	The authors note that retirees are concerned about the risk of outliving their portfolio (shortfall probability) and the severity of the shortfall conditioned upon its occurrence (Shortfall magnitude). They ask what asset allocation, given return, variance and correlation assumptions, minimizes the risk of ruin and what allocation minimizes the magnitude of ruin. They find that limiting the magnitude of ruin results in extending the length of time a portfolio survives in worst case scenarios. They note that allocations required to limit shortfall probability differ from allocations required to mitigate worst-case outcomes. Given their focus on downside risk metrics, the authors assert that the optimal weighting of equity within a retirement income portfolio is rarely greater than 25% and may be as low as 5%. The method used to determine the financial health of a portfolio is termed the Retirement Present Value [RPV] Process. Assuming no additional contributions are forthcoming, the RPV is calculated by multiplying, on a year-by-year basis, (the target cash flow) times (the probability of survival) times (the present value factor) The present value factor is 1 ÷(1+realized yearly return of the portfolio). Summing the mortality-adjusted present valued cash flows over the applicable planning horizon—to age 110—gives the RPV of the retirement portfolio. If the dollar value of the RPV is positive, this indicates that the present value of a bequest; if negative, this indicates the present value of additional retirement funds needed. There is a correspondence among the lower partial moments of a distribution and the downside risk measures with which the authors are concerned. Assuming that the target portfolio value must remain above zero throughout the investor's lifetime, the zero order lower partial moment is the number of trials in a simulation of a portfolio operating with a set asset allocation vector and with a fixed withdrawal amount divided by the total number of trials less one. Expressed differently, the total number of negative outcomes divided by the total number of trials:	A given allocation may, when simulated, produce an expected shortfall of $1,000 at a 10% probability. If shortfall avoidance is the sole preferencing criteria, this portfolio is preferred to one that, when simulated, produces an expected shortfall of $100,000 at a 9.99% probability. Both probability and magnitude are important to investor choice. The equation used to calculate RPV is isomorphic to that used to calculate an annuity pricing factor. The numerator is the amount of cash flow adjusted by the probability that the investor is alive to receive it. The denominator is the 'term structure' of realized portfolio real returns. The annuity pricing factor denominator is the term structure of interest rates. The RPV denominator is the 'term structure' of each simulated trial's sequence of realized real returns. The authors explore a MiniMax strategy where the investor minimizes the maximum losses that could occur. Yaari notes [p. 138] that such a strategy is inappropriate for cases seeking to maximize the utility of consumption under an expected utility based model.

		$$\text{LPM}_0 = \sum_{RPV_j<\tau}(\tau - RPV_j^0\ /(n-1)$$ The expected shortfall can be expressed by the first order lower partial moment which preserves the dollar value of the average shortfall as well as the probability of a shortfall over the total number of trials: $$\text{LPM}_1 = \sum_{RPV_j<\tau}(\tau - RPV_j^1\ /(n-1)$$ Finally, insight into the magnitude of a shortfall is given by the variance of the conditional shortfall, or, the square root of second order lower partial moment [also known as semi-deviation]: $$\text{LPM}_2 = \sum_{RPV_j<\tau}\{(\tau - RPV_j^2\ /(n-1)\}^{1/2}$$ The optimal downside risk-based allocation has a higher probability of ruin (9.96%) than the optimal shortfall risk minimization allocation (7.39%); but in the worst case scenarios (trials in which the RPV is negative) produces portfolios that last approximately two years longer. The allocation of this "worst-case" minimization portfolio for a 65 year-old male is 11% stocks, 24% bonds and 65% cash given the distributional parameters used to generate simulated trials over the investor's lifetime. The authors note that the equity weight within the portfolio does not increase beyond 20% even assuming that the return on cash approaches zero.	
2014	"How Does Household Expenditure Change with Age for Older Americans?" Sudipto Banerjee, Employee Benefit Research	The author explores the consumption behavior of older Americans from data taken from the Health and Retirement Study and the Consumption and Activities Mail Survey. Home-related expenses are the largest expenditure category with health expenses in second place: "In 2011, households with at least one member between ages 50 and 64 spent 8 percent of their total budget on health items, while those ages 85 or over spent 19 percent of their budget on health items." Health-related	A relatively small portion of the distribution of retired investors aged 90 or over show a large increase in spending as a result of high health-care costs.

	Institute Notes Vol. 35, No. 9. (September 2014), pp. 1 – 11.	expenditures are heavily skewed towards the end of life. "By age 90, health care expenses account for more than 20 percent of the households' entire budget." "…at age 90 and older, the 90th and 95th percentiles of health care expenses show abrupt increases. This shows that for some people, end–of-life health care spending can be very high." For older age households, income tends to fall faster than expenditures: "Taking age 65…as the benchmark, the average household income is 20 percent less by age 75 and 50 percent less by age 85. In comparison, average household expenditures….are 16 percent lower by age 75 and 40 percent lower by age 85." However, spending does not fall uniformly across the distribution with age. "…the median drops from $43,580 (for those between ages 50-64) to $19,681 (for those age 90 or older), a 55 percent drop. For the same age groups, the 90th percentile drops from $100,400 to $65,289, a 35 percent drop." For all percentiles there is a decline in expenditures as a function of age.	
2014	Konicz, Agnieszka K. and Pisinger, David and Weissensteiner, Alex, "Optimal Retirement Planning with a Focus on Single and Joint Life Annuities" (April 18, 2014). Quantitative Finance, 16:2, 275-295, 2016. Available at SSRN: http://ssrn.com/abstract=2426544	The paper employs a multi-stage stochastic programming approach where the optimization goal is to maximize the expected utility of retirement consumption given bequest motives, life expectancy, risk aversion, and personal preferences. The model uses data from the British stock and bond markets. Results indicate that retired individuals benefit from holding a wide variety of both annuities and pure endowments even in the face of loads and high surrender charges. However, "…the right annuity for a household is much more complex than any of those available in the market…." The basic model is a probability-adjusted summation of the utility of consumption and bequest assuming constant relative risk aversion with a time preference factor ρ. The model scales the utility of bequeathing wealth (W), by a scaler (k) indicating the strength of the bequest motive: $$\max E\left[\sum_{t=t_0}^{T} tPxu(t,C_t) + \sum_{t=t_0}^{T-1} tPxq_{x+t}ku(t,W_t)\right]$$ Where P is survival probability and q is probability of death during the following period. The authors extend the basic model to two-person	

		households where husband and wife may have different motives and preferences. Given that a zero-coupon bond pays a guaranteed amount at maturity [M] regardless of whether an investor is alive, while a pure endowment (fixed or variable rate, single annuitant, or joint and survivor) pays contingent upon survival of at least one annuitant, the authors claim that "…all annuity payout options can be replicated by a combination of four basic financial products…." The price of a zero-coupon bond is the payout discounted by the appropriate interest rate raised to a power equal to the number of periods until maturity: $$price_t^b = \left(\frac{1}{1 + r(t, M)}\right)^{M-t}$$ The price of a fixed payment annuity adjusts the basic pricing formula for either male, female, or joint mortality (assumes independence of lives); and the price of a variable payment annuity is the ratio of the risky asset benchmark performance [h] to the appropriate interest rate [r] for the period under consideration: $$1 * \left(\frac{1 + h(t, M)}{1 + r(t, M)}\right)^{M-t}$$ The model reflects costs for buying and selling bonds. Additionally, it incorporates a complex structure of ongoing annuity fees as well as surrender charges for withdrawals in excess of those "…exceeding a certain free amount specified in the contract." The authors develop a table of graded surrender charges reflective of the type of annuity or pure endowment contract, and of the period of time over which the contract extends. They remark: "…we intentionally choose high surrender charges for our model." Investment returns—"time-varying investment opportunities" —are modeled with a Vector Autoregressive (1) model which combines the evolution of interest rates (historical zero coupon interest rates from the Bank of England) and the FTSE 100 index. They utilize the Nelson/Siegel	

		model for the evolution of the yield curve. This is a three-factor model which expresses the spot rate at maturity time M as a function of the level, slope, and curvature of interest rates: Spot rate (M) = level + slope + curvature, or $$y(\beta_t, M) = \beta_{1,t} + \beta_{2,t}\left(\frac{1-e^{-\lambda_t M}}{\lambda_t M}\right) + \beta_{3,t}\left(\frac{1-e^{-\lambda_t M}}{\lambda_t M} - e^{-\lambda_t M}\right)$$ Where lambda is fixed to minimize the mean squared error in the data set. Estimation of the Beta parameters is performed by Ordinary Least Squares analysis. A linear algebra matrix models the combined evolution of interest rates and equity returns. Innovations are assumed normal and are considered to be homoscedastic and independently distributed over time. The eigenvalues of the slope matrix have modulus less than one indicating a stable process with unconditional expected mean [μ] and covariance [Γ] for the steady state t = infinity. The authors approximate a discrete-time multivariate process by constructing a scenario tree with a set of nodes for sub-periods, the set of stock and bond parameters, and the set of pure endowment and annuity products that are priced according to the Beta factors derived from the three-factor Nelson/Siegel model. The model is self-financing (no short positions allowed) with rebalancing (purchase of new assets) funded by money remaining after period-by-period consumption. The authors characterize the model as follows: “The presented MSP formulation is a convex optimization problem with a nonlinear objective function and linear constraints.” There are 4 periods between nodes each of which span a 5-year period. This means that each decision step occurs at the end of 5 years—a structure which keeps the branching structure manageable. If an individual survives the entire period (to age 85), he or she invests all remaining wealth in a lifetime, fixed, nominal annuity. The time preference factor is 5%. Not surprisingly, optimal consumption and asset allocation differ according to the investor’s level of risk aversion and bequest motive. Bequest motives result in higher allocations to stocks and bonds. The more risk averse the investor, the lower the bequest amount to heirs. In	

		addition to stocks and bonds, the optimal asset allocations across single and two-person households at various risk-aversion levels indicate a complex mix of pure endowment and annuity contracts. The central finding of the study is "Despite high surrender charges, life contingent products are the primary asset class in the portfolio for the period after retirement."	
2014	"Are Retirees Falling Short? Reconciling the Conflicting Evidence," Alicia H. Munnell, Matthew S. Rutledge, and Anthony Webb. Center for Retirement Research at Boston College CRR WP 2014-16 (November 2014)	Although the National Retirement Risk Index indicates that approximately half of today's working households will be unable to maintain their pre-retirement standards of living, one study suggests that the majority of pre-retirement households are saving optimally (only 16% of households had less wealth than optimal); another points to only small declines in consumption for households who retired between2001 and 2007. The authors of this study argue that "…retirees are falling short and will fall increasingly short over time." The conclusion that pre-retirement households are saving optimally rests, according to the authors, on two key assumptions: "…households are content with declining levels of consumption in retirement;" and "households reduce their consumption when children leave home." Although the ratio of net wealth to income which is tracked in the Federal Reserve's Survey of Consumer Finances has remained stable from 1983 through 2010, this is not a favorable development: "…five major developments should have led to higher ratios of wealth to income." 1. Life expectancy has increased; 2. Social Security replacement rates have declined; 3. Pensions have shifted from defined benefit plans to 401(k)s; 4. Health Care costs have increased; 5. Lower interest rates produce less retirement income from savings. Why do the studies produce differing conclusions? "The notion that households accept declining consumption in retirement means that	The National Retirement Risk Index calculates the ability of working-age households to maintain their current standard of living in retirement. It assumes age 65 retirement, annuitization of all wealth, and a reverse mortgage on a home. Retirees purchase an inflation-adjusted annuity. Evidence indicates "…that people tend to maintain their pre-retirement spending when they first retire, but then cut back sharply thereafter." The primary reason is a lack of "adequate resources to maintain their initial levels of consumption throughout the retirement period."

		households must accumulate much less wealth to maintain their standards of living. The assumption that parents reduce their consumption once children leave home means that a household has a modest target to replace and saves more between the emptying of the nest and retirement, yielding few at risk. These two assumptions are the levers that allow one to toggle between the two models." The authors acknowledge: "No one really knows the answer to either question." The study then turns its attention to the observation that retirement spending declines only modestly—one to six percent. The observation of steady consumption in retirement suggests that retirees have adequate financial resources. The authors launch three tests of this hypothesis: 1. Do households possess sufficient resources in their first year of retirement to maintain their current consumption for life—"The household with the average income and consumption in each of the bottom seven deciles of the income distribution will not have enough money to maintain its first year's consumption." 2. Did households continue to maintain their immediate post-retirement consumption after six to ten years?—"Our results show a sharp decline in total spending among those who retired. By the end of the period, median spending is 23 percent lower…." 3. Compare the trajectories of households with sufficient resources to maintain consumption with those of households with insufficient resources—"…spending for those with insufficient resources declined by 31 percent compared to 16 percent for those with sufficient resources." The authors argue that "…the fact that consumption does not decline early in retirement ignores the fact that many people do not have the resources to consume at that pace over their entire retirement."	
2014	"A Better Systematic Withdrawal Strategy—the Actuarial Approach,"	The author, a retired FSA member, outlines a five-step approach to implementing "The Actuarial Approach" to retirement income planning: 1) Gather Data 2) Make Relevant Assumptions—he points out that higher-	The article begins with a good summary of potential conflicts in setting financial goals in retirement: • Spend enough each year to maintain a certain standard of living

	Ken Steiner Journal of Personal Finance Vol. 13, No. 2 (2014), pp. 51 – 56.	expected return assets require a higher discount rate to reflect their risk: "Some actuaries and economists believe that assumed investment return should approximate a risk-free interest rate, as the higher expected returns associated with investment in riskier assets, such as equities, also carry a higher risk of volatility…." The 'relevant assumptions' section divides into several parts: (1) Assumed Investment Return—"The investment return assumption should generally reflect investment return assumptions inherent in immediate annuity contract with perhaps some small adjustment for greater expected returns on equities." (2) Expected Payout Period Recommendation:—suggests using 95 minus current age or life expectancy, if greater. (3) Inflation—suggests 3% per year ,or "…the investment return assumption minus 2% if the investment return assumption is lower than 5% per annum." 3) Perform Calculations to Determine Preliminary Spendable Amount. This is defined as "What total spendable amount (from accumulated savings and annuities) may be spent in the current year, to be increased each subsequent year by a constant percentage so that accumulated assets will exactly equal the amount desired to be left to heirs at the end of the expected payout period?" 4) Apply a Budgetary Smoothing Algorithm. Provides for a more stable and predictable spending pattern. The algorithm presented by the author sets an upper and lower bound corridor around the preliminary calculated spending amount. After adjusting for inflation, current spending will either fall within the corridor; or, if the spending target lies outside the corridor, the amount of adjustment is limited to the upper or lower limit. 5) Store Results for the Next Year. "The author recommends that retirees revisit their spending budget at the beginning of each year." The article compares "The Actuarial Approach" to (1) the 4% of initial portfolio value spending rule; (2) the 4% of current year value spending	• Not spend so much that accumulated savings run out prior to death • Having relatively predictable and stable inflation-adjusted spendable income from year to year • Having flexibility to meet unforeseen expenses • Maximizing the general level of spendable income • Not leaving too much unspent at death. The author points out the benefits of using a 'life expectancy calculator' found on the Society of Actuaries website at www.soa.org/research/software-tools/research-simple-life-calculator.aspx Note: the similarities in the calculation methodology advocated by Steiner to that advocated by Gordon Pye.

		rule; and (3) the Required Minimum Distribution rule. The 4% rule "…does not adjust for actual experience" and offers little spending flexibility. The benefit of the rule is highly dependent on the year of retirement. The constant 4% rule adjusts for actual experience but may result in a substantial variance in year-to-year spending. Finally, the RMD rule "…tends to understate withdrawals, particularly in the early years of retirement."	
2014	Robert C. Merton, "The Crisis in Retirement Planning," <u>Harvard Business Review</u> (July-August, 2014), pp. 1401 – 1408.	The premise of the article is that "…an investment that is risk-free from an asset value standpoint may be very risky in income terms." However, most investors focus primarily on asset value—i.e., the value of their 401(k) portfolio—as opposed to the level of income that the asset produces. A disconnect in thinking occurs when the investor confuses stability in asset value with stability and sustainability in lifetime income. Merton uses a deferred annuity to illustrate his points. The market value of an annuity is, to a great extent, a function of interest rates. However, rates can fluctuate enormously: "In 2012, for instance, there was a 30% range between the highest and lowest market value of the annuity for the 45-year-old over the 12 months. However, the income that the annuity will provide in retirement does not change at all." For many investors, the relevant goal is income stability not stability in the value of principal. Merton asks the reader to consider a risk-averse investor who owns a portfolio holding only T-bills. Over time, the investor incurs only a slight risk of loss of principal. However, the income produced by the all T-bill portfolio changes dramatically depending on the date when the investor decides to convert the portfolio to retirement income by, for example, acquiring an inflation-adjusted annuity: "…if the goal is income for life after age 65, the relevant risk is retirement income uncertainty, not portfolio value….if we look at the unit of measure that matters to our consumer—how much the saver would receive if the investment were converted into an income stream—then T-bills are shown to be very risky, nearly as volatile as the stock market." Merton executes an intellectual pivot by redefining the Modern Portfolio	Although this short article is primarily focused on pre-retirement savings and investment strategies, it makes a number of interesting observations regarding choices faced by retiring investors. Merton opines: "…putting relatively complex investment decisions in the hands of individuals with little or no financial expertise is problematic. Research demonstrates that decision making is pervaded with behavioral biases." The essay sheds light on the merits of a credible portfolio monitoring program from a top-down perspective. Although the prudent investor takes care to assure that the portfolio is sufficient to fund threshold income needs—e.g., portfolio value equals or exceeds cost of essential lifetime income, actual purchase of annuity or fixed income (TIPS) instruments need not occur so long as wealth exceeds the current cost of the target floor income. The critical distinction is between 'being able to buy an annuity' and 'actually buying an annuity.' The first puts the investor in a position to exercise an option to secure an income floor if and only if it becomes necessary. Otherwise, the investor is free to earn the expected risk premiums provided in the capital markets. The second path commits the investor to exchange wealth either for illiquid annuity contracts or for low-yield principal-

		Theory optimization goal in terms of income rather than wealth. This pivot is achieved, in part, by redefining both risk and the risk-free asset: "…risk should be defined from an income perspective, and the risk-free assets should be deferred inflation-indexed annuities." Given an investor close to retirement, Merton recommends dividing income needs into three categories: Minimum guaranteed lifetime income: shielding the retiree from "…longevity risk, interest rate fluctuations, and inflation." Conservatively flexible income: "…more flexible but still relatively safe alternative to annuities…." Desired additional income: investing in risky assets holds out "…the possibility of earning sufficient returns to permit achieving the desired higher retirement income."	guaranteed investments. For example, in terms of pension fund management, "It's important to note that the fund manager need not actually commit the employee to purchasing a deferred annuity but should manage the risk-free part of the portfolio in such a way that, on retirement, the employee would be able to purchase an annuity that would support the target standard of living…." This suggests that the concept of 'flooring' or 'a floor income' can/should be 'virtual' as opposed to 'actual' whenever the portfolio monitoring program indicates the presence of a 'surplus.'
2014	Massi De Santis and Samuel Y. Wang, "Retirement Planning: Balancing Wealth and Income Considerations," Dimensional Fund Research paper (April 2014).	The authors assert: "…different fixed income strategies can have very different volatility profiles when measured in terms of wealth or in terms of income." Strategies designed to facilitate the goal of capital preservation may not be effective strategies to facilitate the goal of lifetime retirement income. 'Units of wealth—i.e., nominal or constant dollars—can be converted into appropriate 'units of income' by dividing wealth by the price of an annuity paying a lifetime income: "…the price of an annuity is a market price (updated continually) that reflects the present discounted value of future income weighted by the probability of being alive to receive that income." Extending this line of analysis, the authors point out that the cost of lifetime income should also be inflation adjusted. The authors examine three portfolio strategies: A TIPS portfolio with maturities of 1, 5, and 15 years. The one year TIPS portfolio is an example of a capital preservation strategy; the five and fifteen year TIPS portfolios act as "…component pieces of a liability management strategy where the critical input is the duration of the asset and the liability." In terms of capital preservation, the one-year TIPS portfolio exhibits the lowest volatility; the fifteen-year TIPS portfolio exhibits the highest volatility. However, when volatility is measured in terms of 'income	Not surprisingly, the Dimensional paper parallels the paper written by Robert C. Merton, "The Crisis in Retirement Planning," Harvard Business Review (July-August, 2014), pp. 1401 – 1408. Merton is on the board of directors for the Dimensional Funds. Several articles from the 'goals-based' school of portfolio investment selection advocate matching 'low-risk' assets with critical goals. This is somewhat common in the Maslowian portfolio construction school of thought. The Dimensional paper provides a nice counterpoint by demonstrating that low-risk assets may be high risk when savings towards retirement income goals. In an ALM context, the low-risk asset is the liability mimicking asset not the low volatility asset.

		units,' the findings are reversed. The analysis is conducted in terms of the effectiveness of a given strategy in terms of its ability to hedge future income—that is, given a period of time prior to retirement, what strategy is most promising in terms of its ability to meet the future cost of annuitized income. "…investments that, on expectations, co-move closely with lifetime income (the investor's liability) is a form of liability-driven investing." The authors conclude: "…managing wealth variability and income variability are two different investment goals. Investments with low wealth volatility can have high income volatility and vice versa. The key to identifying the right investment strategy is deciding what matters most—wealth or lifetime income."	
2014	Agnieszka Karolina Konicz Bell, David Pisinger, and Alex Weissensteiner, "Optimal retirement planning with a focus on single and multilife annuities," Technical University of Denmark, Department of Management Engineering (April, 2014), pp. 1 – 31.	The paper proposes: "…an optimization model that helps individuals to invest their retirement savings." Investment choices include stocks, bonds, and annuities on both single and joint lives. The investment return process is modelled by means of a vector-autoregressive model with time varying investment opportunities. Scenario trees are used within a multi-stage stochastic programming approach. The annuities under consideration encompass a variety of contract types including both single and joint-life contracts, fixed and variable benefit payments, term and "whole life" annuities and guaranteed minimum term or life contingent payments. The analysis incorporates annuity M&E costs (estimated at 1%), annuity contract surrender charges (graded downwards according to a time-to-maturity schedule), and stock and bond transaction costs (1%). The investor's objective is "…to maximize the expected utility of consumption (for either a one-person or a two-person household) during the retirement period, given the individual's risk aversion, personal preferences, lifetime expectancy, and the bequest motive." The investor has a utility function for both consumption and bequests that assumes Constant Relative Risk Aversion. The time preference factor (impatience) is set to a constant .05. Each household maximizes expected total expected utility throughout the applicable planning horizon where the horizon is stochastic and depends on survival probabilities. Available annuity contracts include single and joint-life contracts (with	The analysis tracks the British retirement structure which compels complete, late-in-life, annuitization of pension accounts. In this article, the designated age for mandatory annuitization of remaining wealth is 85. The age-85 annuity is an immediate life-only contract providing constant nominal benefits for the remainder of life. The article develops and explores several interesting concepts: 1. The portfolio's asset allocation during the period from retirement onset to the age of mandatory annuitization is particularly interesting. The assets at retirement consist of stocks, bonds, a deferred "whole-life" annuity, and a sequence of "pure endowments" (with both fixed and variable maturity payments) with staggered 5-year maturity dates. The actuarial 'elements' are available as both single- or joint-life products. The portfolio elements are combined to replicate various payment patterns, where payments are either life contingent (i.e., a pure endowment), or guaranteed (i.e., a zero-coupon bond). By

		various percentage payouts to the survivor), whole life annuities (with and without guaranteed payment periods), term annuities, immediate or deferred annuities, and fixed or variable payment contracts. The article demonstrates how various combinations of actuarial instruments and securities can replicate an annuity contract's payout structure. For example, purchase of a 10-year guaranteed payment immediate life annuity is equivalent to purchase of (1) a sequence of 10-year zero-coupon bonds with maturities ranging from 1 to 10 years, and (2) a series of pure endowments (payments are life-contingent) thereafter with the first pure endowment maturity date set to year 11. The multi-stage model divides the planning period from retirement age 65 (single households or two-person households) through mandatory annuitization age of 85 into four stages each lasting 5 years. A decision tree structure spaces each tree node at 5-year intervals. During each discrete time interval, the model generates the joint structure of stock and bond returns. The portfolio is self-financing—i.e., there are no exogenous sources of income or consumption other than portfolio values. Unspent annuity income is used for new asset purchases. Negative positions are disallowed. The price of a zero-coupon bond is simply the discounted value of the proceeds at maturity date: $$price_t^b = \left(\frac{1}{1+r(t,M)}\right)^{M-t}$$ Where 'r' is the term structure of interest rates. The price of a pure endowment paying out conditionally at maturity date 'M' adjusts the price of a zero-coupon bond by multiplying it by a survival probability factor (based on British mortality tables). This adjustment, of course, lowers the cost of the pure endowment instrument. Joint pure endowment instruments are adjusted by joint survival probability over the relevant maturity horizons (5, 10, 15, and 20 years). Variable pay pure endowments are priced like pure fixed-pay endowments. Their payout, however, depends on the realized returns of securities within an	combining portfolio elements, the authors identify optimal asset allocations even if such allocations are not readily achievable by combining products sold in the retail financial marketplace. By combining actuarial elements, such as annuity contracts and pure endowments, with stocks and bonds, the multistage stochastic programming model produces optimal portfolios providing complex payment patterns that retirees cannot replicate. This leads the authors to conclude that retail products with greater payout flexibility should be offered to retirees. 2. Immediate annuitization of wealth at the beginning of retirement is sub-optimal. The model's outputs suggest that "bucketing" approaches should be avoided. If anything, the results imply a form of reverse-bucketing strategy focused on late-in-life annuity income.

		underlying portfolio. The evolution of the yield curve uses the Nelson/Siegel three-parameter model which calculates spot rates as a function of the level, slope, and curvature of the term structure of interest rates. Each parameter coefficient (β) is a component in a three-factor model and impacts the yield curve's shape and evolution. Time varying investment realizations are solved in a VAR(1) model in which the combined equity and bond returns form a 2x1 vector where this two-component vector [**ξ**] at time t-1 is included in the following expression for the VAR process: $\boldsymbol{\xi}_t = c + A\, \boldsymbol{\xi}_{t-1} + \mathbf{u}_t$ Where **c** is the vector of intercepts, **A** is the matrix of slope coefficients and **u** is a vector of i.i.d. innovations with zero mean and non-zero variance. Empirical return data indicate a stable process: "All modulus of the eigenvalues of the characteristic polynomial are below one." The steady-state expected annual return on equity equals 5.7%; the term structure of interest rates is increasing and concave, as expected. The decision tree structure starts with the unconditional expected values. Future annuity prices correspond with the yields extracted from the β coefficients. For both one and two-person households, the objective is "…to maximize the expected utility of consumption and bequest given uncertain lifetime." The results section focuses on a single female age 65. Although the decision tree nodes are set at five-year intervals, any available asset can be traded at any time prior to age 85. The authors solve for the optimal allocation given (1) various weightings on the consumption and bequest preferences, and (2) various risk-aversion parameters. Investors with lower risk aversion load for more risky assets and, therefore, increase consumption opportunities. In general, an investor gradually purchases a deferred annuity contract with payments beginning at age 85. Remaining portfolio amounts are distributed among stocks, a 15-year pure endowment, and zero-coupon bonds with long-dated maturities. Of some interest is the finding that "The investor	

		has no need to purchase an immediate whole life annuity; she finances consumption during the first 15 years of retirement from the capital gains, and afterwards from pure endowments and deferred life annuities." The stronger the bequest motive, the greater the allocation to stocks. When the choice of assets allows for purchase of variable payout pure endowments, all investments in stocks and bonds disappear in the absence of a bequest motive: "The return on these products is equal to the return on stocks plus the survival credit, while the volatility is the same as for stocks. This makes variable pure endowments an attractive investment asset, and consequently the optimal portfolio includes these assets with all possible maturities."	
2014	"On improving pension product design," Agnieszka Karolina Konicz and John M. Mulvey, 30th International Congress of Actuaries, 30 March – 4 April 2014 Washington, DC, pp. 1 – 27.	The authors "combine" a multi-period stochastic programming [MSP] approach along a scenario tree structure with a stochastic optimal control [SOC] approach which comes into play at the 'end of the tree.' The MSP covers the first several time steps and calculates the optimal solution at each step node; the SOC covers the remaining stochastic life span over the period from the final node to the end of the investor's life. The MSP observes the fully revealed information set (probability = 1) at node zero. Future evolutions of random variables (e.g., stock returns) are anticipated but, of course, the realizations are not yet known. Decisions (spending, rebalancing, asset allocation, etc.) are made in a vector of second-stage decisions, where the second-stage decisions depend on the realization of random variable vector xi [ξ]. Given the realized results obtained from the first-stage decision, the second-stage decision incorporates the updated information set (results plus additional insight into the process driving random variables) and seeks a decision vector that optimizes the next stage results. First-stage decisions are anticipative; second-stage decisions are adaptive. The authors point out that they can only push a MSP model so far. The tree-structure exponentially grows into a number of nodes that renders computations impossible. To remedy this difficulty, the model introduces an "end effect." The end effect is a SOC model which (1) covers the investor's remaining lifespan, and (2) makes a number of simplifying assumptions in the objective function (continuous time	This presentation is, in many respects, a companion piece to Agnieszka Karolina Konicz Bell, David Pisinger, and Alex Weissensteiner, "Optimal retirement planning with a focus on single and multilife annuities," Technical University of Denmark, Department of Management Engineering (April, 2014), pp. 1 – 31 which is also discussed in this bibliography. The article blends methods taken from operations research and actuarial science. It presents an interesting analysis of a Utility-based benefit of annuitization (immediate and deferred). Utility is a function of: (1) the sequence of realized investment returns; and, (2) the shape of the investor's optimal payout (withdrawal) curve as determined by impatience, risk aversion, mortality assumptions (subjective v. objective), health conditions, bequest motives, and expected portfolio return/volatility. Changes in these parameters have significant

		wealth evolutions under a standard Brownian motion process and martingale probability, no upper or lower bounds on the value attainable by variables, a risk-free rate, and a deterministic death benefit). Introducing these assumptions into the model allows the authors to formulate and solve for the Hamilton-Jacobi-Bellman value function's optimal controls—e.g., optimal allocation to risky assets, bequest targets, etc. [Mortality rates are calibrated to Danish mortality statistics]. The advice model incorporates information concerning the investor's economic circumstances (current wealth, savings contributions, and pension entitlements); personal preferences (risk aversion [CRRA], life expectancy, payout preferences, bequest motive, and asset allocation preferences). The authors briefly discuss scenario generation methods for stochastic programming along a tree structure—sampling, simulation, scenario reduction techniques, moment matching—and elect the moment-matching technique to match the first four moments and correlations of relevant underlying processes. At each node, both asset returns and associated probabilities are decision variables. Moving from node to node, optimal decisions are a function of decisions made at the previous-stage node and of future mortality and asset return realizations. Fees, expenses, and taxes are ignored. The optimal amount to invest at each node depends on risk aversion, updated information regarding market parameters, expected returns, volatilities and correlations, the dollar value of current wealth, and pension entitlements. An investor lacking any retirement pension should follow the Merton fixed-mix portfolio advice. As the pension entitlement grows, the proportion of wealth invested in risky assets declines. The optimal payout decision—assuming that objective and subjective mortality rates are equal—shows a flat payout curve if the 'impatience factor' equals the risk-free rate ($\rho = r$); front-loaded (i.e., decreasing) if $\rho > r$, and back-loaded (i.e., increasing) if $\rho < r$. The risk aversion parameter controls the slope of the payout curve. The authors	consequences for how an investor values the option to annuitize. The number and complexity of factors determining an investor's retirement income preferences are sometimes not sufficiently appreciated in the literature. The model structure—multi-period stochastic programming followed by an "end effect" period—bears interesting similarities to the 2011 model presented in "Asset-liability management under time-varying investment opportunities," Robert Ferstl and Alex Weissensteiner, <u>Journal of Banking & Finance</u> Vol. 35 (2011), pp. 182 – 192 which is also discussed in this bibliography.

		emphasize that there is no optimal payout structure. Rather, the investor must select the payout profile yielding the greatest personal utility. Differing levels of risk aversion impact asset allocation which, in turn, influences the amount of the expected payout. The study presents a matrix of payouts (optimal withdrawal rates) for differing values of risk aversion, impatience, and expected returns. Finally, differences between subjective and objective mortality expectations are important factors in determining the optimal payout. Payout curves are also analyzed for various annuity types under a variety of health states and bequest motives. For example, the study concludes that "…it is optimal for the person to invest in deferred life annuities only if the payout curve is decreasing. The optimal withdrawal rate is proportional to the probability of survival, therefore she should spend more savings in the beginning of retirement."	
			Updated October 6, 2017